



Pseudo-differential conference

BE HARMONIC WITH ANALYSIS



$$\int_{\widehat{G}} \text{Tr}(\pi(x)\sigma(x, \pi)\hat{f}(\pi))d\mu(\pi)$$

Make a difference: pseudo-differentiate
integrate with mind but pseudo-differentiate with heart



International Conference on Pseudo-differential Operators 7-8 July 2020

- Paul Alphonse (Université de Rennes 1, France)
- Duván Cardona (Ghent University, Belgium)
- Paula Cerejeiras (University of Aveiro, Portugal)
- Elena Cordero (University of Torino, Italy)
- Marianna Chatzakou (Imperial College London, UK)
- Julio Delgado (Universidad del Valle, Cali, Colombia)
- Gerd Grubb (Copenhagen University, Denmark)
- Claudia Garetto (University of Loughborough, UK)
- Arash Ghaani Farashahi (University of Leeds, UK)
- Serena Federico (Ghent University, Belgium)
- Bernard Helffer (Université de Nantes, France)
- Marius Mantoiu (University of Chile, Chile)

International Conference on Pseudo-differential Operators 7-8 July 2020

- Vishvesh Kumar (Ghent University, Belgium)
- Gerardo Mendoza (Temple University, USA)
- Rakesh Kumar Parmar (Bikaner Technical University, India)
- Alberto Parmeggiani (University of Bologna, Italy)
- Sylvie Paycha (University of Potsdam, Germany)
- Karel Pravda-Starov (Université de Rennes 1, France)
- Luigi Rodino (University of Torino, Italy)
- David Rottensteiner (Ghent University, Belgium)
- Michael Ruzhansky (Ghent University, Belgium and Queen Mary University of London, UK)





International Conference on Pseudo-differential Operators 7-8 July 2020

- Elmar Schrohe (University of Hannover, Germany)
- Johannes Sjöstrand (University of Burgundy in Dijon, France)
- Joachim Toft (Linnaeus University, Sweden)
- Ville Turunen (Aalto University, Finland)
- Gunther Uhlmann (University of Washington, USA)
- Jens Wirth (Stuttgart University, Germany)
- Man Wah Wong (York University, Canada)
- Ingo Witt (University of Göttingen)
- Maciej Zworski (University of California, Berkley, USA)

$$Op(\hat{G})f(x) = \int Tr \left(\pi(x) \hat{G}(x, \pi) \hat{f}(\pi) \right) d\mu(\pi)$$

Tuesday, 7 July

Gerd Grubb
Copenhagen University, Denmark



*Fractional-order pseudodifferential operators
on nonsmooth domains*

Tuesday, 7 July

10:10 - 10:40



Abstract: Fractional-order pseudodifferential operators, in particular non-integer powers of the Laplacian, have recently come into focus because of their interest in financial theory and probability (they also occur in differential geometry and mathematical physics). In spite of their pseudodifferential nature, the use of ps.d.o. methods has been somewhat scarce, maybe because the ps.d.o. apparatus is technically complicated and is not part of the general curriculum in the teaching of Analysis.

For boundary value problems it was possible in 2014/15 to use ps.d.o. methods to get fine regularity results, obtained earlier only in low-order Hölder spaces by potential-theoretic methods. When lecturing on these things to applications-oriented people, I have been met by the criticism, that ps.d.o. methods require a high regularity of the set-up, C^∞ coefficients and C^∞ domains.

There do exist pseudodifferential theories where symbols $p(x, \xi)$ are only Hölder-smooth in x , but the question of how a non-smooth change of variables affects the operators has not been satisfactorily dealt with up to now.

Very recently I have completed a joint work with Helmut Abels, where we address this question, and work out conclusions for suitable fractional-order boundary problems.

Gerd Grubb

Copenhagen University, Denmark



Johannes Sjöstrand
University of Burgundy in
Dijon, France

Bergman kernels in the analytic case

Tuesday, 7 July

10:50 - 11:20



Johannes Sjöstrand

University of Burgundy in Dijon
France

Abstract: We explain a result on the precise asymptotics of the Bergman kernel for exponentially weighted spaces and for high powers of a complex line bundle in the real analytic framework, due to Sj., Rouby, Ngoc and also to A. Deleporte. We give some ideas from our proof, based on an adapted calculus of Fourier integral operators, and also discuss a recent simplification by A. Deleporte, M. Hitrik, Sj.





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Presentation:
Book of abstracts.

Elena Cordero
University of Torino, Italy

*Properties of Eigenfunctions of pseudodifferential
operators via Gabor frames*

Tuesday, 7 July

11:40 - 12:10



Elena Cordero

University of Torino, Italy

Abstract: We study the smoothness and the decay properties for eigenfunctions of compact pseudodifferential operators with symbols in modulation spaces. In particular, we focus on localization and Weyl operators. Localization operators have become popular with the papers by I. Daubechies in 1988 and from then widely investigated by several authors in different fields of mathematics and physics: in Signal Analysis, Quantum Mechanics and in the framework of pseudodifferential calculus. Localization operators can be introduced via the time-frequency representation known as short-time Fourier transform. Recall first the modulation M_ω and translation T_x operators of a function f on $\mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i t \cdot \omega} f(t), \quad T_x f(t) = f(t - x), \quad x, \omega \in \mathbb{R}^d.$$

Fix a non-zero window function ψ in the Schwartz class $S(\mathbb{R}^d)$. The short-time Fourier transform of a tempered distribution $f \in S'(\mathbb{R}^d)$, is given by

$$V_\psi f(x, \omega) = \langle f, \pi(x, \omega) \psi \rangle = F(f T_x \psi)(\omega) = \int_{\mathbb{R}^d} f(y) \overline{\psi(y - x)} e^{-2\pi i y \cdot \omega} dy.$$

The localization operator $A_a^{\psi_1, \psi_2}$ with symbol a and windows ψ_1, ψ_2 is formally defined by

$$A_a^{\psi_1, \psi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x, \omega) V_{\psi_1} f(x, \omega) M_\omega T_x \psi_2(t) dx d\omega.$$

A localization operator $A_a^{\psi_1, \psi_2}$ can be represented as a Weyl pseudodifferential operator L_σ with symbol σ as

$$A_a^{\psi_1, \psi_2} = L_{a * W(\psi_2, \psi_1)}$$

Where $\sigma = a * W(\psi_2, \psi_1)$ and $W(\psi_2, \psi_1)$ is the cross-Wigner distribution:

$$W(\psi_2, \psi_1) = \int_{\mathbb{R}^d} \psi_2\left(x + \frac{t}{2}\right) \overline{\psi_1\left(x - \frac{t}{2}\right)} e^{-2\pi i y \cdot \omega} dt.$$

Considering symbols a in the wide modulation space $M^{p, \infty}$ (containing the Lebesgue space $L^p()$, $p < \infty$, and two general windows ψ_1, ψ_2 in the Schwartz class $S(\mathbb{R}^d)$), we show that $L^2(\mathbb{R}^d)$ -eigenfunctions with non-zero eigenvalue are indeed highly compressed onto a few Gabor atoms.

The key idea is to study first smoothness and decay properties for eigenfunctions of compact Weyl operators. We shall show that under suitable assumptions on the Weyl symbol, any L^2 -eigenfunction with non-zero eigenvalue is highly concentrated on the time-frequency space.

An important role in the proofs is played by quasi-Banach Wiener amalgam and modulation spaces. As a byproduct, new convolution relations for modulation spaces in the quasi-Banach setting are exhibited. Finally, we will show conjectures and partial results for eigenfunctions of localization and Kohn-Nirenberg operators on locally compact abelian (LCA) groups.



Paul Alphonse
Université de Rennes 1
France

*Smoothing properties of semigroups generated
by accretive quadratic operators*

Tuesday, 7 July

12:20 - 12:30



Abstract: Accretive quadratic operators are differential operators defined by the Weyl quantization of complex-valued quadratic forms defined on the phase space with non-negative real parts. The smoothing properties of the semigroups generated by this class of operators on $L^2(\mathbb{R}^n)$ are encoded in the structure of a vector subspace of the phase space intrinsically linked to the quadratic operator at play (called its singular space) introduced by M. Hitrik and K. Pravda-Starov. This talk is devoted to present this result in details and to give the sketch of its proof. The method is based on a polar decomposition approach, that is, on describing the decomposition of the evolution operators generated by accretive quadratic operators as the product of a selfadjoint operator and a unitary operator on $L^2(\mathbb{R}^n)$. Such a decomposition is obtained by exploiting the Fourier integral operator structure of these evolution operators given in a paper by L. Hörmander. This is a joint work with J. Bernier.

Paul Alphonse
Université de Rennes 1, France



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G H E N T A N A L Y S I S P D E

Lunch break: 12:30 - 14:20
Presentation of GAP



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Karel Pravda-Starov
Université de Rennes 1, France

*Uncertainty principles and null-controllability
of evolution equations enjoying Gelfand-Shilov
smoothing effects*

Tuesday, 7 July

14:20 - 14:50



Abstract: We discuss uncertainty principles for finite combinations of Hermite functions and establish some spectral inequalities for control subsets that are thick with respect to some unbounded densities growing almost linearly at infinity. These spectral inequalities allow to derive the null-controllability in any positive time for evolution equations enjoying specific regularizing effects in Gelfand-Shilov spaces.

This is a joint work with Jérémy Martin (Université de Rennes 1).

Karel Pravda-Starov

Université de Rennes 1, France



Alberto Parmeggiani
University of Bologna, Italy

On the solvability of degenerate PDEs

Tuesday, 7 July

15:00 - 15:30



Abstract: In this talk I will survey some solvability results related to a class of degenerate second order PDEs which are generalizations of the (formal adjoint of the) Kannai operator.

Alberto Parmeggiani
University of Bologna, Italy





Ingo Witt
University of Göttingen
Germany



Hyperbolic problems with totally characteristic boundary

Tuesday, 7 July

15:40 - 16:10

Abstract: We consider first-order $N \times N$ hyperbolic system

$$(1) \quad \begin{cases} \partial_t u + \mathcal{A}(t, x, y, xD_x, D_y)u = f(t, x, y) & \text{in } (0, T) \times \mathbb{R}_+^{1+d} \\ u|_{t=0} = u_0(x, y) & \text{on } \mathbb{R}_+^{1+d} \end{cases}$$

with variables $(t, x, y) \in (0, T) \times \mathbb{R}_+ \times \mathbb{R}^d$. The lateral boundary at $x = 0$ is totally characteristic due to the presence of the derivative $x D_x$. As the lateral boundary is totally characteristic, no boundary conditions are required.

We show that the Cauchy problem **(1)** has a solution u in a class of Sobolev spaces with asymptotics, where

$$u(t, x, y) \sim \sum_{(p,k)} x^p \log^k x \, u_{pk}(t, y) \quad \text{as } x \rightarrow +0$$

with $(p, k) \in \mathbb{C} \times \mathbb{N}_0$, $\Re p \rightarrow \infty$ as $|p| \rightarrow \infty$, provided that f and u_0 have asymptotics of the same type. Furthermore, we show that the boundary traces u_{pk} solve hyperbolic Cauchy problems on $(0, T) \times \mathbb{R}^d$. A special case arises when $u(t, x, y) \sim \sum_{l \in \mathbb{N}_0} x^l u_l(t, y)$ as $x \rightarrow +0$ in which case we solve in the standard Sobolev spaces $H^s((0, T) \times \mathbb{R}_+^{1+d}; \mathbb{C}^N)$.

This is a joint work with Zhuoping Ruan (Nanjing University).





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Presentation:
Book of abstracts.

Maciej Zworski
University of California
Berkeley, USA



Analytic hypoellipticity of Keldysh operators

Tuesday, 7 July

16:30 - 17:00



Abstract: For operators modeled by $P = x_1 D_{x_1}^2 + D_{x_2}^2 + a D_{x_1}$ we show that if u is smooth and $P u$ is analytic then u is analytic. This is motivated by the question of analyticity of quasinormal modes of black holes across event horizons and is a consequence of a general microlocal result. Joint work with J. Galkowski.

