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P D E

Think fractional, gain integral

# INTERNATIONAL CONFERENCE ON FRACTIONAL CALCULUS 9-10 JUNE 2020

Nasser AL-SALTI (Oman)

Ravshan ASHUROV (Uzbekistan)

Allaberen ASHYRALYEV (Turkey)

Teodor M. ATANACKOVIĆ (Serbia)

Jaan JANNO (Estonia)

Erkinjon KARIMOV (Uzbekistan)

Virginia KIRYAKOVA (Bulgaria)

Anatoly KOCHUBEI (Ukraine)

Francesco MAINARDI (Italy)

Stevan PILIPOVIĆ (Serbia)

Arsen PSKHU (Russia)

Michael RUZHANSKY (Belgium, and UK)

Marián SLODIČKA (Belgium)

Živorad TOMOVSKI (Czech Republic)

Berikbol TOREBEK (Belgium)

Sabir UMAROV (USA)

Enrico VALDINOCI (Australia)

Masahiro YAMAMOTO (Japan)



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## Workshop: Fractional Calculus

*Think fractional, gain integral*



$$\rightarrow {}^{RL}_0 D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tilde{t} = \underbrace{\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tilde{t}}_{= {}^C_0 D_t^\alpha f(t)} + \frac{f(0)}{\Gamma(1-\alpha)}$$

- Liouville  
derivative

↑ Euler gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

$$= {}^C_0 D_t^\alpha f(t)$$

← Caputo fractional  
derivative

Set  $u^{(k)} := {}^{RL}_0 D_t^\alpha u$  and consider equation  $u^{(k)} + u = f$ .  
 Apply the Laplace transform  $\Rightarrow (s^\alpha + 1) \mathcal{L}u = \mathcal{L}f$   
 $\mathcal{L}(u^{(k)}) = s^\alpha \mathcal{L}u$   
 $\mathcal{L}u = \frac{1}{s^\alpha + 1} \mathcal{L}f$   
 Inverse Laplace transform  $\Rightarrow$

Mittag-Leffler 1 parameter  
function

$$e_{\alpha,1}(t) = E_{\alpha,1}(-t^\alpha)$$

where

$$\frac{1}{z^k}$$

$$u = -e'_{\alpha,1} * f$$

**TUESDAY 9. JUNE**



# FRANCESCO MAINARDI

(UNIVERSITY OF BOLOGNA, ITALY)

*Fractional Calculus: What is it?  
What is for?*

*Tuesday, 9. June  
10:05-11:00*



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## Workshop: Fractional Calculus

*Think fractional, gain integral*

**Francesco Mainardi**  
**Fractional Calculus: What Is It? And What Is It for?**

**Abstract:** In 1695 L'Hospital inquired to Leibniz what meaning could be given to the symbol  $d^n y/dx^n$  when  $n = 1/2$ . In a letter dated September 30, 1695 Leibniz replied "It will lead to a paradox, from which one day useful consequences will be drawn". This discussion led to a new branch of mathematics which deals with derivatives and integrals of arbitrary order and is known as Fractional Calculus. Of course this is a misnomer kept only for historical reasons. It can be considered as a branch of mathematical analysis which deals with integro-differential operators and equations where the integrals are of convolution type and exhibit (weakly singular) kernels of power-law type. It is strictly related to the theory of pseudo-differential operators. Fractional differential and integral equations have gained considerable popularity and importance during the past three decades. The main advantage of the fractional calculus is that provides excellent instruments for the description of memory and non local properties of various materials and processes. The list of applications is huge and includes, just to cite a few, Visco-elasticity, Electrical Circuits, Control theory, intermediate phenomena between Diffusion and Wave propagation, Biology, Bioengineering, Image processing, Finance, Stochastic processes.....



