Smoothing and Strichartz estimates for Degenerate Schrödinger Operators

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Abstract

In what follows we shall present some recent results about the validity of smoothing and Strichartz-type estimates for time-degenerate Schrödinger operators. These results have important applications in the study of the local well-posedness of the initial value problem (IVP) associated with the operators under consideration.

Time degenerate Schrödinger-type Operators

We shall consider the following classes of degenerate Schrödinger-type operators

$$\mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x + b(t,x)\cdot\nabla_x, \tag{1}$$

$$\mathcal{L}_c = i\partial_t + c(t)\Delta_x, \tag{2}$$

where $\alpha > 0$, $b(t,x) = (b_1(t,x), ..., b_n(t,x))$ is such that, for all $j = 1, ..., n$, $b_j \in C([0,T], C_b^\infty(\mathbb{R}^n))$, while $c \in C^\infty(\mathbb{R}^n)$.

The class (1) was considered in [1] where both homogeneous and inhomogeneous weighted local smoothing estimates are derived. These estimates are also employed to obtain local well-posedness results for the associated nonlinear IVP (see [1]).

The class (2) was studied in [2] where global weighted homogeneous smoothing estimates are proved by means of comparison principles. For the class (2) weighted Strichartz estimates are proved in [2] where the application to the local well-posedness of the semilinear IVP is given.

The main difference between the operators of the form (1) and (2) and the other Schrödinger operators studied so far **is the presence of degeneracies**. Specifically, the degeneracies are given by the coefficient t^α and $c(t)$ in (1) and (2) respectively.

Why to study smoothing and Strichartz estimates?

These estimates give us important information about the regularity properties of the solution of the IVP. In particular:

- **The homogeneous smoothing** effect describes a gain of smoothness of the homogeneous solution of the IVP with respect to the smoothness of the initial data.
- **The inhomogeneous smoothing** effect describes a gain of smoothness of the solution of the inhomogeneous IVP with respect to the regularity of the inhomogeneous data.
- **Strichartz estimates** describe a gain of integrability instead of a gain of smoothness of the solution of the IVP.

Additionally, **these estimates are fundamental in order to prove the local well-posedness of the corresponding semilinear and nonlinear IVP** through the standard fixed point argument.

What are comparison principles? (Following [3])

Question: Given two operators $P_a(t,x,D_t,D_x)$ and $P_b(t,x,D_t,D_x)$ depending on two functions a and b respectively, is it possible to compare (in a suitable sense) the solutions of the HIVP (homogeneous IVP) for P_a and P_b if a and b are comparable (in a suitable sense)?

This is essentially what the **comparison principles** we refer to do, that is, **they translate a relation between a and b in a relation between the solutions of the HIVP for P_a and P_b.**

Example (see [3]). Let $a, \tilde{a} \in C^1(\mathbb{R})$ be real valued and strictly monotone on the support of a measurable function χ, and let $\sigma, \tau \in C^0(\mathbb{R})$. Then, if $\forall \xi \in \mathrm{supp}\chi$ we have

$$\frac{|\sigma(\xi)|}{|a'(\xi)|^{1/2}} \le C\frac{|\tau(\xi)|}{|\tilde{a}'(\xi)|^{1/2}}, \quad \text{then} \quad \|\chi(D_x)\sigma(D_x)e^{ita(D)}\varphi\|_{L^2(\mathbb{R}^2)} \le C\|\chi(D_x)\tau(D_x)e^{it\tilde{a}(D)}\varphi\|_{L^2(\mathbb{R}^2)}$$

for all $\varphi = \varphi(x)$ smooth enough.

Weighted local smoothing effect for $\mathcal{L}_\alpha$

We consider the IVP

$$\begin{cases} \partial_t u = it^\alpha \Delta_x u + ib(t,x)\cdot\nabla_x u + f(t,x) \\ u(0,x) = u_0(x). \end{cases} \tag{3}$$

When $b \equiv 0$ one can proceed by using Fourier analysis. However, in the general case $b \not\equiv 0$, the use of pseudo-differential calculus is needed.