Maciej Zworski
University of California
Berkeley, USA





Gerardo Mendoza
Temple University, USA



$\mathbb{R}$ -actions and invariant differential operators

Tuesday, 7 July

17:10 - 17:40

Gerardo Mendoza
Temple University, USA



Abstract: A compact manifold together with a nowhere vanishing real vector field that leaves invariant some Riemannian metric behaves in many senses like a circle bundle over a compact base. To illustrate the point, I will discuss a theorem concerning the action of the vector field on the kernel of an invariant differential operator, the latter acting on sections of a Hermitian vector bundle and assumed to be normal with respect to the natural Hilbert space structure, the former acting as a formally self adjoint operator (all under certain hypoellipticity/ellipticity condition). As an application I will discuss how this generalizes Kodaira's vanishing theorem.



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Presentation:
Book of abstracts.

Ville Turunen
Aalto University, Finland



Time-frequency analysis on locally compact groups

Tuesday, 7 July

18:00 - 18:30



Ville Turunen
Aalto University, Finland

Abstract: Time-frequency analysis can be described as Fourier analysis simultaneously both in time and in frequency. Its origins are in quantum mechanics, in signal processing in Euclidean spaces, and in pseudo-differential operators. In this presentation we show how to generalize time-frequency analysis to those locally compact groups that allow a nice-enough Fourier transform: wide families of non-commutative groups can be treated. Our results on locally compact groups shed new light also on the analysis in Euclidean spaces.

Ville Turunen
Aalto University, Finland



$$\operatorname{Re} (Tf, f)_{L^2(G)} \geq -C \|f\|_{L^2_{(m-(s-\delta))/2}(G)}^2$$

Wednesday, 8 July



Bernard Helffer
Université de Nantes, France

*Spectral analysis near a Dirac type crossing in
a weak non-constant magnetic field (after H.
Cornean, B. Helffer and R. Purice)*



Wednesday, 8 July

10:00 - 10:30

Bernard Helffer
Université de Nantes, France



Abstract: This is the continuation of a series of works devoted to the justification of the Peierls substitution in the case of a weak magnetic field. Here we deal with two 2d Bloch eigenvalues which have a conical crossing. It turns out that in the presence of an almost constant weak magnetic field, the spectrum near the crossing develops gaps which remind of the Landau levels of an effective mass-less magnetic Dirac operator. This involves the semi-classical analysis for the Peierls-Onsager effective Hamiltonian which is done through the combination of different pseudo-differential calculi.



Claudia Garetto
University of Loughborough
UK



Hyperbolic Cauchy problems with multiplicities

Wednesday, 8 July

10:40 - 11:10

Claudia Garetto

University of Loughborough, UK



Abstract: In this talk we give a survey on some recent results, obtained in collaboration with Michael Ruzhansky and Christian Jäh, on hyperbolic Cauchy problems with multiplicities. In particular, we will explain how the theory of pseudo-differential operators is employed in order to get well-posedness.



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Presentation:
Book of abstracts.

Joachim Toft
Linnaeus University, Sweden

*Liftings for ultra-modulation spaces, and
one-parameter groups of Gevrey type
pseudo-differential operators*

Wednesday, 8 July

11:30 - 12:00



Joachim Toft

Linnaeus University, Sweden

Abstract: Let $\Gamma_s^{(\omega_0)}$ be the set of all $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$|\partial^\alpha a(X)| \leq C_\alpha h^{|\alpha|} \alpha^s \omega_0(X), \quad \omega_0(X+Y) \leq C \omega_0(X) e^{r|Y|^{\frac{1}{s}}},$$

for some $h > 0$ and every $r > 0$. We deduce one-parameter group properties for pseudo-differential operators in $Op(\Gamma_s^{(\omega_0)})$. We use this to show that

$$Op(a) \circ Op(b) = Op(b) \circ Op(a) = Id \text{ for some } a \in Op(\Gamma_s^{(\omega_0)}), \quad b \in Op(\Gamma_s^{(1/\omega_0)}).$$

By combining these results with the techniques [4], we explain how to deduce lifting property for modulation spaces and construct explicit isomorphisms between them. Some ideas goes back [2,4]. Especially we prove that for ω, ω_0 in a general class of weight functions, the Toeplitz (or localization) operator $Tp(\omega_0)$ is an isomorphism from $M_\omega^{p,q}$ onto $M_\omega^{p,q}$ for every $p, q \in (0, \infty]$. We also give some examples on specific liftings, e.g. that the operators

$$(1 + |x|^{2N_1})^r + (1 + \Delta^{N_2})^\rho \quad \text{and} \quad Tp \left(\exp \left((1 + |x|^2)^r + (1 + |\xi|^2)^\rho \right)^t \right),$$

$t < 1/2$, are bijective mappings between suitable modulation spaces. This can be used to deduce that the Weyl operator $Op^\omega \left(\exp \left((1 + |x|^2)^r + (1 + |\xi|^2)^\rho \right)^t \right)$ has index zero as a map between these suitable modulation spaces. The talk is based on joint work with A. Abdeljawad and S. Coriasco.



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Joachim Toft

Linnaeus University, Sweden

References

- [1] A. Abdeljawad, S. Coriasco, J. Toft. Liftings for ultra-modulation spaces, and one-parameter groups of Gevrey type pseudo-differential operators, *Anal. Appl.*, appeared online 2019.
- [2] P. Boggiatto, J. Toft. Embeddings and compactness for generalized Sobolev-Shubin spaces and modulation spaces, *Appl. Anal.* 84 (2005), 269–282.
- [3] K. Gröchenig, J. Toft. Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces, *J. Anal. Math.* 114 (2011), 255–283.
- [4] K. Gröchenig, J. Toft. The range of localization operators and lifting theorems for modulation and Bargmann-Fock spaces, *Trans. Amer. Math. Soc.* 365 (2013), 4475–4496.



Paula Cerejeiras
Universidad de Aveiro
Portugal



*Discrete Pseudo-differential Operators and
Boundary Value Problems in Hypercomplex
Function Theory*

Wednesday, 8 July

12:10 - 12:40

Paula Cerejeiras
Universidad de Aveiro, Portugal



Abstract: During the last decades one can observe an increasing interest in discrete structures due to a fast growing computational power. This evolution in computational power also allows to replace the traditional approach of discretisation of partial differential equations via a variational formulation by a finite element modelation directly on the mesh. This also opens the way for discrete analogues to the usual continuous structures, in particular a discrete function theory. While there is a deep and long standing theory in the two-dimensional case, which has applications to problems in probability and statistical physics (see D. Chelkak, R. Novikov, S. Smirnov, just to name a few) the higher dimensional case is only recently being developed. One of the drawbacks of the discrete case is the lack of explicit expressions for the kernels of the corresponding discrete operators. In most cases only their Fourier symbols are known. In this talk we adapt the recent theories of discrete pseudo-differential operators by Botchway/Kibiti/Ruzhansky and Rabinovich/Roch/ Silberman to the case of symbols with values in a complexified Clifford algebras and construct the necessary tools for a discrete hypercomplex function theory, like discrete Cauchy kernels, discrete Hilbert and Riesz-transforms, and Hardy spaces. Special emphases will be given to the study and application of discrete Riemann-Hilbert boundary value problems.

Lunch break: 12:50 - 14:20



G H E N T A N A L Y S I S P D E



Poster session, Part I.

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Poster session Part II (Blitztalks)

$$Af(x) = \sum_{[\xi] \in \hat{G}} d_{\xi} \operatorname{tr} [\xi(x) G(x, \xi) \hat{f}(\xi)]$$

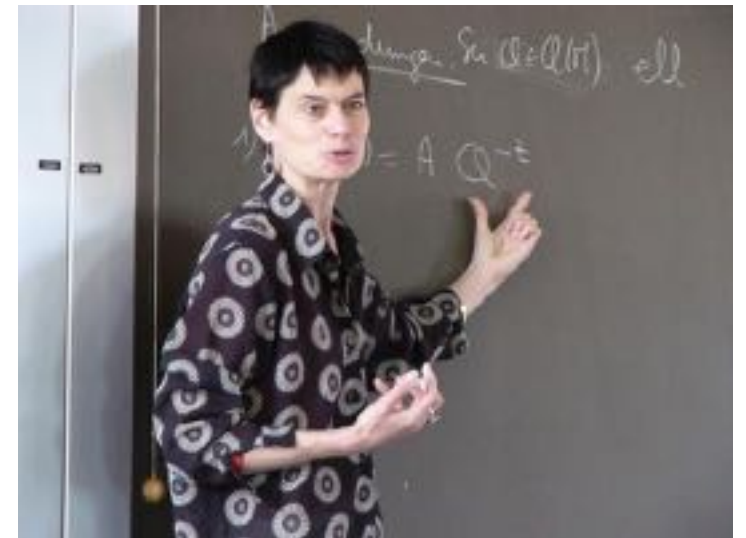
14:20 - 15:00

Sylvie Paycha
University of Potsdam
Germany

Regularised traces and Getzler's rescaling

Wednesday, 8 July

15:00 - 15:30



Abstract: Inspired by Gilkey's invariance theory, Connes' deformation to the normal cone and Getzler's rescaling method, we single out a class of geometric operators among pseudodifferential operators acting on sections of a class of natural vector bundles, to which we attach a rescaling degree. This degree is then used to express regularised traces of geometric operators in terms of a rescaled limit of Wodzicki residues. When applied to complex powers of the square of a Dirac operator, this amounts to expressing the index of a Dirac operator in terms of a local residue involving the Getzler rescaled limit of its square.

Sylvie Paycha
University of Potsdam
Germany





Elmar Schrohe
University of Hannover
Germany

*Degenerate Elliptic Boundary Value Problems
with Non-smooth Coefficients*

Wednesday, 8 July

15:40 - 16:10



Abstract: On a manifold of bounded geometry with boundary we consider a uniformly strongly elliptic second order operator A that locally is of the form

$$A = - \sum_{j,k} a_{jk} \partial_{x_j} \partial_{x_k} + \sum_j b_j \partial_{x_j} + c,$$

together with a degenerate boundary operator T of the form

$$T = \phi_0 \gamma_0 + \phi_1 \gamma_1,$$

where γ_0 and γ_1 , denote the evaluation of a function and its exterior normal derivative, respectively, at the boundary, and ϕ_0, ϕ_1 are smooth functions on the boundary with $\phi_0 \geq 0$, $\phi_1 \geq 0$, and $\phi_0 + \phi_1 \geq c_0 > 0$. Unless either $\phi_0 \equiv 0$, or $\phi_1 \equiv 0$ this problem is not elliptic in the sense of Lopatinskij and Shapiro. We show that the realization A_T of A in $L^p(\Omega)$, has a bounded H^∞ -calculus whenever the a_{jk} are Hölder continuous and b_j as well as c are L^∞ . For the proof we first treat the operator with smooth coefficients on $\mathbb{R}_+^n$. Here we rely on an extension of Boutet de Monvel's calculus to operator-valued symbols of Hörmander type $(1, \delta)$. We then use H^∞ -perturbation techniques in order to treat the nonsmooth case. As an application we study the porous medium equation. (Joint work with Thorben Krietenstein, Hannover).



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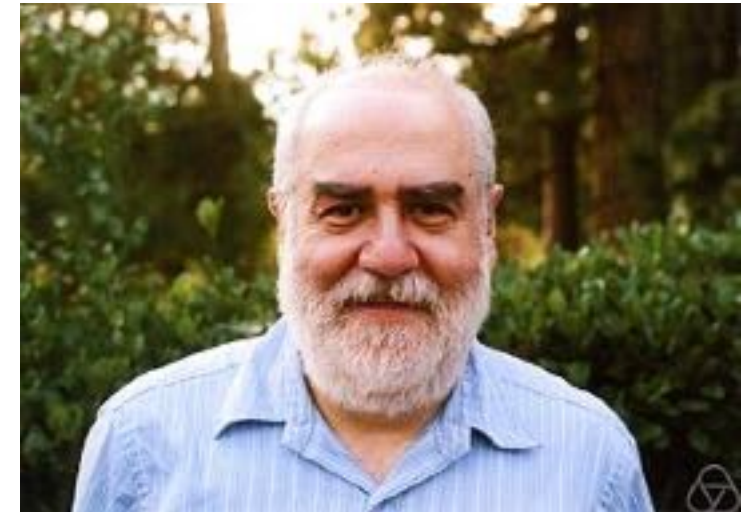
Presentation:
Book of abstracts.