# ENRICO VALDINOCI

(THE UNIVERSITY OF WESTERN AUSTRALIA)



*Nonlocal logistic equations with  
Neumann conditions*

*Tuesday, 9. June  
11:05-11:50*



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## Workshop: Fractional Calculus

*Think fractional, gain integral*

**Enrico Valdinoci**

**Nonlocal logistic equations with Neumann conditions**

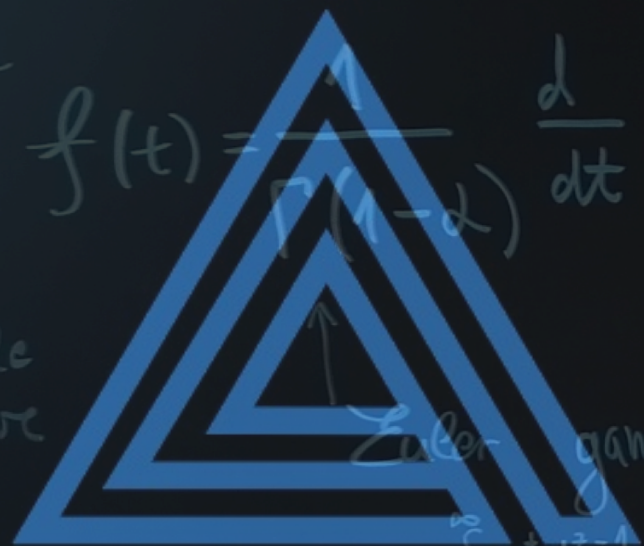
**Abstract:** We consider a problem of population dynamics modeled on a logistic equation with both classical and nonlocal diffusion, possibly in combination with a pollination term. The environment considered is a niche with zero-flux, according to a new type of Neumann condition. We discuss the situations that are more favorable for the survival of the species, in terms of the first positive eigenvalue. The eigenvalue analysis for the one dimensional case is structurally different than the higher dimensional setting, and it sensibly depends on the nonlocal character of the dispersal. We also analyze the role played by the optimization strategy in the distribution of the resources, also showing concrete examples that are unfavorable for survival, in spite of the large resources that are available in the environment.



## Workshop: Fractional Calculus

*Think fractional, gain integral*

10 min break:  
Presentation Book  
of abstract



# GHENT ANALYSIS PDE

THINK FRACTIONAL  
GAIN INTEGRAL

Laplace transform

Inverse Laplace transform

Mittag-Leffler 1 parameter function

$${}^{\text{RL}}_0 D_t^\alpha f(t) = \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau = \frac{f'(t)}{\Gamma(1-\alpha)} + \frac{f(0)}{\Gamma(1-\alpha)}$$

$$= {}^{\text{Caputo}}_0 D_t^\alpha f(t)$$

Set  $u^{(\alpha)} := {}^{\text{RL}}_0 D_t^\alpha u$  and consider equation  $u^{(\alpha)} + u = f$

Apply the Laplace transform  $\Rightarrow (s^\alpha + 1) \tilde{u} = \tilde{f}$

$\tilde{u} = \frac{1}{s^\alpha + 1} \tilde{f}$

$\Rightarrow u = \tilde{L}^{-1} \left( \frac{1}{s^\alpha + 1} \tilde{f} \right)$

Euler gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

$${}^{\text{RL}}_0 D_t^\alpha f(t) = \frac{d}{dt} \int_0^t \frac{f(\tau)}{\Gamma(1-\alpha)} d\tau$$

Liouville derivative

Caputo derivative



# ZIVORAD TOMOVSKI

University Ostrava, Czech Republic

## *Hilfer-Prabhakar derivatives and their applications*

*Tuesday, 9. June  
12:00-12:30*



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## Workshop: Fractional Calculus

*Think fractional, gain integral*

**Živorad Tomovski**  
**Hilfer-Prabhakar derivatives and their applications**

**Abstract:** We analyze and discuss properties of Hilfer-Prabhakar derivatives and compare them with some "new" definitions of fractional derivatives. Further, we show some applications of these generalized Hilfer-Prabhakar derivatives in classical equations of mathematical physics, like the heat and the difference-differential equations governing the dynamics of generalized renewal stochastic processes.



