The Harmonic Oscillator on The Heisenberg Group

David Rottensteiner[1] and Michael Ruzhansky[1,2]

Department of Mathematics: Analysis, Logic and Discrete mathematics, and Ghent Analysis & PDE Center (Ghent-Belgium)[1]
School of Mathematical Sciences, Queen Mary University of London (London-UK)[2]

Abstract

We present a notion of harmonic oscillator on the Heisenberg group $\mathbf{H}_n$. This operator forms a natural analogue of the harmonic oscillator on $\mathbb{R}^n$.

Introduction

Our ansatz is based on a few reasonable assumptions: the harmonic oscillator on $\mathbf{H}_n$ should be a negative sum of squares of operators related to the sub-Laplacian on $\mathbf{H}_n$, essentially self-adjoint with purely discrete spectrum, and its eigenvectors should be smooth functions and form an orthonormal basis of $L^2(\mathbf{H}_n)$. This leads to a differential operator on $\mathbf{H}_n$ which is determined by the Dynin-Folland Lie algebra, a stratified 3-step nilpotent Lie algebra.

Ansatz

Our approach is motivated by the following three realizations of the classical harmonic oscillator $\mathcal{Q}_{\mathbb{R}^n}$ on $\mathbb{R}^n$:

- (R1) the negative sum of squares $-\Delta + |x|^2$ of partial derivatives of order 1 and coordinate multiplication operators;
- (R2) the Weyl and Kohn-Nirenberg quantizations on $\mathbb{R}^n$ of the symbol $\sigma(x,\xi) := |\xi|^2 + |x|^2$ with $x, \xi \in \mathbb{R}^n$;
- (R3) the image $d\rho_1(-\mathcal{L}_{\mathbf{H}_n})$ of the negative sub-Laplacian $-\mathcal{L}_{\mathbf{H}_n}$ on $\mathbf{H}_n$ under the infinitesimal Schrödinger representation $d\rho_1$ (of Planck's constant equal to 1) of the Heisenberg Lie algebra $\mathfrak{h}_n$.

The Schrödinger representation ρ_1 of $\mathbf{H}_n$ acting on $L^2(\mathbb{R}^n)$ and the associated Lie algebra representation, naturally acting on $\mathcal{S}(\mathbb{R}^n)$, clearly relate each of the realisations (R1) - (R3) to the others. One can expect that similar realisations should be available for the canonical harmonic oscillator on $\mathbf{H}_n$.

Main Result

The harmonic oscillator $\mathcal{Q}_{\mathbf{H}_n}$ on the Heisenberg group $\mathbf{H}_n$ has a purely discrete spectrum $\operatorname{spec}(\mathcal{Q}_{\mathbf{H}_n}) \subset (0,\infty)$. The number of its eigenvalues, counted with multiplicities, which are less or equal to $\lambda > 0$ is asymptotically (as $\lambda \to \infty$) given by

$$N(\lambda) \sim \lambda^{\frac{6n+3}{2}},$$

and the magnitude of the eigenvalues is asymptotically equal to

$$\lambda_k \sim k^{\frac{2}{6n+3}} \text{ for } k = 1, 2, \ldots .$$

Moreover, the eigenvectors of $\mathcal{Q}_{\mathbf{H}_n}$ are in $\mathcal{S}(\mathbf{H}_n)$ and form an orthonormal basis of $L^2(\mathbf{H}_n)$.

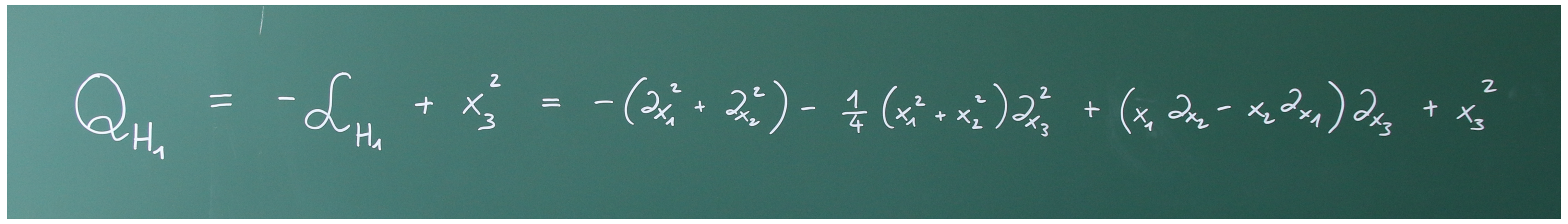

Figure 1: The Harmonic Oscillator on $\mathbf{H}_1$.

Definition

The Dynin-Folland Lie group $\mathbf{H}_{n,2} = \mathbb{R}^{2n+2} \rtimes \mathbf{H}_n$ acts on $f \in L^2(\mathbf{H}_n)$ by the unitary irreducible representation

$$\big(\pi(z,y,x)f\big)(t) = e^{iz} e^{i\langle t \cdot \frac{1}{2}x, y\rangle} f(t \cdot x),$$

where $t \cdot \frac{1}{2}x$ and $t \cdot x$ denote the $\mathbf{H}_n$-group products of the corresponding coordinate vectors.
For the basis $\{X_1, \ldots, Y_{2n+1}, Z\}$ of its Lie algebra $\mathfrak{h}_{n,2}$ we define the **harmonic oscillator** on $\mathbf{H}_n$ by

$$\mathcal{Q}_{\mathbf{H}_n} := d\pi\big(-\mathcal{L}_{\mathbf{H}_{n,2}}\big) = -d\pi\big(X_1\big)^2 - \ldots - d\pi\big(X_{2n}\big)^2 - d\pi\big(Y_{2n+1}\big)^2,$$

where $-\mathcal{L}_{\mathbf{H}_{n,2}}$ is the sub-Laplacian on $\mathbf{H}_{n,2}$. Its natural domain includes the smooth vectors $\mathcal{H}_\pi^\infty \cong \mathcal{S}(\mathbf{H}_n)$.