Gunther Uhlmann
University of Washington, USA

Microlocal Analysis and the X-ray Transform

Wednesday, 8 July

16:40 - 17:10



Abstract: We will consider recent applications of microlocal analysis to the study of the geodesic X-ray transform which consists of integrating a function along geodesics. This type of transforms have applications to many fields including medical imaging and geophysics.

Gunther Uhlmann

University of Washington, USA





Man Wah Wong

York University, Canada



*The Heat Semigroup of the Twisted Laplacian
on the Non-Isotropic Heisenberg Group with
Multi-Dimensional Center*

Wednesday, 8 July

17:20 - 17:50

Man Wah Wong
York University, Canada



Abstract: We give a formula for the one-parameter strongly continuous semigroup $e^{-\tau L^\lambda}$, $\tau > 0$, generated by the twisted Laplacian L^λ , $\lambda \in \mathbb{R}^m \setminus \{0\}$, on the non-isotropic Heisenberg group $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ in terms of Weyl transforms on these groups, and use it to obtain an L^2 -estimate for the solution of the initial value problem for the heat equation governed by L^λ in terms of the L^p norm of the initial data for $1 \leq p \leq \infty$. This is joint work with Ms. Shengwen Yang of York University.

Marius Mantoiu
University of Chile

*Quantizations on Nilpotent Lie Groups and
Algebras Having Flat Coadjoint Orbits*

Wednesday, 8 July

18:00 - 18:30



Abstract: For a connected simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$ and unitary dual G' one has (a) a global quantization of operator-valued symbols involving the representation theory of the group, (b) a quantization of scalar-valued symbols defined on $G \times \mathfrak{g}^*$, taking the group structure into account and (c) Weyl-type quantizations of all the coadjoint orbits. We show how these quantizations are related, by a careful analysis of the composition of two different types of Fourier transformations, interesting in itself. We also describe the concrete form of the operator-valued symbol quantization, by using *Kirillov theory* and the Euclidean version of the unitary dual and *Plancherel measure*. In the case of the Heisenberg group, this corresponds to the known picture, presenting the representation theoretical pseudo-differential operators in terms of families of Weyl operators depending on a parameter. For illustration, we work out a couple of examples and put into evidence some specific features of the case of Lie algebras with one-dimensional center. When G is also graded, we make a short presentation of the symbol classes S^m , transferred from $G \times G'$ to $G \times \mathfrak{g}^*$ by means of the connection mentioned above. Joint work with *Michael Ruzhansky*.

Marius Mantoiu
University of Chile, Chile





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Organisers:

Ghent Analysis & PDE Center

Description

This open access book provides an extensive treatment of Hardy inequalities and closely related topics from the point of view of Folland and Stein's homogeneous (Lie) groups. The place where Hardy inequalities and homogeneous groups meet is a beautiful area of mathematics with links to many other subjects. While describing the general theory of Hardy, Rellich, Caffarelli-Kohn-Nirenberg, Sobolev, and other inequalities in the setting of general homogeneous groups, the authors pay particular attention to the special class of stratified groups. In this environment, the theory of Hardy inequalities becomes intricately intertwined with the properties of sub-Laplacians and subelliptic partial differential equations. These topics constitute the core of this book and they are complemented by additional, closely related topics such as uncertainty principles, function spaces on homogeneous groups, the potential theory for stratified groups, and the potential theory for general Hörmander's sums of squares and their fundamental solutions.

Content

- Analysis on Homogeneous Groups;
- Hardy Inequalities on Homogeneous Groups;
- Rellich, Caffarelli-Kohn-Nirenberg, and Sobolev Type Inequalities;
- Fractional Hardy Inequalities;
- Integral Hardy Inequalities on Homogeneous Groups;
- Horizontal Inequalities on Stratified Groups;
- Hardy-Rellich Inequalities and Fundamental Solutions;
- Geometric Hardy Inequalities on Stratified Groups;
- Uncertainty Relations on Homogeneous Groups;
- Function Spaces on Homogeneous Groups;
- Elements of Potential Theory on Stratified Groups;
- Hardy and Rellich Inequalities for Sums of Squares of Vector Fields.

Hardy Inequality

Classical Hardy inequality in the Euclidean space $\mathbb{R}^n$: for all $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2} dx,$$

where the constant $(n-2)^2/4$ is sharp, $|\cdot|_E$ is the standard Euclidean norm, and ∇ is the standard gradient on $\mathbb{R}^n$.

Hardy inequality on homogeneous Lie groups G : for all $f \in C_0^\infty(G \setminus \{0\})$, we have

$$\int_G |\mathcal{R}f(x)|^2 dx \geq \left(\frac{Q-2}{2}\right)^2 \int_G \frac{|f(x)|^2}{|x|^2} dx,$$

where the constant $(Q-2)^2/4$ is sharp, Q is the homogeneous dimension of G , $|\cdot|$ is any homogeneous quasi-norm on homogeneous Lie groups G , and $\mathcal{R}$ is the radial derivative, $\mathcal{R}f(x) := \frac{\partial f(x)}{\partial |x|}$.

100 Years of Hardy Inequalities



G.H. Hardy and Harald Bohr (from Wikipedia page)

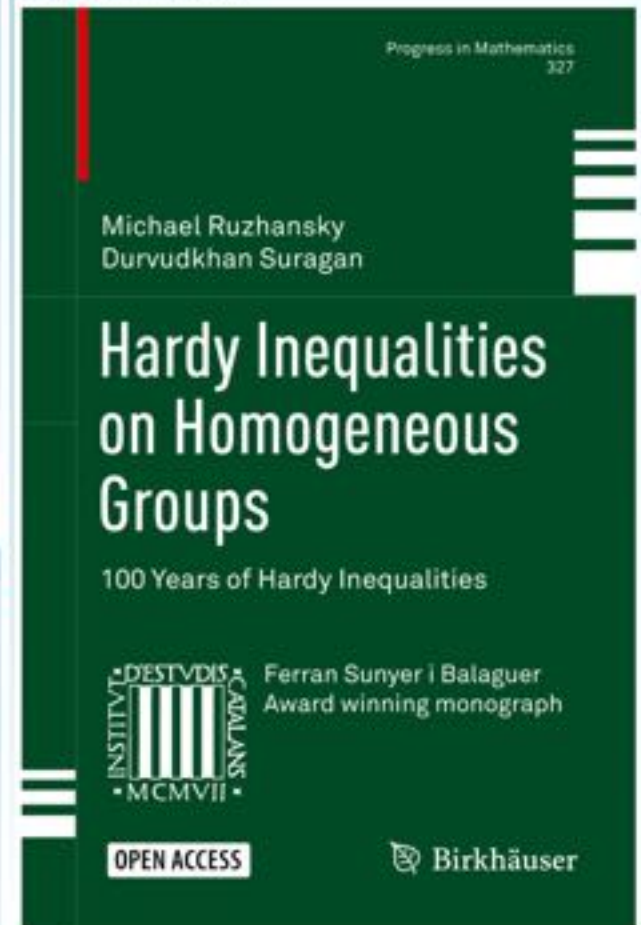
G.H. Hardy reported Harald Bohr as saying 'all analysts spend half their time hunting through the literature for inequalities which they want to use but cannot prove'.

Ferran Sunyer i Balaguer Prize

This monograph is the winner of the 2018 Ferran Sunyer i Balaguer Prize, a prestigious award for books of expository nature presenting the latest developments in an active area of research in mathematics. As can be attested as the winner of such an award, it is a vital contribution to literature of analysis not only because it presents a detailed account of the recent developments in the field, but also because the book is accessible to anyone with a basic level of understanding of analysis. Undergraduate and graduate students as well as researchers from any field of mathematical and physical sciences related to analysis involving functional inequalities or analysis of homogeneous groups will find the text beneficial to deepen their understanding.



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L^p -bounds for pseudo-differential operators on graded Lie groups

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Abstract

We present the sharp L^p -estimates for pseudo-differential operators on arbitrary graded Lie groups proved by the authors in [1]. The results are presented within the setting of the global symbolic calculus on graded Lie groups by using the Fourier analysis associated to every graded Lie group which extends the usual one due to Hörmander on $\mathbb{R}^n$. The main result extends the classical Fefferman's sharp theorem on the L^p -boundedness of pseudo-differential operators for Hörmander classes on $\mathbb{R}^n$ to general graded Lie groups, also adding the borderline $p = \delta$.

Introduction

- The investigation of the L^p boundedness of pseudo-differential operators is a crucial task for a large variety of problems in mathematical analysis and its applications, mainly due to its consequences for the regularity, approximation and existence of solutions on L^p -Sobolev spaces.
- There is an extensive literature on the subject, in particular, devoted to operators associated with symbols belonging to the Hörmander classes $S_{\rho,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$, (see for instance, J.J. Kohn and L. Nirenberg [3], L. Hörmander [4] and C. Fefferman [2]).
- Our main goal is to extend a classical and sharp result by C. Fefferman [2] and to provide a critical order for the L^p boundedness of pseudo-differential operators on graded Lie groups based on the quantization procedure developed by the third author and V. Fischer in [5].
- Our main estimate recover the sharp Fefferman theorem on $\mathbb{R}^n$, adding the critical case $p = \delta$.

Fourier analysis on nilpotent Lie groups

- Let G be a simply connected nilpotent Lie group and let G be its unitary dual.
- Let $\exp_G : \mathfrak{g} \rightarrow G$, be the exponential mapping on G . The Schwartz class on G , is defined by those $f \in C^\infty(G)$, such that $f \circ \exp_G \in \mathcal{S}(\mathfrak{g})$, with $\mathfrak{g} \simeq \mathbb{R}^{\dim(G)}$. The Fourier transform of $f \in \mathcal{S}(G)$, at $\pi \in G$, is defined by:

$$\widehat{f}(\pi) = \int_G f(x) \pi(x)^* dx$$



Pseudo-differential operators on graded Lie groups

- Roughly speaking, a pseudo-differential operator is a continuous linear operator on $\mathcal{S}'(G)$, defined by the (quantization) formula:

$$\text{Op}(\sigma)(f)(x) = \int_G f(y) \pi(x \cdot y^{-1}) \sigma(\pi) d\mu(\pi)$$

In such a case, we say that σ is the symbol associated with $\text{Op}(\sigma)$. We have denoted by $d\mu(\pi)$ the Plancherel measure on G .

- A Rockland operator is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree $\nu = \nu_{\mathcal{R}}$ and such that, for every unitary irreducible non-trivial representation $\pi \in \widehat{G}$, $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$: $\sigma_{\mathcal{R}}(\pi) = \pi(\mathcal{R})$ is the symbol associated to $\mathcal{R}$.
- It can be shown that a Lie group G is graded if and only if there exists a differential Rockland operator on G .
- The basic example of graded Lie group is the Heisenberg group $\mathbb{H}^n$.

Hörmander classes on graded Lie groups

Hörmander classes on the phase space $G \times G$ can be defined by using Rockland operators. Indeed, the Hörmander class of order m , and of type (ρ, δ) , $S_{\rho,\delta}^m(G \times G)$, is defined by those symbols σ satisfying symbol inequalities of the kind:

$$\left| \frac{\partial^{\alpha} \sigma}{\partial x^{\alpha}} \right| \leq C_{\alpha} (1 + |\xi|)^{m - \rho|\alpha|} (1 + |\eta|)^{\delta|\alpha|} \quad \forall \alpha \in \mathbb{N}^n$$

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L^p , H^1 - L^1 and L^∞ -BMO-boundedness of pseudo-differential operators

Theorem (Fefferman type estimates on Graded Lie groups)

Let G be a graded Lie group of homogeneous dimension Q . Let $A = \text{Op}(\sigma) : C^\infty(G) \rightarrow \mathcal{S}'(G)$ be a pseudo-differential operator with symbol $\sigma \in S_{\rho,\delta}^m(G \times G)$, $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$. Then,

- (a) if $m = \frac{Q(1-\rho)}{2}$, then A extends to a bounded operator from $L^\infty(G)$ to $BMO(G)$, from the Hardy space $H^1(G)$ to $L^1(G)$, and from $L^p(G)$ to $L^p(G)$ for all $1 < p < \infty$.
- (b) If $m \geq m_p := Q(1-\rho)_+ - \frac{Q}{p}$, $1 < p < \infty$, then A extends to a bounded operator from $L^p(G)$ into $L^p(G)$.