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**Workshop: Fractional Calculus**

*Think fractional, gain integral*



# MICHAEL RUZHANSKY AND BERIKBOL TORENBEEK



*Van der Corput lemmas  
for Mittag-Leffler  
functions*

*12:30 – 13:00*



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## Workshop: Fractional Calculus

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**Michael Ruzhansky and Berikbol Torebek**  
**Van der Corput lemmas for Mittag-Leffler functions**

**Abstract** In this paper, we study analogues of the van der Corput lemmas involving Mittag-Leffler functions. The generalisation is that we replace the exponential function with the Mittag-Leffler-type function, to study oscillatory type integrals appearing in the analysis of time-fractional partial differential equations. Several generalisations of the first and second van der Corput lemmas are proved. As an application of the above results, the generalised Riemann-Lebesgue lemma and the Cauchy problem for the time-fractional Schrödinger equation are considered.



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**Workshop: Fractional Calculus**

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Lunch break: 13:00 – 15:00

Presentation of GAP



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**T H I N K F R A C T I O N A L**

**G A I N I N T E G R A L**



**TEODOR M. ATANACKOVIC**

(UNIVERSITY OF NOVI SAD, SERBIA)

*On the Thermodynamical  
restrictions  
in isothermal deformations*

*Tuesday, 9 June  
15:00 – 15:45*



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**Workshop: Fractional Calculus**

*Think fractional, gain integral*

**Teodor M. Atanacković**  
**On the Thermodynamical restrictions in isothermal deformations**

**Abstract:** We investigate restrictions on a constitutive equation for a fractional viscoelastic bodies that follow from the weak form of Entropy inequality under isothermal conditions. Our results extend the ones obtained by Bagley-Torvik. We present restrictions for several types of constitutive equations.

# STEVAN PILIPOVIC

(UNIVERSITY OF NOVI SAD, SERBIA)

*Some mathematical methods in  
fractional calculus*

*Tuesday, 9 June  
15:45 – 16:30*



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## Workshop: Fractional Calculus

*Think fractional, gain integral*

**Stevan Pilipović**  
**Some mathematical methods in fractional calculus**

**Abstract:** Our main results are related to the solvability, in the weak sense, that is, in the framework of distribution theory, of a Cauchy problem which corresponds to a Zener model. The solvability and the regularity of a solution is the essential part of the talk. Next, we allow a stochastic excitation in the body force term  $f$  and give several examples which illustrate such perturbations.



# VIRGINIA KIRYAKOVA

(BULGARIAN ACADEMY OF SCIENCES, BULGARIA)

*The multi-index Mittag-Leffler functions  
as Special Functions of Fractional  
Calculus: Examples and Applications*

*Tuesday, 9 June*

*16:30-17:00*



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## Workshop: Fractional Calculus

*Think fractional, gain integral*



**Virginia Kiryakova**

**The multi-index Mittag-Leffler functions as Special Functions of Fractional Calculus: Examples and Applications**

**Abstract** The developments in theoretical and applied science require knowledge of the properties of mathematical functions, from elementary trigonometric functions to the multitude of Special Functions (SF), appearing whenever natural and social phenomena are studied, engineering problems are formulated, and numerical simulations are performed. This survey aims to attract attention to classes of SF that were not so popular (or some of them not introduced) until Fractional Calculus (FC) gained the important role, related to the boom of applications of fractional order models. The so-called **Special Functions of Fractional Calculus (SF of FC)** are basically Fox H-functions, among them are the generalized Wright hypergeometric functions  ${}_p\Psi_q$  and in particular, the classical Mittag-Leffler (M-L) functions and their various extensions. The SF of FC are now unavoidable tool in solutions of fractional order integral and differential equations and systems.



