L^p-bounds for pseudo-differential operators on graded Lie groups

Duván Cardona[1], Julio Delgado[3] and Michael Ruzhansky[1,2]

Department of Mathematics: Analysis, Logic and Discrete mathematics, and Ghent Analysis & PDE Center (Ghent-Belgium)[1]

School of Mathematical Sciences, Queen Mary University of London, (London-UK)[2]

Department of Mathematics, Universidad del Valle, (Cali-Colombia)[3]

Abstract

We present the sharp L^p-estimates for pseudo-differential operators on arbitrary graded Lie groups proved by the authors in [1]. The results are presented within the setting of the global symbolic calculus on graded Lie groups by using the Fourier analysis associated to every graded Lie group which extends the usual one due to Hörmander on $\mathbb{R}^n$. The main result extends the classical Fefferman's sharp theorem on the L^p-boundedness of pseudo-differential operators for Hörmander classes on $\mathbb{R}^n$ to general graded Lie groups, also adding the borderline $\rho = \delta$.

Introduction

- The investigation of the L^p boundedness of pseudo-differential operators is a crucial task for a large variety of problems in mathematical analysis and its applications, mainly due to its consequences for the regularity, approximation and existence of solutions on L^p-Sobolev spaces.
- There is an extensive literature on the subject, in particular, devoted to operators associated with symbols belonging to the Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, (see for instance, J.J. Kohn and L. Nirenberg [5], L. Hörmander [4] and C. Fefferman [2]).
- Our main goal is to extend a classical and sharp result by C. Fefferman [2] and to provide a critical order for the L^p-boundendess of pseudo-differential operators on graded Lie groups based on the quantization procedure developed by the third author and V. Fischer in [3].
- Our main estimate recover the sharp Fefferman theorem on $\mathbb{R}^n$, adding the critical case $\rho = \delta$.

Fourier analysis on nilpotent Lie groups

- Let G be a simply connected nilpotent Lie group and let $\widehat{G}$ be its unitary dual.
- Let $\exp_G : \mathfrak{g} \to G$, be the exponential mapping on G. The Schwartz class on G, is defined by those $f \in C^\infty(G)$, such that $f \circ \exp_G \in \mathscr{S}(\mathfrak{g})$, with $\mathfrak{g} \simeq \mathbb{R}^{\dim(G)}$.

  The Fourier transform of $f \in \mathscr{S}(G)$, at $\pi \in \widehat{G}$, is defined by:

$$\widehat{f}(\pi) = \int_G f(x)\pi(x)^* dx$$

Pseudo-differential conference

BE HARMONIC WITH ANALYSIS

$$\int_{\widehat{G}} \mathrm{Tr}(\pi(x)\sigma(x,\pi)\hat{f}(\pi))d\mu(\pi)$$

Make a difference: pseudo-differentiate

Integrate with mind but pseudo-differentiate with heart

Pseudo-differential operators on graded Lie groups

- Roughly speaking, a pseudo-differential operator is a continuous linear operator on $\mathscr{S}(G)$, defined by the (quantization) formula:

$$\mathrm{Op}(\sigma)f(x) = \int_{\widehat{G}} \mathrm{Tr}\left(\pi(x)\sigma(x,\pi)\hat{f}(\pi)\right) d\mu(\pi)$$

  In such a case, we say that σ is the symbol associated with $\mathrm{Op}(\sigma)$. We have denoted by $d\mu(\pi)$ the Plancherel measure on $\widehat{G}$.
- A Rockland operator is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree $\nu = \nu_{\mathcal{R}}$ and such that, for every unitary irreducible non-trivial representation $\pi \in \widehat{G}$, $\pi(\mathcal{R})$ is injective on $\mathcal{H}^\infty_\pi$; $\sigma_{\mathcal{R}}(\pi) = \pi(\mathcal{R})$ is the symbol associated to $\mathcal{R}$.
- It can be shown that a Lie group G is graded if and only if there exists a differential Rockland operator on G.
- The basic example of graded Lie group is the Heisenberg group $\mathbb{H}^n$.

Hörmander classes on graded Lie groups

Hörmander classes on the phase space $G \times \widehat{G}$ can be defined by using Rockland operators. Indeed, the Hörmander class of order m, and of type (ρ, δ), $S^m_{\rho,\delta}(G \times \widehat{G})$, is defined by those *symbols* σ satisfying symbol inequalities of the kind:

$$\sup_{x \in G, \pi \in \widehat{G}} \left\| \pi(I+R)^{\frac{\rho[\alpha]-m+\delta[\beta]+\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi)\, \pi(I+R)^{-\frac{\gamma}{\nu}} \right\|_{\mathcal{L}(\mathcal{H}_\pi)} < \infty$$

L^p-L^p, H^1-L^1 and L^∞-BMO-boundedness of pseudo-differential operators

Theorem (Fefferman type estimates on Graded Lie groups)

Let G be a graded Lie group of homogeneous dimension Q. Let $A \equiv Op(\sigma) : C^\infty(G) \to \mathscr{D}'(G)$ be a pseudo-differential operator with symbol $\sigma \in S^{-m}_{\rho,\delta}(G \times \widehat{G})$, $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$. Then,

- *(a) if $m = \frac{Q(1-\rho)}{2}$, then A extends to a bounded operator from $L^\infty(G)$ to $BMO(G)$, from the Hardy space $H^1(G)$ to $L^1(G)$, and from $L^p(G)$ to $L^p(G)$ for all $1 < p < \infty$.*
- *(b) If $m \geq m_p := Q(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$, $1 < p < \infty$, then A extends to a bounded operator from $L^p(G)$ into $L^p(G)$.*

References

[1] Cardona D. Delgado J. Ruzhansky M., *L^p-bounds for pseudo-differential operators on graded Lie groups.* arXiv:1911.03397

[2] Fefferman C., *L^p-bounds for pseudo-differential operators.* Israel J. Math. 14, 1973, 413–417.

[3] Fischer V. Ruzhansky M., *Quantization on nilpotent Lie groups.* Progress in Mathematics, 2016.

[4] Hörmander L., *The Analysis of the linear partial differential operators.* Springer-Verlag, 1985.

[5] Kohn J.J. Nirenberg L., *An algebra of pseudo-differential operators.* Commun. Pure and Appl. Math. 18, 1965, 269-305.

[6] Taylor M., *Pseudodifferential Operators.* Princeton Univ. Press, Princeton, 1981.

D. C. was supported by the FWO Odysseus Project. J. D. was supported by the Leverhulme Grant RPG-2017-151. M. R. was supported in parts by the FWO Odysseus Project and also by the Leverhulme Grant RPG-2017-151 and EPSRC grant EP/R003025/1.