The Harmonic Oscillator on The Heisenberg Group

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Abstract

We present a notion of harmonic oscillator on the Heisenberg group $\mathbf{H}_n$. This operator forms a natural analogue of the harmonic oscillator on $\mathbb{R}^n$.

Introduction

Our ansatz is based on a few reasonable assumptions: the harmonic oscillator on $\mathbf{H}_n$ should be a negative sum of squares of operators related to the sub-Laplacian on $\mathbf{H}_n$, essentially self-adjoint with purely discrete spectrum, and its eigenvectors should be smooth functions and form an orthonormal basis of $L^2(\mathbf{H}_n)$. This leads to a differential operator on $\mathbf{H}_n$ which is determined by the Dynin-Folland Lie algebra, a stratified 3-step nilpotent Lie algebra.

Ansatz

Our approach is motivated by the following three realizations of the classical harmonic oscillator $\mathcal{Q}_\mathbb{R}$ on $\mathbb{R}^n$:

- (R1) the negative sum of squares $-\Delta + |x|^2$ of partial derivatives of order 1 and coordinate multiplication operators;
- (R2) the Weyl and Kohn-Nirenberg quantizations on $\mathbb{R}^n$ of the symbol $\sigma(x, \xi) := |\xi|^2 + |x|^2$ with $x, \xi \in \mathbb{R}^n$;
- (R3) the image $d\rho_1(-\mathcal{L}_{\mathbf{H}_n})$ of the negative sub-Laplacian $-\mathcal{L}_{\mathbf{H}_n}$ on $\mathbf{H}_n$ under the infinitesimal Schrödinger representation $d\rho_1$ (of Planck's constant equal to 1) of the Heisenberg Lie algebra $\mathfrak{h}_n$.

The Schrödinger representation ρ_1 of $\mathbf{H}_n$ acting on $L^2(\mathbb{R}^n)$ and the associated Lie algebra representation, naturally acting on $\mathcal{S}(\mathbb{R}^n)$, clearly relate each of the realisations (R1) - (R3) to the others. One can expect that similar realisations should be available for the canonical harmonic oscillator on $\mathbf{H}_n$.

Main Result

The harmonic oscillator $\mathcal{Q}_{\mathbf{H}_n}$ on the Heisenberg group $\mathbf{H}_n$ has a purely discrete spectrum $\text{spec}(\mathcal{Q}_{\mathbf{H}_n}) \subset (0, \infty)$. The number of its eigenvalues, counted with multiplicities, which are less or equal to $\lambda > 0$ is asymptotically (as $\lambda \rightarrow \infty$) given by

$$N(\lambda) \sim \lambda^{\frac{n+1}{2}},$$

and the magnitude of the eigenvalues is asymptotically equal to

$$\lambda_k \sim k^{\frac{n+1}{2}} \text{ for } k = 1, 2, \dots$$

Moreover, the eigenvectors of $\mathcal{Q}_{\mathbf{H}_n}$ are in $\mathcal{S}(\mathbf{H}_n)$ and form an orthonormal basis of $L^2(\mathbf{H}_n)$.

Figure 1: The Harmonic Oscillator on $\mathbf{H}_1$

Definition

The Dynin-Folland Lie group $\mathbf{H}_{n,2} = \mathbb{R}^{2n+2} \times \mathbf{H}_n$ acts on $f \in L^2(\mathbf{H}_n)$ by the unitary irreducible representation

$$(\pi(z, y, x)f)(t) = e^{iz \cdot t} e^{i(y \cdot t)} f(t \cdot x),$$

where $t \cdot \frac{1}{2}x$ and $t \cdot x$ denote the $\mathbf{H}_n$ -group products of the corresponding coordinate vectors.

For the basis $\{X_1, \dots, Y_{2n+1}, Z\}$ of its Lie algebra $\mathfrak{h}_{n,2}$ we define the **harmonic oscillator on $\mathbf{H}_n$** by

$$\mathcal{Q}_{\mathbf{H}_n} := d\pi(-\mathcal{L}_{\mathbf{H}_{n,2}}) = -d\pi(X_1)^2 - \dots - d\pi(X_{2n})^2 - d\pi(Y_{2n+1})^2,$$

where $-\mathcal{L}_{\mathbf{H}_{n,2}}$ is the sub-Laplacian on $\mathbf{H}_{n,2}$. Its natural domain includes the smooth vectors $\mathcal{H}_2^n \cong \mathcal{S}(\mathbf{H}_n)$.

Interpretation

The essentially self-adjoint differential operator $\mathcal{Q}_{\mathbf{H}_n}$ on $\mathbf{H}_n$ admits analogues of (R1) - (R3):

- (R1') the differential operator $-\mathcal{L}_{\mathbf{H}_n} + x^2$;
- (R2') i) the Dynin-Weyl quantization on $\mathbf{H}_n$ of the symbol $\sigma(x, \xi) := \xi_1^2 + \dots + \xi_{2n}^2 + x_{2n+1}^2$ with $(x, \xi) \in \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$;
ii) the Kohn-Nirenberg quantization, in the sense of [25], of the operator-valued symbol on $\mathbf{H}_n \times \mathbf{H}_n$ $\sigma(x, p_x) := -p_x(X_1)^2 - \dots - p_x(X_{2n})^2 + x_{2n+1}^2$;
- (R3') the image $d\pi(-\mathcal{L}_{\mathbf{H}_{n,2}})$ of the sub-Laplacian $\mathcal{L}_{\mathbf{H}_{n,2}}$ on $\mathbf{H}_{n,2}$, here by definition.

Remark

The power $\frac{n+1}{2}$ intimately related to the homogeneous structure of $\mathfrak{h}_{n,2}$: the nominator $6n + 3$ is the homogeneous dimension of the first two strata $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \subseteq \mathfrak{h}_{n,2}$, while the denominator 2 is the homogeneous degree of $-\mathcal{L}_{\mathbf{H}_{n,2}}$.

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Abstract

In what follows we shall present some recent results about the validity of smoothing and Strichartz-type estimates for time-degenerate Schrödinger operators. These results have important applications in the study of the local well-posedness of the initial value problem (IVP) associated with the operators under consideration.

Time degenerate Schrödinger-type Operators

We shall consider the following classes of degenerate Schrödinger-type operators

$$\mathcal{L}_0 = i\partial_t + P^2 \Delta_x + A(t, x) \cdot \nabla_x, \quad (1)$$

$$\mathcal{L}_1 = i\partial_t + P^2 \Delta_x, \quad (2)$$

where $a > 0$, $k, \ell, x = (k_1(t, x), \dots, k_\ell(t, x))$ is such that, for all $j = 1, \dots, \ell$, $k_j \in C([0, T], C_b^\infty(\mathbb{R}^{n_j}))$, while $v \in C^1(\mathbb{R})$.

The class (1) was considered in [1] where both homogeneous and inhomogeneous weighted local smoothing estimates are derived. These estimates are also employed to obtain local well-posedness results for the associated nonlinear IVP (see [1]).

The class (2) was studied in [2] where global weighted homogeneous smoothing estimates are proved by means of comparison principles. For the class (2) weighted Strichartz estimates are proved in [2] where the application to the local well-posedness of the semilinear IVP is given.

The main difference between the operators of the form (1) and (2) and the other Schrödinger operators studied so far is the presence of degeneracies. Specifically, the degeneracies are given by the coefficient P^2 and $v(t)$ in (1) and (2) respectively.

Why to study smoothing and Strichartz estimates?

These estimates give us important information about the regularity properties of the solution of the IVP. In particular:

- The **homogeneous smoothing** effect describes a gain of smoothness of the homogeneous solution of the IVP with respect to the smoothness of the initial data.
- The **inhomogeneous smoothing** effect describes a gain of smoothness of the solution of the inhomogeneous IVP with respect to the regularity of the inhomogeneous data.
- **Strichartz estimates** describe a gain of integrability instead of a gain of smoothness of the solution of the IVP.

Additionally, these estimates are fundamental in order to prove the local well-posedness of the corresponding semilinear and nonlinear IVP through the standard fixed point argument.

What are comparison principles? (Following [3])

Question: Given two operators $P_1(t, x, D_x, D_t)$ and $P_2(t, x, D_x, D_t)$ depending on two functions a and b respectively, is it possible to compare (in a suitable sense) the solutions of the HVP (homogeneous IVP) for P_1 and P_2 if a and b are comparable (in a suitable sense)?

This is essentially what the comparison principles we refer to do, that is, they translate a relation between a and b in a relation between the solutions of the HVP for P_1 and P_2 .

Example (see [3]). Let $a, b \in C^1(\mathbb{R})$ be real-valued and strictly monotone on the support of a measurable function χ , and let $a, b \in C^1(\mathbb{R})$. Then, if $a \leq b$ on $\text{supp } \chi$, we have

$$\frac{\|u\|_{L^2(\mathbb{R}^n)}}{\|v\|_{L^2(\mathbb{R}^n)}} \leq C \frac{\|a\|_{L^2(\mathbb{R}^n)}}{\|b\|_{L^2(\mathbb{R}^n)}} \quad \text{then} \quad \|u\|_{L^2(\mathbb{R}^n)} \leq C \|v\|_{L^2(\mathbb{R}^n)} \quad \text{for all } u = P_1(t, x, D_x, D_t) \chi, v = P_2(t, x, D_x, D_t) \chi$$

for all $\mu = P_1(t, x, D_x, D_t) \chi$ smooth enough.

Weighted local smoothing effect for $\mathcal{L}_0$

We consider the IVP

$$\begin{cases} i\partial_t u = P^2 \Delta_x u + A(t, x) \cdot \nabla_x u + f(t, x) \\ u(0, x) = u_0(x). \end{cases} \quad (3)$$

When A is 0 one can proceed by using Fourier analysis. However, in the general case $A \neq 0$ is the use of pseudo-differential calculus is needed.

Theorem (F.-Staffilani)

Let $u_0 \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Assume that, for all $j = 1, \dots, \ell$, k_j is such that $k_j \in C([0, T], C_b^\infty(\mathbb{R}^{n_j}))$ and there exists $\alpha > 1$ such that

$$|\text{Im } k_j(t, x)| \leq C |k_j(t, x)|^{\alpha-1}, \quad x \in \mathbb{R}^{n_j}, \quad (4)$$

and denote by $k(t, x) = (k_1(t, x), \dots, k_\ell(t, x))$.

Then

(i) If $f \in L^2([0, T], H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0, T], H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \leq C_1 e^{C_2 \int_0^T \|k(t)\|_{L^\infty} dt} \left(\|u_0\|_{H^s} + \int_0^T \|f(t)\|_{L^2} dt \right).$$

(ii) If $f \in L^2([0, T], H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0, T], H^s(\mathbb{R}^n))$ and there exist two positive constants C_1, C_2 such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s}^2 + \int_0^T \int_{\mathbb{R}^n} |k|^{s+1} |u|^2 dx dt \leq C_1 e^{C_2 \int_0^T \|k(t)\|_{L^\infty} dt} \left(\|u_0\|_{H^s}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt \right).$$

(iii) If $k^{s+1/2} f \in L^2([0, T], \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ then the IVP (3) has a unique solution $u \in C([0, T], H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s}^2 + \int_0^T \int_{\mathbb{R}^n} |k|^{s+1} |u|^2 dx dt \leq C_1 e^{C_2 \int_0^T \|k(t)\|_{L^\infty} dt} \left(\|u_0\|_{H^s}^2 + \int_0^T \int_{\mathbb{R}^n} |k|^{s+1} |f|^2 dx dt \right).$$

Local well-posedness results for $\mathcal{L}_0$

We consider the nonlinear IVPs

$$\begin{cases} \text{IVP1} = \begin{cases} i\partial_t u = \pm u |u|^2 \\ u(0, x) = u_0(x). \end{cases} & \text{IVP2} = \begin{cases} i\partial_t u = \pm u \nabla_x u \cdot x^{\beta-1} \\ u(0, x) = u_0(x). \end{cases} \end{cases} \quad \beta \geq \alpha > 0.$$

Theorem (F.-Staffilani). Let $\mathcal{L}_0$ be such that condition (4) is satisfied. Then the IVP1 is locally well posed in H^s for $s > n/2$.