**Workshop: Fractional Calculus**

*Think fractional, gain integral*

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau = \underbrace{\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau}_{= {}_0^C D_t^\alpha f(t)} + \frac{f(0)}{\Gamma(1-\alpha)}$$

Liouville derivative

Euler gamma-function

Caputo fractional derivative

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Set  $u^{(\alpha)} := {}_0^C D_t^\alpha u$  and consider equation  $u^{(\alpha)} + u = f$

Apply the Laplace transform  $\Rightarrow (s^\alpha + 1) \mathcal{L}u = \mathcal{L}f$

$$\mathcal{L}u = \frac{1}{s^\alpha + 1} \mathcal{L}f$$

Inverse Laplace transform  $\Rightarrow$

$$u = \mathcal{L}^{-1} \left( \frac{1}{s^\alpha + 1} \right) * f$$

Mittag-Leffler 1 parameter function

$$e_{\alpha,1}(t) = E_{\alpha,1}(-t^\alpha)$$

$$u = -e_{\alpha,1} * f$$

WEDNESDAY, 10. JUNE

# ANATOLY KOCHUBEI

(INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY  
OF SCIENCES OF UKRAINE)

*Analysis on radial complex-valued  
functions of  $p$ -adic arguments*

*Wednesday, 10 June*

*10:00 – 10:45*



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## Workshop: Fractional Calculus

*Think fractional, gain integral*

**Anatoly Kochubei**

**Analysis on radial complex-valued functions of p-adic arguments**

**Abstract:** In an earlier paper (A. N. Kochubei, Pacif. J. Math. 269 (2014), 355--369), the author considered a restriction of Vladimirov's fractional differentiation operator  $D^\alpha$ ,  $\alpha > 0$ , to radial functions on a non-Archimedean field. In particular, it was found to possess such a right inverse  $I^\alpha$  that the change of an unknown function  $u = I^\alpha v$  reduces the Cauchy problem for an equation with  $D^\alpha$  (for radial functions) to an integral equation whose properties resemble those of classical Volterra equations. In other words, we found, in the framework of non-Archimedean pseudo-differential operators, a counterpart of ordinary differential equations. Later we studied nonlinear equations of this kind, found conditions of their local and global solvability. In this talk, I plan to give an introduction to the above research area, the p-adic fractional calculus.



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## Workshop: Fractional Calculus

*Think fractional, gain integral*



# MASAHIRO YAMAMOTO

(THE UNIVERSITY OF TOKYO, JAPAN)

*Some inverse problems for  
fractional diffusion equations*

*Wednesday, 10 June*

*10:45 – 11:30*



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## Workshop: Fractional Calculus

*Think fractional, gain integral*

**Masahiro Yamamoto**  
**Some inverse problems for fractional diffusion equations**

**Abstract:** I will present recent results on the uniqueness and the stability results for inverse problems for fractional equations, which have been obtained by me and my colleagues.

10 min break  
Book of abstracts



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GAIN INTEGRAL

# RAVSHAN ASHUROV

(INSTITUTE OF MATHEMATICS, UZBEKISTAN)



# SABIR UMAROV

(UNIVERSITY OF NEW HAVEN, USA)



*Determination of the order  
of fractional derivative for  
subdiffusion equations*

*Wednesday, 10 June*

*11:40 – 12:25*



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## Workshop: Fractional Calculus

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**Ravshan Ashurov and Sabir Umarov**  
**Determination of the order of fractional derivative for subdiffusion equations**

**Abstract:** In this talk, we consider an inverse problem for determining the order of time fractional derivative in a subdiffusion equation with an arbitrary second order elliptic differential operator. We prove that the additional information about the solution at a fixed time instant at a monitoring location, as "the observation data", identifies uniquely the order of the fractional derivative.