Interpretation

The essentially self-adjoint differential operator $\mathcal{Q}_{\mathbf{H}_n}$ on $\mathbf{H}_n$ admits analogues of (R1) – (R3):

- (R1') the differential operator $-\mathcal{L}_{\mathbf{H}_n} + x_{2n+1}^2$;
- (R2') i) the Dynin-Weyl quantization on $\mathbf{H}_n$ of the symbol $\sigma(x,\xi) := \xi_1^2 + \cdots + \xi_{2n}^2 + x_{2n+1}^2$ with $(x,\xi) \in \mathbb{R}^{2n+1} \times \widehat{\mathbb{R}}^{2n+1}$;
  ii) the Kohn-Nirenberg quantization, in the sense of [3], of the operator-valued symbol on $\mathbf{H}_n \times \widehat{\mathbf{H}}_n$ $\sigma(x,\rho_\lambda) := -\rho_\lambda(X_1)^2 - \ldots - \rho_\lambda(X_{2n})^2 + x_{2n+1}^2$;
- (R3') the image $d\pi(-\mathcal{L}_{\mathbf{H}_{n,2}})$ of the sub-Laplacian $\mathcal{L}_{\mathbf{H}_{n,2}}$ on $\mathbf{H}_{n,2}$, here by definition.

Methods

$\mathcal{Q}_{\mathbf{H}_n}$ has purely discrete spectrum in $(0,\infty)$ by [5]. The asymptotic growth rate of its eigenvalues is obtained via a powerful method developed in [8]. The number of eigenvalues (with multiplicities), asymptotically behaves like the volumes of certain subsets of the coadjoint orbit $\mathcal{O}_\pi \subset \mathfrak{h}_{n,2}^*$ corresponding to representation $\pi \in \widehat{\mathbf{H}}_n$. The subsets in question are determined (up to a multiplicative constant) by a (any) homogeneous quasi-norm on $\mathfrak{g}^*$. The flatness of $\mathcal{O}_\pi$ and a convenient choice of quasi-norm facilitate the computations substantially.

Remark

The power $\frac{6n+3}{2}$ intimately related to the homogeneous structure of $\mathfrak{h}_{n,2}$: the nominator $6n+3$ is the homogenous dimension of the first two strata $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \subseteq \mathfrak{h}_{n,2}$, while the denominator 2 is the homogeneous degree of $-\mathcal{L}_{\mathbf{H}_{n,2}}$.

References

[1] Alexander S. Dynin.
Pseudodifferential operators on the Heisenberg group.
Dokl. Akad. Nauk SSSR, 225:1245–1248, 1975.

[2] Véronique Fischer, David Rottensteiner, and Michael Ruzhansky.
Heisenberg-Modulation Space on the Crossroads of Coorbit Theory and Decomposition Space Theory.
Preprint, 2018.
https://arxiv.org/abs/1812.07876.

[3] Veronique Fischer and Michael Ruzhansky.
Quantization on nilpotent Lie groups, volume 314 of *Progress in Mathematics*.
Birkhäuser/Springer, [Cham], 2016.

[4] Gerald B. Folland.
Meta-Heisenberg groups.
In *Fourier analysis (Orono, ME, 1992)*, volume 157 of *Lecture Notes in Pure and Appl. Math.*, pages 121–147. Dekker, New York, 1994.

[5] A. Hulanicki, J. W. Jenkins, and J. Ludwig.
Minimum eigenvalues for positive, Rockland operators.
Proc. Amer. Math. Soc., 94(4):718–720, 1985.

[6] David Rottensteiner and Michael Ruzhansky.
Harmonic and Anharmonic Oscillators on the Heisenberg Group.
Preprint, 2018.
https://arxiv.org/abs/1812.09620.

[7] David Rottensteiner and Michael Ruzhansky.
The Harmonic Oscillators on the Heisenberg Group.
Preprint. To appear in C. R. Math. Acad. Sci. Paris., 2020.
https://arxiv.org/abs/2005.12095.

[8] A. F. M. ter Elst and Derek W. Robinson.
Spectral estimates for positive Rockland operators.
In *Algebraic groups and Lie groups*, volume 9 of *Austral. Math. Soc. Lect. Ser.*, pages 195–213. Cambridge Univ. Press, Cambridge, 1997.

Michael Ruzhansky was supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations, by the EPSRC grant EP/R003025/1 and by the Leverhulme Grant RPG-2017-151.

David Rottensteiner was supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations and the Austrian Science Fund (FWF) project [I 3403].

Contact Information

- https://analysis-pde.org/david-rottensteiner/
- https://analysis-pde.org/pseudo-differential-conference/