Theorem (F.-Staffilani). Let $\mathcal{L}_0$ be such that condition (4) is satisfied with $\beta = 2N$ (thus $k(t, x) = (x_1)^{2N}$) for some $N \geq 1$, and $\alpha > n + 2N + 3$ such that $\alpha - 1/2 \in \mathbb{N}$. Let $H_x^s = \{u_0 \in H^s(\mathbb{R}^n); u_0 \in H^s(\mathbb{R}^n)\}$, then the IVP2 with $\beta \geq \alpha > 0$ is locally well posed in H_x^s .

Global weighted smoothing and Strichartz estimates for $\mathcal{L}_1$

For operators of the form $\mathcal{L}_1$, several comparison principles are proved in [2]. These are used to obtain global smoothing estimates. We state below only one of them, namely the one corresponding to the suitable generalization of the standard (corresponding to the case $v(t) = 1$) global homogeneous smoothing estimate. For more global smoothing estimates see [2].

Theorem (F.-Ruzhansky)

Let $v \geq 1, v \in C^1(\mathbb{R})$ be such that it vanishes at 0. Then, for $v \in \mathbb{R}^n$,

(i) If v is such that $\{t \in \mathbb{R}; v(t) = 0\}$ is finite, then there exist a constant $C > 0$ such that, for all f ,

$$\sup_{x_j} \|v^{1/2} |f|^{1/2} \|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in L^2(\mathbb{R}^n).$$

(ii) If v is such that the set $\{t \in \mathbb{R}; v(t) = 0\}$ is countable, then there exists a function $\lambda \in C(\mathbb{R})$, and a positive constant C such that, for all f ,

$$\sup_{x_j} \|v^{1/2} |f|^{1/2} \|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in L^2(\mathbb{R}^n).$$

where $x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$.

Both global and local Strichartz estimates are satisfied by $\mathcal{L}_1$. Below we give the statement of the local ones which are those employed to prove the local well-posedness of the semilinear IVP.

Theorem (F.-Ruzhansky)

Let $v \in C^1([0, T])$ be vanishing at 0 and such that $\{t \in [0, T]; v(t) = 0\} = \{t = 0\}$. Then, on denoting by $L^2([0, T], L^p(\mathbb{R}^n))$, we have that for any (q, p) admissible pair $(\frac{2}{q} + \frac{n}{p} = \frac{n}{2})$ such that $2 < q, p < \infty$, the following estimates hold

$$\|v^{1/2} |u|^{1/2} \|_{L^2([0, T], L^p(\mathbb{R}^n))} \leq C(n, q, p, T) \|u\|_{L^2([0, T], L^p(\mathbb{R}^n))},$$

$$\|v^{1/2} |u|^{1/2} \|_{L^2([0, T], L^p(\mathbb{R}^n))} \leq C(n, q, p, T) \|u\|_{L^2([0, T], L^p(\mathbb{R}^n))},$$

$$\|v^{1/2} |u|^{1/2} \|_{L^2([0, T], L^p(\mathbb{R}^n))} \leq C(n, q, p, T) \|u\|_{L^2([0, T], L^p(\mathbb{R}^n))},$$

$$\|v^{1/2} |u|^{1/2} \|_{L^2([0, T], L^p(\mathbb{R}^n))} \leq C(n, q, p, T) \|u\|_{L^2([0, T], L^p(\mathbb{R}^n))}.$$

Local well-posedness of the semilinear IVP for $\mathcal{L}_1$

We can now apply the previous results to obtain the local well-posedness of the semilinear IVP

$$\begin{cases} i\partial_t u + v \Delta_x u = \mu |u|^2 u \\ u(0, x) = u_0(x). \end{cases} \quad \mu \in \mathbb{R}. \quad (5)$$

Theorem (F.-Ruzhansky). Let $1 < p < \frac{2}{n} + 1$ and $v \in C^1([0, \infty))$ be vanishing at 0 and it is either strictly monotone or such that $\{t \in [0, T]; v(t) = 0\} = \{t = 0\}$ for any $T < \infty$. Then for all $u_0 \in L^2(\mathbb{R}^n)$ there exists $T = T(\|u_0\|_{L^2}, n, \mu, p) > 0$ such that there exists a unique solution u of the IVP (5) in the time interval $[0, T]$ with $u \in C([0, T], L^2(\mathbb{R}^n)) \cap L^p([0, T], L^2(\mathbb{R}^n))$ and $q = \frac{2n}{n-2}$. Moreover the map $u_0 \mapsto u(\cdot, t)$, locally defined from $L^2(\mathbb{R}^n)$ to $C([0, T], L^2(\mathbb{R}^n))$, is continuous.

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Hardy-Littlewood inequality and L^p - L^q Fourier multipliers on compact hypergroups

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Aim

- To prove Hardy-Littlewood inequality and Paley inequality for compact hypergroups [6].
- To establish Hörmander's L^p - L^q Fourier multiplier theorem on compact hypergroups for all $1 < p \leq 2 \leq q < \infty$ [9].

Hypergroups: What & why?

- Roughly, a hypergroup K is a locally compact Hausdorff space with a convolution on the space $M_b(K)$ of regular bounded measures on K with properties similar to those of group convolution.
- In non-commutative setting, the analysis on hypergroups provides a natural extension of analysis on locally compact groups. After its commutative setting, they extend the theory of spherical functions and Gelfand pairs.
- Some of important examples are the Heisenberg group, the space of conjugacy classes on Lie groups and the space of group orbits.
- In particular, the results presented here are true for several interesting examples including Jacobi hypergroups with Jacobi polynomials as characters, compact hypergroups structure on the fundamental domain with Hecke-Opfer polynomials as characters and addition on club hypergroups.
- A compact hypergroup can be naturally realized also (K, τ) . This property distinguishes them from conjugacy groups.
- Unlike locally compact groups, the support of the Plancherel measure on the dual space may not be full space in the case of commutative hypergroups.
- For more details on analysis on hypergroups and several interesting examples, see [4, 20, 43].

Fourier analysis on compact hypergroups

- Let K be compact hypergroup with normalized Haar measure λ . Denote by $\hat{K}$ the dual space consisting of irreducible inequivalent continuous representations of K equipped with the discrete topology.
- Every $\pi \in \hat{K}$ is finite dimensional but may not be unitary in contrast to compact groups case.
- In commutative setting also, the dual space $\hat{K}$ may not have a hypergroup structure, as in the case of solvable groups.
- Denote the convolution and hyperderivation of $\pi \in \hat{K}$ by α_π and δ_π .
- For each $x \in K$, the Fourier transform f of $f \in L^1(K)$ is defined as

$$\hat{f}(\pi) = \int_K f(x)\pi(x) d\lambda(x),$$

- where λ is the conjugate representation of π .
- We refer to [10, 4, 43] for more details on Fourier analysis and representation theory of compact hypergroups.

Methods

- To establish Hardy-Littlewood-Paley inequality we first prove Paley inequality [2, 11] for compact hypergroups [8] and then use weighted interpolation with Hausdorff-Young inequality [5].
- An application of Paley inequality gives Hardy-Littlewood inequality for compact hypergroups.
- We obtain Hörmander's L^p - L^q multiplier of Fourier multiplier [7] by means of compact hypergroup with the help of the Hausdorff-Young-Paley inequality.



Literature

- Hardy-Littlewood inequality [3] was recently extended to compact conjugacy groups [8] and to compact quantum groups [1, 3].
- L^p - L^q boundedness of Fourier multiplier on locally compact quantum groups was proved in [1] using von Neumann algebra techniques.

Hausdorff-Young-Paley inequality on compact hypergroups

Theorem (Hausdorff-Young-Paley (Pitt) inequality)

Let K be a compact hypergroup and let $1 < p \leq q \leq p' < \infty$. If a function satisfies $\|f\|_p \leq A$, consider the condition

$$M_p = \sup_{\alpha \in \hat{K}} \sum_{\beta \in \hat{K}} \alpha_\beta < \infty,$$

then we have

$$\left(\sum_{\beta \in \hat{K}} \alpha_\beta \left(\frac{\|\hat{f}\|_q}{\sqrt{\alpha_\beta}} \alpha_\beta \right)^p \right)^{1/p} \leq M^{1/p} \|f\|_p.$$

Non-commutative version of Hardy-Littlewood inequality

Theorem (Hardy-Littlewood inequality for compact hypergroups)

Let $1 < p \leq 2$ and let K be a compact hypergroup. Assume that a sequence $\{\alpha_\pi\}_{\pi \in \hat{K}}$ grows sufficiently fast, that is,

$$\sum_{\pi \in \hat{K}} \frac{\alpha_\pi}{(\alpha_\pi)^{1/p}} < \infty \quad \text{for some } p \geq 0.$$

Then we have

$$\sum_{\pi \in \hat{K}} \alpha_\pi^{1/p} \left(\frac{\|\hat{f}\|_q}{\sqrt{\alpha_\pi}} \right)^p \leq \|f\|_p^p.$$

L^p - L^q boundedness of Fourier multipliers on compact hypergroups

Theorem (Hörmander theorem for Fourier multipliers)

Let K be a compact hypergroup and let $1 < p \leq 2 \leq q < \infty$. Let A be a left Fourier multiplier with symbol a_π . And let δ_π satisfy

$$A(f)(x) = a_\pi(x)f(x), \quad x \in K.$$

Then we have

$$\|A(f)\|_q \leq C \sup_{\pi \in \hat{K}} \left(\sum_{\beta \in \hat{K}} \alpha_\beta \right)^{1/p} \|f\|_p.$$

H-L inequality for Conj(SU(2))

Let $1 < p \leq 2$ and $f \in L^p(\text{Conj}(\text{SU}(2)))$, then we have

$$\sum_{\alpha \in \hat{K}} (\alpha + \alpha^*)^{-1/p} \|\hat{f}\|_q^p \leq C \|f\|_p^p.$$

H-L for D-II hypergroups [8]

Let $1 < p \leq 2$ then there exists a constant $C = C(p)$ such that

$$\|f\|_q^p \leq C \sum_{\alpha \in \hat{K}} (\alpha + \alpha^*)^{-1/p} \|\hat{f}\|_q^p + C \|f\|_p^p.$$

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Introduction

In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals. This estimate was first obtained by the Dutch mathematician Johannes Gauthier van der Corput [1] and named in his honour. While the paper [1] was published in *Mathematische Annalen* in 1921, he submitted it there on 17 December 1920 (from Utrecht). Therefore, it seems appropriate to us to dedicate this paper to the 100th anniversary of this lemma. Let us state the classical van der Corput lemmas as follows:

• **van der Corput lemma.** Suppose ϕ is a real-valued and smooth function in $[a, b]$. If ψ is a smooth function and $|\phi^{(k)}(x)| \geq 1$, $k \geq 1$, for all $x \in (a, b)$, then

$$\left| \int_a^b \exp(i\lambda\phi(x))\psi(x)dx \right| \leq \frac{C}{\lambda^{1/k}}, \quad \lambda \rightarrow \infty, \quad (1)$$

for $k = 1$ and ϕ' is monotonic, or $k \geq 2$. Here C does not depend on λ .