# ALLABEREN ASHYRALYEV

(NEAR EAST UNIVERSITY, TURKEY)

*Bounded Solutions of Semilinear  
Fractional Schrödinger  
Differential and Difference  
Equations*

*Wednesday, 10 June*

*12:25 – 13:10*



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## Workshop: Fractional Calculus

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**Allaberen Ashyralyev**  
**Bounded Solutions of Semilinear Fractional Schrodinger Differential  
and Difference Equations.**

**Abstract:** This is a discuss, the initial value problem for semilinear fractional Schordinger integro-differential equation

$$i \frac{du}{dt} + Au = \int_0^t f(s, D_s^\alpha u(s)) ds, 0 < t < T, u(0) = 0$$

is considered in a Hilbert space  $H$  with a self-adjoint positive definite operator  $A$ . A First and second order of accuracy difference schemes for the approximate solution of differential problem are presented. Theorems on existence and uniqueness of the bounded solutions of these semilinear Schrodinger differential and difference problems are established. In practice, existence and uniqueness theorems for a bounded solution of the initial boundary value one-dimensional problem with nonlocal condition and multi-dimensional problem with local condition on the boundary are proved. Numerical results and explanatory illustrations are presented on one and multi-dimensional problems to show the validation of the theoretical results.



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**Workshop: Fractional Calculus**

*Think fractional, gain integral*

Lunch break: 13:00 – 15:00

Presentation of GAP



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# JAAK JANNU

(TALLINN UNIVERSITY OF TECHNOLOGY, ESTONIA)

*Reconstruction of time-dependent  
sources and scalar parameters of  
fractional diffusion equations from final  
measurements*

*Wednesday, 10 June  
15:00 – 15:30*



300 x 449



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## Workshop: Fractional Calculus

*Think fractional, gain integral*



**Jaan Janno**

**Reconstruction of time-dependent sources and scalar parameters of fractional diffusion equations from final measurements**

**Abstract:** In this talk, We consider the time fractional diffusion equation with the time derivative of the order  $\beta \in (0,1)$ , the diffusion coefficient  $\lambda > 0$  and the source function of the form  $f(x)q(t)$ . We complement the equation by initial, boundary and final conditions and formulate the following inverse problems:

- 1) determine  $q(t)$ ;
- 2) determine  $\beta$  and  $q(t)$ ;
- 3) determine  $\lambda$  and  $q(t)$ ;
- 4) determine  $\beta$ ,  $\lambda$  and  $q(t)$ .

We establish sufficient conditions of uniqueness for these problems.

# ARSEN PSKHU

(KABARDIN-BALKAR SCIENTIFIC CENTER OF RUSSIAN ACADEMY OF SCIENCES, RUSSIA)

*Nakhushev extremum principle for a  
class of integro-differential operators*

*Wednesday, 10 June  
15:30 – 16:00*



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## Workshop: Fractional Calculus

*Think fractional, gain integral*

**Arsen Pshku**  
**Nakhushev extremum principle for a class of integro-differential operators.**

**Abstract:** We prove a principle of extremum for a class of integro-differential operators with a kernel of a general form. This statement generalizes an analogue of Fermat's extremum theorem for the Riemann-Liouville fractional derivative, established by Adam Nakhushev in the 70s of the last century. As corollaries we formulate extremal properties of some integro-differential operators of fractional calculus.



10 min break

**CHEN T**

**ANALYSIS**

**PDE**

**THINK FRACTIONAL**

**GAIN INTEGRAL**

$f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left( \frac{f(\tau)}{(t-\tau)^\alpha} \right) d\tilde{t} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau + \frac{f(0)}{\Gamma(1-\alpha)}$

$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

Euler gamma function

${}_0^C D_t^\alpha f(t)$  Caputo fractional derivative

Set  $u^{(2)} := {}_0^L D_t^\alpha u$  and consider equation  $u^{(2)} + u = f$

Apply the Laplace transform  $\Rightarrow (s^\alpha + 1) \mathcal{L}u = \mathcal{L}f$

$\mathcal{L}u = \frac{1}{s^\alpha + 1} \mathcal{L}f$

Inverse Laplace transform  $\Rightarrow$

Mittag-Leffler 1 parameter function

$u = \mathcal{L}^{-1} \left( \frac{1}{s^\alpha + 1} \right) * f$

Laplace transform

**NASSER AL-SALTI**

(SULTAN QABOOS UNIVERSITY, OMAN)