Theorem (F.-Staffilani)

Let $u_0 \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Assume that, for all $j = 1, ..., n$, b_j is such that $b_j \in C([0,T], C_b^\infty(\mathbb{R}^n))$ and there exists $\sigma > 1$ such that

$$|\mathrm{Im}\, \partial_x^\gamma b_j(t,x)|, |\mathrm{Re}\, \partial_x^\gamma b_j(t,x)| \lesssim t^\alpha \langle x\rangle^{-\sigma-|\gamma|}, \quad x \in \mathbb{R}^n, \tag{4}$$

and denote by $\lambda(|x|) := \langle x\rangle^{-\sigma}$ and by $\Lambda := \langle \xi \rangle$.
Then

(i) If $f \in L^1([0,T]; H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0\le t\le T} \|u(t)\|_s \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)}\left(\|u_0\|_s + \int_0^T \|f(t)\|_s dt\right);$$

(ii) If $f \in L^2([0,T]; H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist two positive constants C_1, C_2 such that

$$\sup_{0\le t\le T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)dx\, dt \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)}\left(\|u_0\|_s^2 + \int_0^T \|f(t)\|_s^2 dt\right);$$

(iii) If $\Lambda^{s-1/2} f \in L^2([0,T]\times\mathbb{R}^n; t^{-\alpha}\lambda(|x|)^{-1}dtdx)$ then the IVP (3) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0\le t\le T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)dx\, dt \le C_1 e^{C_2\frac{T^{\alpha+1}}{\alpha+1}}\left(\|u_0\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^{-\alpha}\lambda(|x|)^{-1}\left|\Lambda^{s-1/2}f\right|^2 dx\, dt\right).$$

Local well-posedness results for $\mathcal{L}_\alpha$

We consider the nonlinear IVPs

$$\text{IVP1} = \begin{cases} \mathcal{L}_\alpha u = \pm u|u|^{2k} \\ u(0,x) = u_0(x), \end{cases} \qquad \text{IVP2} = \begin{cases} \mathcal{L}_\alpha u = \pm t^\beta \nabla u \cdot u^{2k}.\ \beta \ge \alpha > 0, \\ u(0,x) = u_0(x). \end{cases}$$

Theorem (F.-Staffilani). Let $\mathcal{L}_\alpha$ be such that condition (4) is satisfied. Then the IVP1 is locally well posed in H^s for $s > n/2$.

Theorem (F.-Staffilani). Let $\mathcal{L}_\alpha$ be such that condition (4) is satisfied with $\sigma = 2N$ (thus $\lambda(|x|) = \langle x\rangle^{-2N}$) for some $N \ge 1$, and $s > n + 4N + 3$ such that $s - 1/2 \in 2\mathbb{N}$. Let $H^s_\lambda := \{u_0 \in H^s(\mathbb{R}^n); \lambda(|x|)u_0 \in H^s(\mathbb{R}^n)\}$, then the IVP2 with $\beta \ge \alpha > 0$, is locally well posed in H^s_λ.

Global weighted smoothing and Strichartz estimates for $\mathcal{L}_c$

For operators of the form $\mathcal{L}_c$ several comparison principles are proved in [2]. These are used to obtain global smoothing estimates. We state below only one of them, namely the one corresponding to the suitable generalization of the standard (corresponding to the case $c(t) = 1$) global homogeneous smoothing estimate. For more global smoothing estimates see [2].

Theorem (F.-Ruzhansky)

Let $n \ge 1$, $c \in C^1(\mathbb{R})$ be such that it vanishes at 0. Then, $\forall x \in \mathbb{R}^n$,

(i) If c is such that $\{t \in \mathbb{R}; c(t) = 0\}$ is finite, then there exist a constant $C > 0$ such that, for all j,

$$\sup_{x_j} \||c'(t)|^{1/2}|D_{x_j}|^{1/2}e^{ic(t)\Delta_x}\varphi\|_{L^2(R^{n-1}_{x'}\times\mathbb{R}_t)} \le C\|\varphi\|_{L^2_x(\mathbb{R}^n)}, \quad \forall\varphi \in L^2_x(\mathbb{R}^n);$$