Formulation of problem

The main goal of the present paper is to study van der Corput lemmas for the oscillatory integrals defined by (see [2], [3]) respectively:

$$I_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx, \quad (2)$$

and

$$\mathcal{I}_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx, \quad (3)$$

where $\alpha > 0$, $\beta > 0$, ϕ is a phase and ψ is an amplitude, and λ is a positive real number that can vary. Here $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R},$$

with the property that

$$E_{1,1}(z) = e^z, \quad (4)$$

Main results for $I_{\alpha,\beta}$

van der Corput lemmas on $\mathbb{R}$: consider $I_{\alpha,\beta}$ defined by (2).

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha < 1$, $\beta > 0$, and $m = \operatorname{ess\,inf}_{x \in \mathbb{R}} |\phi'(x)| > 0$, then

$$|I_{\alpha,\beta}(\lambda)| \leq \frac{M}{1 + \lambda m} \|\psi\|_{L^1(\mathbb{R})}, \quad \lambda \geq 1,$$

where M does not depend on ϕ , ψ and λ .

van der Corput lemmas on $I = [a, b] \subset \mathbb{R}$: $-\infty < a < b < +\infty$.

• Let $0 < \alpha < 1$, $\beta > 0$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-1} \log(1 + \lambda), \quad \lambda \geq 1,$$

where M_k does not depend on λ .

• Let $-\infty < a < b < +\infty$ and $I = [a, b] \subset \mathbb{R}$. Let $0 < \alpha \leq 1$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$. Then

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-1/k}, \quad \lambda \geq 1, \quad (5)$$

for $k = 1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ .

Main results for $\mathcal{I}_{\alpha,\beta}$

van der Corput lemmas on $\mathbb{R}$: consider $\mathcal{I}_{\alpha,\beta}$ defined by (3).

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha \leq 2$, $\beta > 1$, and $m = \operatorname{ess\,inf}_{x \in \mathbb{R}} |\phi'(x)| > 0$, then

$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq \frac{M}{1 + \lambda m} \|\psi\|_{L^1(\mathbb{R})}, \quad \lambda \geq 1, \quad 0 < \alpha < 2, \quad \beta \geq \alpha + 1,$$

$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq \frac{M}{(1 + \lambda m)^{1/\alpha}} \|\psi\|_{L^1(\mathbb{R})}, \quad \lambda \geq 1, \quad 0 < \alpha \leq 2, \quad 1 < \beta < \alpha + 1,$$

where M does not depend on ϕ , ψ and λ .

van der Corput lemmas on $I = [a, b] \subset \mathbb{R}$: $-\infty < a < b < +\infty$.

• Let $0 < \alpha < 2$, $\beta > 1$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then

$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-1} \log(1 + \lambda), \quad \lambda \geq 1, \quad 0 < \alpha < 2, \quad \beta \geq \alpha + 1,$$

$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-1} (1 + \lambda)^{-1/\alpha}, \quad \lambda \geq 1, \quad 0 < \alpha < 2, \quad 1 < \beta < \alpha + 1,$$

where M_k does not depend on λ .

• Let $-\infty < a < b < +\infty$ and $I = [a, b] \subset \mathbb{R}$. Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$. Then

$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-1/k}, \quad \lambda \geq 1, \quad (6)$$

for $k = 1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ .

Remark

The case of $\alpha = 1$ in (5) and (6) corresponds to the classical van der Corput lemma (1).

Conclusion

The main goal of the paper was to study van der Corput lemmas for the integrals defined by (2) and (3). Van der Corput type lemmas were obtained, for the different cases of parameters α and β . As an immediate application of the obtained results, time-estimates of the solutions of time-fractional Klein-Gordon and Schrödinger equations and generalisations of the Riemann-Lebesgue lemma were also considered in [2] and [3].

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Abstract

A Gelfand triple is a triple of spaces (E, H, E') such that H is a Hilbert space and there are continuous dense embeddings $E \hookrightarrow H \hookrightarrow E'$. A classical example is $(\mathcal{S}(\mathbb{R}), L^2(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ in which $\mathcal{S}'(\mathbb{R})$ is the space of rapidly decreasing functions on a nilpotent Lie group G . But characterizing the image of $\mathcal{S}'(\mathbb{R})$ under the group Fourier transform $\mathcal{F}_G$ is difficult and uneasy. For this purpose we develop and study new Gelfand triplets for the Fourier transform and the Kohn-Nirenberg quantization. The new Gelfand triplets contain distribution spaces for which we may define multiplication on the Fourier side for a large class of smooth covariant valued functions. In this paper we may also show that the formula $\sigma = \sigma^* - (A \otimes 1) \sigma$ for the Kohn-Nirenberg symbol σ holds for any operator $A \in \mathcal{L}(G, \mathbb{C})$.

Gelfand triples for the Fourier transform

Homogeneous Lie groups

Let G be a homogeneous Lie group, i.e. G is a nilpotent Lie group together with an invertible derivation A on the Lie algebra $\mathfrak{g}$ with only positive eigenvalues. The group of automorphisms

$$\delta^s := (1 + sA)^{-1} A, \quad s \in \mathbb{R}, \quad \text{where } A \cdot x = e^{A \cdot x} \partial_x(x)$$

are the dilations on G . The trace $\text{tr}(A)$ is the homogeneous dimension of G . Let $\tilde{G}$ be the dual of G , i.e. the set of unitary equivalence classes of irreducible unitary representations. Since G is nilpotent we may use Kirillov's correspondence between elements of $\tilde{G}$ and coadjoint orbits.

The group Fourier transform for $\dim Z(G) = 1$ and $\mathcal{S}(G) \neq \emptyset$

Let ρ be a fixed irreducible unitary representation that is square integrable modulo the center (in symbols $\rho \in \mathcal{I}(G)$). Furthermore we choose a real structure G_ρ on $(H^\rho, \langle \cdot, \cdot \rangle, M, \tau)$, where M is the representation space of ρ . H^ρ is the space of smooth vectors of ρ and $M^{\text{reg}} := (M^\rho)^\tau$. We will write $\chi_\lambda(x) := \rho(x)\chi$, $\lambda \in \mathbb{R}$ and $\chi_\lambda(x) = G_\rho(x)\chi(x)G_\rho^{-1}$, $\lambda \in \mathbb{R}^+ := \mathbb{R} \setminus \{0\}$. Using this definition, we may define the following decomposition of invariant spaces

$$(\mathcal{S}^{\text{reg}}_\rho)_\pm(\mathcal{S}^{\text{reg}}_\rho)^{\text{reg}}(G) \rightarrow (\tilde{\mathcal{S}}^{\text{reg}}_\rho)_\pm(G), \quad \chi \mapsto [\chi] \quad (1)$$

in which $\tilde{\mathcal{S}}^{\text{reg}}_\rho \subset \tilde{G}$ are the equivalence classes corresponding to generic orbits and μ is the Plancherel measure. Since $\mathcal{I}(G) \cap \tilde{\mathcal{S}}^{\text{reg}}_\rho = \emptyset$, we may use (1) and Pedersen's machinery [3] to construct a unitary isomorphism

$$\tilde{\mu} : L^2(\mathbb{R}^+; \rho)(\mathcal{S}^{\text{reg}}_\rho)^{\text{reg}}(G) \xrightarrow{\sim} L^2(\tilde{G}) = \mathcal{F}_G L^2(G).$$

We will use $\tilde{\mu}$ to define an adjoint Fourier transform as described in Figure 1.

Defining new Gelfand triples

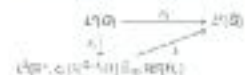


Figure 1: Commutative diagram describing $\tilde{\mu}$.

But it is rather inconvenient that the Fourier image of the space of Schwartz functions $\mathcal{S}(G)$ does not decompose into a tensor product of a functions space and an operator space. In order to bypass the problematic behaviour of $\mathcal{F}_G \mathcal{S}(G)$ in $\lambda \rightarrow 0$ for $\rho \in \mathcal{I}(G)$, we will use operator valued functions that vanish in $\lambda \rightarrow 0$. Denote by $\mathcal{S}_0(G)$ the space of Schwartz functions on $\mathbb{R}$ that are orthogonal to all polynomials. We will also denote by $\mathcal{S}_0(\mathbb{R})$ the set of Schwartz functions on $\mathbb{R}$ that vanish of arbitrarily high order in $\lambda \rightarrow 0$. We split the Lie algebras into vector subspaces $\mathfrak{g} = Z(G) \oplus \mathfrak{g}_0$ and define

$$\mathcal{S}_0(G) := \{ \rho \in \mathcal{S}_0(G) \mid \|\rho\|_0 \leq \rho \leq \sup_{\lambda \in \mathbb{R}^+} \lambda^2 \rho \in \mathfrak{g}_0 \in \mathcal{A}(\mathcal{H}^{\rho}_0, \mathcal{H}^{\rho}_0) \}$$

for any $Z \in Z(G) \setminus \{0\}$. Using this definition we may formulate the following theorem.

Theorem

The map $\tilde{\mu}$ is an isomorphism in each case of

$$\tilde{\mu} : \begin{pmatrix} \mathcal{S}_0(G) \\ L^2(\mathbb{R}^+; \rho)(\mathcal{S}^{\text{reg}}_\rho)^{\text{reg}}(G) \\ \mathcal{S}_0(G) \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{S}_0(\mathbb{R}^+; \mathbb{C}M^{\text{reg}}_\rho(A)) \\ L^2(\mathbb{R}^+; \rho)(\mathcal{S}^{\text{reg}}_\rho)^{\text{reg}}(A) \\ \mathcal{S}_0(\mathbb{R}^+; \mathbb{C}M^{\text{reg}}_\rho(A)) \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{S}_0(\mathbb{R}^+; \mathbb{C}) \\ L^2(\mathbb{R}^+; \rho) \\ \mathcal{S}_0(\mathbb{R}^+; \mathbb{C}) \end{pmatrix}$$

Gelfand triples for the Kohn-Nirenberg quantization on a homogeneous Lie group

Jonas Brinkon
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Multiplication operators on the Fourier side

Due to [1] we may identify a large class of operator valued functions that act as multiplication operators on $\mathcal{S}_0(\mathbb{R}^+; \mathbb{C})$ and $\mathcal{S}_0(\mathbb{R}^+; \mathbb{C})$. In the following $\mathcal{O}_0(\mathbb{R}^+)$ denotes the space of multiplication operators on $\mathcal{S}_0(\mathbb{R}^+)$.

Then we

The multiplication $(f, g) \mapsto f \cdot g$, defined by pointwise composition, is an isomorphism on linear maps

$$\mathcal{O}_0(\mathbb{R}^+; \mathbb{C}, \mathcal{O}_0(\mathbb{R}^+; \mathbb{C})) \rightarrow \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \rightarrow \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \quad (2)$$

and by changing order there is an isomorphism on multiplication

$$\mathcal{O}_0(\mathbb{R}^+; \mathbb{C}, \mathcal{O}_0(\mathbb{R}^+; \mathbb{C})) \rightarrow \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \rightarrow \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \quad (3)$$

Gelfand triples for the Kohn-Nirenberg quantization

We may use the Kohn-Nirenberg quantization as the map

$$\text{Op} : L^2(\tilde{G}) \rightarrow L^2(\tilde{G}) \rightarrow \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}), \quad \text{Op}(f) = \mathcal{F}_G^{-1} \tilde{\mu}^{-1}(f) \in \mathcal{O}_0(\mathbb{R}^+; \mathbb{C})$$

in which $\mathcal{F}_G(x, y) = \langle x, y \rangle$ and $\mathcal{F}_G : \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \rightarrow L^2(\tilde{G} \times \tilde{G})$ is the canonical isomorphism with $(A, \psi) \mapsto \langle \psi, A \rangle \in \mathcal{O}_0(\mathbb{R}^+; \mathbb{C})$. Let us call Op_0 the map we obtain by substituting $\mathcal{F}_G$ by $\mathcal{F}_0$. Similar to Figure 1 we get the commutative diagram



Figure 2: Commutative diagram describing Op_0 .