**ERKINJON KARIMOV**

(INSTITUTE OF MATHEMATICS, UZBEKISTAN)



*Inverse source problems  
for degenerate fractional  
PDEs*

*Wednesday, 10 June*

*16:10 – 16:40*



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**Workshop: Fractional Calculus**

*Think fractional, gain integral*

**Nasser Al-Salti and Erkinjon Karimov**  
**Inverse Source Problems for Degenerate Time-Fractional PDE**

**Abstract** In this talk, we present two inverse source problems for degenerate time-fractional partial differential equations in rectangular domains. The first problem involves a space-degenerate partial differential equation and the second one involves a time-degenerate partial differential equation. Solutions to both problems are expressed in series expansions. For the first problem, we obtained solutions in the form of Fourier-Legendre series. Convergence and uniqueness of solutions have been discussed. Solutions to the second problem are expressed in the form of Fourier-Sine series and they involve a generalized Mittag-Leffler type function. Moreover, we have established a new estimate for this generalized Mittag-Leffler type function. The obtained results are illustrated by providing example solutions using certain given data at the initial and final times.



# MARIÁN SLODIČKA

(Ghent University, Belgium)

*Some inverse source problems  
in semilinear fractional PDEs*

*Wednesday, 10 June*

*16:40 – 17:10*



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## Workshop: Fractional Calculus

*Think fractional, gain integral*

**Marián Slodička**

**Some inverse source problems in semilinear fractional PDEs**

**Abstract:** The talk deals with some inverse source problems in time fractional evolutionary partial differential equations.

First, a short introduction to inverse source problems will be given.

We discuss the appropriate choice of the side condition needed for the recovery of a solely space dependent source.

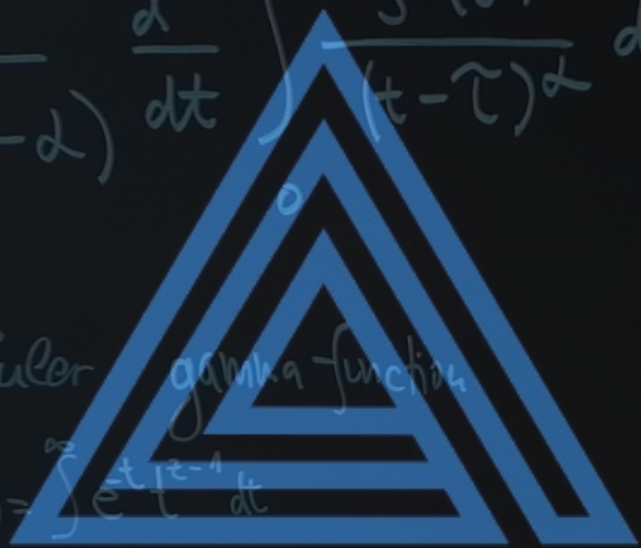
This will be explained on a simple 1D-example (heat equation) and the corresponding spectral analysis. It will be shown that a single point measurement of a solution could lead to multiple solutions.

The second part of the lecture is an introduction to fractional derivatives and positive definite convolution kernels.

Finally, some inverse source problems for time-fractional (parabolic and/or hyperbolic) partial differential equations will be addressed.



Closing



$f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau$

$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

Euler gamma function

${}_0^C D_t^\alpha f(t)$

Caputo fractional derivative

**CHENT**

**ANALYSIS**

**PDE**

Set  $u^{(k)} := {}_0^R D_t^k u$  and consider equation  $u^{(k)} + u = f$

Apply the Laplace transform  $\Rightarrow (s^k + 1) \mathcal{L}u = \mathcal{L}f$

$\mathcal{L}(u^{(k)}) = s^k \mathcal{L}u$

Laplace transform

**THINK FRACTIONAL**

**GAIN INTEGRAL**

Inverse Laplace transform  $\Rightarrow$

Mittag-Leffler 1 parameter function

$u = \mathcal{L}^{-1} \left( \frac{1}{s^k + 1} \right) * f$