(ii) If c is such that the set $\{t \in \mathbb{R}; c(t) = 0\}$ is countable, then there exists a function $\tilde{c} \in C(\mathbb{R})$, and a positive constant C such that, for all j,

$$\sup_{x_j} \||\tilde{c}(t)c'(t)|^{1/2}|D_{x_j}|^{1/2}e^{ic(t)\Delta_x}\varphi\|_{L^2(R^{n-1}_{x'}\times\mathbb{R}_t)} \le C\|\varphi\|_{L^2_x(\mathbb{R}^n)}, \quad \forall\varphi \in L^2_x(\mathbb{R}^n);$$

where $x' = (x_1, ..., x_{j-1}, x_{j+1}, ...x_n)$.

Both global and local Strichartz estimates are satisfied by $\mathcal{L}_c$. Below we give the statement of the local ones which are those employed to prove the local well-posedness of the semilinear IVP.

Theorem (F.-Ruzhansky)

Let $c \in C^1([0,T])$ be vanishing at 0 and such that $\sharp\{t \in [0,T]; c(t) = 0\} = k \ge 1$. Then, on denoting by $L^q_t L^p_x := L^q([0,T]; L^p(\mathbb{R}^n))$, we have that for any (q,p) admissible pair $\left(\frac{2}{q} + \frac{n}{p} = \frac{n}{2}\right)$ such that $2 < q, p < \infty$, the following estimates hold

$$\||c'(t)|^{1/q}e^{ic(t)\Delta}\varphi\|_{L^q_t L^p_x} \le C(n,q,p,k)\|\varphi\|_{L^2_x(\mathbb{R}^n)},$$

$$\|e^{ic(t)\Delta}\varphi\|_{L^\infty_t L^2_x} \le \|\varphi\|_{L^2_x(\mathbb{R}^n)},$$

$$\||c'(t)|^{1/q}\int_0^t |c'(s)|e^{i(c(t)-c(s))\Delta}g(s)ds\|_{L^q_t L^p_x} \le C(n,q,p,k)\||c'|^{1/q'}g\|_{L^{q'}_t L^{p'}_x},$$

$$\|\int_0^t |c'(s)|e^{i(c(t)-c(s))\Delta}g(s)ds\|_{L^\infty_t L^2_x} \le C(n,q,p,k)\||c'|^{1/q'}g\|_{L^{q'}_t L^{p'}_x}.$$

Local well-posedness of the semilinear IVP for $\mathcal{L}_c$

We can now apply the previous results to obtain the local well-posedness of the semilinear IVP

$$\begin{cases} \partial_t u + ic'(t)\Delta u = \mu|c'(t)||u|^{p-1}u, \quad \mu \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \tag{5}$$

Theorem (F.-Ruzhansky). Let $1 < p < \frac{4}{n} + 1$ and $c \in C^1([0,+\infty))$ be vanishing at 0 and it is either strictly monotone or such that $\sharp\{t \in [0,\tilde{T}]; c(t) = 0\}$ is finite for any $\tilde{T} < \infty$. Then for all $u_0 \in L^2(\mathbb{R}^n)$ there exists $T = T(\|u_0\|_2, n, \mu, p) > 0$ such that there exists a unique solution u of the IVP (5) in the time interval $[0,T]$ with $u \in C([0,T]; L^2(\mathbb{R}^n)) \bigcap L^q_t([0,T]; L^{p+1}_x(\mathbb{R}^n))$ and $q = \frac{4(p+1)}{n(p-1)}$. Moreover the map $u_0 \mapsto u(\cdot,t)$, locally defined from $L^2(\mathbb{R}^n)$ to $C([0,T); L^2(\mathbb{R}^n))$, is continuous.

References

[1] S. Federico and G. Staffilani, Smoothing effect for time-degenerate Schrödinger operators, Preprint ArXiv.

[2] S. Federico and M. Ruzhansky, Smoothing and Strichartz estimates for degenerate Schrödinger-type equations, Preprint ArXiv.

[3] M. Ruzhansky and M. Sugimoto, Smoothing properties of evolution equations via canonical transforms and comparison principles, Proc. London Math. Soc (**3**) 105 (2012), 393--423.

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