Notice that $\mathcal{F}_0$ restricts to an isomorphism on $\mathcal{O}_0(\mathbb{R}^+; \mathbb{C}, \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}))$ with that $(f, g) \mapsto (f, \mathcal{F}_0^{-1}g)$. By (4) we may formulate the next theorem.

Theorem

The Kohn-Nirenberg quantization, Op_0 , is an isomorphism in each case of

$$\text{Op}_0 : \begin{pmatrix} \mathcal{S}_0(G) \\ L^2(\mathbb{R}^+; \rho)(\mathcal{S}^{\text{reg}}_\rho)^{\text{reg}}(G) \\ \mathcal{S}_0(G) \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \\ L^2(\mathbb{R}^+; \rho)(\mathcal{O}_0(\mathbb{R}^+; \mathbb{C})) \\ \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \end{pmatrix}$$

On the integral formula for Kohn-Nirenberg operators

We may realize that $\rho(x, \lambda) \mapsto \rho(x, \lambda)$ is an operator valued function in $(\mathcal{S}_0(\mathbb{R}^+; \mathbb{C}), \mathcal{L}(H^\rho))$. By using the continuity and invertibility of Op_0 , the continuous dense embedding $\mathcal{S}_0(G) \hookrightarrow L^2(\tilde{G})$ and the hypocontinuous multiplication on $\mathcal{O}_0(G \times \mathbb{R}^+) \subset \mathcal{L}(H^\rho)$, we are able to prove the following theorem.

Theorem

For any $A \in \mathcal{O}_0(\mathbb{R}^+; \mathbb{C})$, $\mathcal{F}_0 : \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \rightarrow \mathcal{O}_0(\mathbb{R}^+; \mathbb{C})$, the equality

$$\sigma = \sigma^* - (A \otimes 1) \sigma = \text{Op}_0(A) \text{ is valid and we may evaluate it by}$$

$$\sigma(x) = \int_{\mathbb{R}^+} \rho(y, \lambda) \cdot \mathcal{F}_0^{-1}(\text{Op}_0(A)) \mathcal{F}_0^{-1}(x) \, d\lambda, \quad \text{for } x \in \mathcal{O}_0(\mathbb{R}^+)$$

where the integral exists weakly in H^ρ . Furthermore

$$\text{Op}_0^{-1} : \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \rightarrow \mathcal{O}_0(\mathbb{R}^+; \mathbb{C}) \rightarrow \mathcal{O}_0(\mathbb{R}^+; \mathbb{C})$$

is continuous.

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Noncommutative Fourier Series on $\Gamma \backslash SE(2)$

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Abstract

This work begins with a systematic study of abstract noncommutative Fourier series on $\Gamma \backslash SE(2)$, where Γ is a discrete and co-compact subgroup of $SE(2)$, the group of handedness-preserving Euclidean isometries of the Euclidean plane.

Introduction

The 2D special Euclidean group, usually denoted as $SE(2)$, is one of the simplest examples of a noncommutative, noncompact real finite-dimensional Lie group. The right coset space of discrete and co-compact subgroups in $SE(2)$, such as $\mathbb{Z}^2 \backslash SE(2)$, appears as the configuration space in many recent applications in computational science and engineering including computer vision, robotics, mathematical crystallography, computational biology, and material science [2, 3, 4].

Preliminaries and Notation

The 2D special Euclidean group, $SE(2)$, is the semidirect product of $\mathbb{R}^2$ with the 2D special orthogonal group $SO(2)$. We denote elements $g \in SE(2)$ as $g = (\mathbf{x}, \mathbf{R})$ where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{R} \in SO(2)$. The classical Fourier Plancherel formula on the group $SE(2)$ is

$$\int_{SE(2)} |f(g)|^2 dg = \int_{\mathbb{R}^2} \|f(p)\|_{L^2(\mathbb{S}^1)}^2 dp,$$

and also the noncommutative Fourier reconstruction formula is

$$f(g) = \int_{\mathbb{R}^2} \text{tr}[f(p)U_p(g)] dp, \quad (1)$$

for $f \in L^1 \cap L^2(SE(2))$ and $g \in SE(2)$, where for each $p > 0$, the irreducible representation $U_p : SE(2) \rightarrow \mathcal{M}(L^2(\mathbb{S}^1))$ is

$$[U_p(g)\varphi](\mathbf{u}) := e^{-i\langle \mathbf{u}, \mathbf{x} \rangle} \varphi(\mathbf{R}^T \mathbf{u}), \quad (2)$$

for all $g = (\mathbf{x}, \mathbf{R}) \in SE(2)$, $\mathbf{x} \in L^2(\mathbb{S}^1)$, $\mathbf{u} \in \mathbb{S}^1$, and the Fourier transform of each $f \in L^1(SE(2))$ at $p > 0$ is given by

$$\hat{f}(p) := \int_{SE(2)} f(g)U_p(g) dg. \quad (3)$$

The standard orthonormal basis for the Hilbert function space $L^2(\mathbb{S}^1)$ is $\mathbf{B} := \{\mathbf{e}_k : k \in \mathbb{Z}\}$, where for each $k \in \mathbb{Z}$, $\mathbf{e}_k : \mathbb{S}^1 \rightarrow \mathbb{C}$ is given by

$$\mathbf{e}_k(\mathbf{u}_\alpha) = \mathbf{e}_k(\cos \alpha, \sin \alpha) := e^{ik\alpha}.$$

For each $g \in SE(2)$ and $p > 0$, the matrix elements of the operator $U_p(g)$ with respect to the basis $\mathbf{B}$, are expressed as

$$u_{mn}(g, p) := \langle U_p(g)\mathbf{e}_m, \mathbf{e}_n \rangle_{L^2(\mathbb{S}^1)}, \quad (4)$$

for each $m, n \in \mathbb{Z}$, see [8].

State of the problem

- Let Γ be a discrete co-compact subgroup of $SE(2)$.
- Suppose μ is the finite $SE(2)$ -invariant measure on the right coset space $\Gamma \backslash SE(2)$ which is normalized with respect to Weil's formula given by

$$\int_{\Gamma \backslash SE(2)} f(\Gamma g) d\mu(\Gamma g) = \int_{SE(2)} f(g) dg, \quad (5)$$

for $f \in L^1(SE(2))$, where

$$f(\Gamma g) = \sum_{\gamma \in \Gamma} f(\gamma \circ g), \quad (6)$$

for $g \in SE(2)$.

- Let $\varphi = \tilde{f} \in L^1(\Gamma \backslash SE(2), \mu)$ with $f \in L^1(SE(2))$ be given.
- How to expand φ using coefficients of Fourier matrix elements of f on $SE(2)$ given by (3)?

Let $f \in L^1(SE(2))$ and $\psi \in L^1(\Gamma \backslash SE(2), \mu)$. We then define the convolution of f with ψ as the function $\psi \circ f : \Gamma \backslash SE(2) \rightarrow \mathbb{C}$ via

$$(\psi \circ f)(\Gamma g) = \int_{SE(2)} \psi(\Gamma h) f(h^{-1} \circ g) dh, \quad (9)$$

for $g \in SE(2)$.

Theorem (GF-Chirikjian)

Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma \backslash SE(2) \rightarrow \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma \backslash SE(2), \mu)$. Let $f_k \in L^1(SE(2))$ with $k \in \{1, 2\}$ such that $[\tilde{f}_k] \in L^1(\Gamma \backslash SE(2), \mu)$. We then have

$$(\tilde{f}_k \circ f_\ell, \psi_\ell) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \sum_{p \in \mathbb{R}^+} \int_{\mathbb{S}^1} \tilde{f}_k(p) u_{nm}(p) \tilde{f}_\ell(p) u_{nm}(p) dp. \quad (10)$$

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Acknowledgements

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Theorem (GF-Chirikjian)

Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma \backslash SE(2) \rightarrow \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma \backslash SE(2), \mu)$. Let $f \in L^1 \cap L^2(SE(2))$ such that $[\tilde{f}] \in L^1(\Gamma \backslash SE(2), \mu)$. We then have

$$(\tilde{f}, \psi_\ell) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^+} f(p) u_{nm}(p) Q^\ell(p) dp, \quad (7)$$

where

$$Q^\ell(p) := \int_{SE(2)} u_{nm}(g, p) \overline{\psi_\ell(\Gamma g)} dg,$$

and

$$f(p)_{nm} = \langle \tilde{f}(p) \mathbf{e}_n, \mathbf{e}_m \rangle,$$

for $p > 0$, $\ell \in \mathbb{I}$ and $m, n \in \mathbb{Z}$.

Proposition (GF-Chirikjian)

Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma \backslash SE(2) \rightarrow \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma \backslash SE(2), \mu)$. Let $f \in L^1(SE(2))$ such that $[\tilde{f}] \in L^1(\Gamma \backslash SE(2), \mu)$. We then have

$$(\tilde{f}, \psi_\ell) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^+} f(p) u_{nm}(p) Q^\ell(p) dp, \quad (8)$$



PROPAGATION OF SINGULARITIES ON HYPO-ANALYTIC STRUCTURES

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GEVREY VECTORS

Let Ω be a smooth manifold of dimension $m > 0$ and consider $\{L_1, \dots, L_m\}$, pairwise commuting complex vector fields defined on Ω , where $n \leq m$. Let $s \geq 1$, we say that a distribution $u \in \mathcal{D}'(\Omega)$ is a **Gevrey- s vector** for $\{L_1, \dots, L_m\}$ if $u \in \mathcal{C}^\infty(\Omega)$ and for every $K \subset \Omega$, compact set, there is a positive constant $C = C_K$ such that

$$\sup_{x \in K} |L^\alpha u(x)| \leq C^{|\alpha|+1} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^m,$$

and we write $u \in G^s(\Omega; L_1, \dots, L_m)$. If $s = 1$ we write $u \in \mathcal{C}^\infty(\Omega; L_1, \dots, L_m)$, and we say that u is an **analytic vector** for $\{L_1, \dots, L_m\}$.

HYPO-ANALYTIC STRUCTURES

Let Ω be a smooth manifold of dimension $n + m$. A hypo-analytic structure on Ω of corank m is a family $\{U_\alpha, (Z_{\alpha,1}, \dots, Z_{\alpha,m})\}_{\alpha \in \Lambda}$ satisfying the following conditions:

- $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open covering of Ω ;
- $Z_{\alpha,j} : U_\alpha \rightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function for every $j = 1, \dots, m$ and $\alpha \in \Lambda$;
- $dZ_{\alpha,1}, \dots, dZ_{\alpha,m}$ are $\mathbb{C}$ -linearly independent at each point of U_α , for every $\alpha \in \Lambda$;
- If $U_\alpha \cap U_\beta \neq \emptyset$ then for every $p \in U_\alpha \cap U_\beta$ there exists a biholomorphic map $F_{\alpha,\beta}^p$ such that $Z_\beta = F_{\alpha,\beta}^p \circ Z_\alpha$ in some neighborhood of p in $U_\alpha \cap U_\beta$.

We call a pair (U, Z) a hypo-analytic chart. We can associate it a locally integrable structure $\mathcal{V}$ setting its orthogonal, $T^\perp$, as the complex bundle defined, locally, by the differentials $dZ_1, \dots, dZ_m$.

EXAMPLE

Let $M \subset \mathbb{C}^N$ be a **generic CR** submanifold of codimension d , so the CR dimension is equal to $N - d$. We can associate a hypo-analytic structure on M setting $Z = z|_M$, for every holomorphic coordinate system (U, z) . In this case the locally integrable structure $\mathcal{V}$ is equal to $T^{0,1}M$, i.e., the collection of all anti-holomorphic vector fields that are tangent to M .

THE LOCAL SET UP AND THE REAL STRUCTURE BUNDLE

For simplicity assume that $0 \in \Omega$. There is a hypo-analytic chart $(U, Z_1(x, t), \dots, Z_m(x, t))$ centered at 0 with $U = V \times W$, where $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are open balls centered at the origin, and the map $Z = (Z_1, \dots, Z_m)$ is given by

$$Z_j(x, t) = x_j + i\phi_j(x, t), \quad j = 1, \dots, m,$$

with $(x, t) \in V \times W$, where the map $\phi(x, t) = (\phi_1(x, t), \dots, \phi_m(x, t))$ is smooth, real-valued, $\phi(0) = 0$ and $d_x \phi(0) = 0$. We associate the complex vector fields $\{M_1, \dots, M_m, L_1, \dots, L_m\}$ with the following properties:

$$\begin{aligned} L_j Z_k &= 0 & M_j Z_k &= \delta_{j,k} \\ L_j t_k &= \delta_{j,k} & M_j t_k &= 0, \end{aligned}$$

For every $t \in W$ the set $\Sigma_t = \{Z(x, t) : x \in V\}$ is a maximally real submanifold of $\mathbb{C}^m$. The **real structure bundle** of Σ_t , $\mathbb{RT}_{\Sigma_t}$, is given by

$$\mathbb{RT}_{\Sigma_t} = \{(Z(x, t), {}^t Z_x(x, t)^{-1} \xi) : x \in V, \xi \in \mathbb{R}^m\}.$$

AN FBI CHARACTERIZATION OF GEVREY VECTORS

Let $\mathcal{H} \subset \mathbb{C}^m$ be a maximally real submanifold and $u \in \mathcal{E}'(\mathcal{H})$. We define the FBI transform of u by

$$\tilde{\mathcal{H}}[u](z, \zeta) \doteq \left\langle u(z'), e^{i\zeta \cdot (z - z') - \langle \zeta, (z - z')^\perp \rangle} \Delta(z - z', \zeta) \right\rangle,$$

for $z \in \mathbb{C}^m$ and $\zeta \in \mathbb{C}_1 = \{\eta \in \mathbb{C}^m : |\operatorname{Im} \eta| < |\operatorname{Re} \eta|\}$, where $\Delta(z, \zeta) = \det(\operatorname{Id} + i(z \otimes \zeta)/\langle \zeta, \zeta \rangle)$. Now if u is a distribution on U (the same as before) such that $L_j u \in G^s(U; L_1, \dots, L_m, M_1, \dots, M_m)$, then actually $u \in \mathcal{C}^\infty(W; \mathcal{D}'(V))$. So we define the FBI transform of $u \in \mathcal{C}^\infty(W; \mathcal{E}'(V))$ by

$$\tilde{\mathcal{H}}[u](t; z, \zeta) = \tilde{\mathcal{H}}[u_t^*](z, \zeta),$$

where u_t^* is the compactly supported distribution on Σ_t defined by $u_t^*(z) = u(x, t)$, for $z = Z(x, t)$.

Theorem 1 (N. Braun Rodrigues, 2020) Let $u \in \mathcal{C}^\infty(W; \mathcal{D}'(V))$ be a solution of

$$\begin{cases} L_1 u = f_1, \\ \vdots \\ L_m u = f_m, \end{cases}$$

where $f_j \in G^s(U; L_1, \dots, L_m, M_1, \dots, M_m)$, $j = 1, \dots, m$. Then are equivalent:

- $u|_{U_\alpha} \in G^s(U; L_1, \dots, L_m, M_1, \dots, M_m)$, for some open neighborhood of the origin U_α ;
- For every $\chi \in \mathcal{C}_0^\infty(V)$, with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in some open neighborhood of the origin, there exist $\tilde{V} \subset V$, $\tilde{W} \subset W$, open balls centered at the origin, constants $C, \epsilon > 0$ such that

$$|\tilde{\mathcal{H}}[\chi u](t; z, \zeta)| \leq C e^{-\epsilon |\zeta|^{\frac{1}{s}}}, \quad \forall t \in \tilde{W}, (z, \zeta) \in \mathbb{RT}_{\Sigma_t}|_{\tilde{V}} \setminus \emptyset,$$

where $(z, \zeta) \in \mathbb{RT}_{\Sigma_t}|_{\tilde{V}}$ means that $z = Z(x, t)$, $\zeta = {}^t Z_x(x, t)^{-1} \xi$, $\xi \in \mathbb{R}^m \setminus 0$ and $x \in \tilde{V}$.

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"PROPAGATORS"

We shall consider only analytic tube structures, i.e., locally the hypo-analytic structure is given by $Z(x, t) = x + i\phi(t)$, defined on $U = V \times W$, and $\phi(t)$ is analytic. One of the reasons we are only dealing with tube structures is that the real structure bundle is trivial, i.e.,

$$\mathbb{RT}_{\Sigma_t} = \{(Z(x, t), \xi) : x \in V, \xi \in \mathbb{R}^m\},$$

for every $t \in W$. Let $\Sigma \subset \Omega$ be a connected submanifold of Ω , satisfying the following properties:

- For every $p \in \Sigma$ there is (U, Z) , a hypo-analytic chart, with $p \in U$, such that $\Sigma \cap U \subset Z^{-1}(0)$;
- In the same situation as above, for every $q \in \Sigma \cap U$, and $\tilde{U}_1 \in U$, an open neighborhood of p , there is $\tilde{U}_2 \in U$, an open neighborhood of q , such that the connected component of the fiber $Z^{-1}(Z(q'))$ that contains q' intersects $\tilde{U}_1$, for every $q' \in \tilde{U}_2$;
- the map $\Sigma \ni p \mapsto \sup\{r > 0 : B_r(p) \subset U\}$ is continuous.

PROPAGATION OF SINGULARITIES

Let $u \in \mathcal{D}'(\Omega)$. We say that $L_j u \in G^s$ if for every (U, Z) described in the local setup section we have that

$$L_j u \in G^s(U; L_1, \dots, L_m, M_1, \dots, M_m),$$

for $j = 1, \dots, m$. Note that since the structure is real-analytic, this is equivalent to $L_j u \in G^s(U)$.

Theorem 2 (N. Braun Rodrigues, 2020)

In the same situation as above, if $u \in \mathcal{D}'(\Omega)$ is such that $L_\alpha u \in G^s(\Omega)$, then $\operatorname{singsupp}_s u \cap \Sigma = \emptyset$ or $\Sigma \subset \operatorname{singsupp}_s u$.

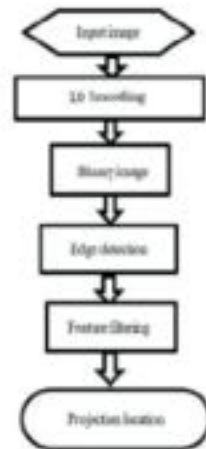
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Overview

Abstract: The license plate location plays an important role in license plate recognition systems. As a premise of character recognition, the accuracy and robustness of license location directly determine the performance of the entire license plate recognition system. In many practical applications, license plate recognition systems mostly work outdoors, and the captured images inevitably suffer from different kinds of degradations caused by lighting, weather, and complex backgrounds, and so on. In general, it is a challenging problem under such a diverse, uncertain and complex environment and has also attracted considerable attention in both academic and industrial fields.

Main Idea: In regard to the complex background in license images, we first use an edge-aware filter, L^0 -norm smoothing to remove the majority background textures but keep the license plate characters. Then, we take a series of feature filtering steps based on the geometrical textures and structures to further reduce the interference of pseudo-licenses. Finally, a simple projection location method is used to extract the position and size of the license plates. The whole location procedure is:



Contributions:

- Propose a practical license plate location system based on a series of feature extraction and filter steps.
- Use L_0 image smoothing algorithm to remove the background noise.
- Use a binarized image for fast multiscale resolution analysis.
- Take full use of textural information and projection location method to extract license plates.

Algorithm

L^0 Smoothing: In license plate images, license characters have high contrast textures, while the image background contains abundant low contrast details. We use the L_0 -norm smoothing filter to suppress the details, which can be phrased as,

$$\arg \min_f E(f) = \|f - g\|_2^2 + \lambda \|\nabla f\|_1, \quad (1)$$

where g and f are the source and target images in $\mathbb{R}^N$, ∇ is gradient operator, $\|\cdot\|_2$ is L^2 -norm, and $\|\cdot\|_1$ is so-called L^1 -norm, counting the number of non-zero elements of a vector, which leads to a sparse regularization, and λ is a weight scalar. **L^0 -norm minimization:** The Eq.(1) is NP-hard to solve, we instead introduce two auxiliary variables h and v for the 2D discrete image, and rewrite it as,

$$\min_{f,h,v} E(f) = \sum_i (f_i - g_i)^2 + \lambda C(h, v) + \beta (\|h\|_1 + \|v\|_1)^2 + (\partial_x f_i - h_i)^2 + (\partial_y f_i - v_i)^2, \quad (2)$$

where $C(h, v) = 1$ if $|h_i| + |v_i| \neq 0$, else 0. It is clear that the solution of Eq. (2) approximates that of Eq. (1) when $\beta \rightarrow +\infty$. We solve the Eq. (2) minimizing $\langle h, v \rangle$ and f alternatively.

Experiments & Results

1. L^0 -norm smoothing



- Left: input image, right: L^0 -norm smoothing result. It is clear that the low-contrast details are significantly suppressed with L^0 -norm smoothing.

2. Binarized image

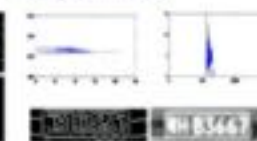


- The binarized image provides a multiscale analysis for license.

3. Feature filtering



4. Projection results



- Edge extraction, removing lines, structured and texture analysis.
- The horizontal and vertical projection histograms, and the output license plate.

1. Computing f : By fixing h and v and minimizing the function,

$$\min_f E(f) = \sum_i (f_i - g_i)^2 + \beta (\|h\|_1 + \|v\|_1)^2 + (\partial_x f_i - h_i)^2 + (\partial_y f_i - v_i)^2. \quad (3)$$

The above function is quadratic and thus has a global minimum. Here we use fast Fourier transform (FFT) to accelerate the solver.

$$f = \mathcal{F}^{-1} \left(\frac{\mathcal{F}(g) + \beta (\mathcal{F}(h) + \mathcal{F}(v)) + \mathcal{F}(\partial_x f) + \mathcal{F}(\partial_y f)}{\mathcal{F}(1) + \beta (\mathcal{F}(h) + \mathcal{F}(v)) + \mathcal{F}(\partial_x f) + \mathcal{F}(\partial_y f)} \right), \quad (4)$$

where $\cdot$ is a component-wise multiplication, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denotes FFT and its inverse operators, and $\mathcal{F}(1)$ is the Fourier Transform of δ function.

2. computing $\langle h, v \rangle$: We solve $\langle h, v \rangle$ by minimizing

$$\min_{h,v} E(f) = \lambda C(h, v) + \beta (\|h\|_1 + \|v\|_1)^2 + (\partial_x f_i - h_i)^2 + (\partial_y f_i - v_i)^2. \quad (5)$$

where $C(h, v)$ returns the number of non-zero elements of $\partial_x f_i$ and $\partial_y f_i$. Eq. (5) can be spatially decomposed and solved fast, because each element h_p and v_p can be estimated individually. It reaches its minimum L^0 under the condition,

$$h_i, v_i = \begin{cases} (h_i, v_i) & \text{if } (\partial_x f_i + \partial_y f_i)^2 \leq \frac{\lambda}{\beta} \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

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Quantitative evaluation:

Method	Precision (%)	Recall (%)	F1-score (%)
Top-Net	77.50	84.50	80.91
MSTAR-SIFT	83.73	90.47	86.97
Wavelet-based	90.30	94.03	92.00
CNN-based	97.80	95.30	96.01
Ours	90.10	95.40	92.30

- Quantitative evaluation on natural image datasets. We collected 1000 images with both simple and complex backgrounds.

Description

The aim of Spectral Geometry of Partial Differential Operators is to provide a basic and self-contained introduction to the ideas underpinning spectral geometric inequalities arising in the theory of partial differential equations. Historically, one of the first inequalities of the spectral geometry was the minimisation problem of the first eigenvalue of the Dirichlet Laplacian. Nowadays, this type of inequalities of spectral geometry have expanded to many other cases with numerous applications in physics and other sciences. The main reason why the results are useful, beyond the intrinsic interest of geometric extremum problems, is that they produce a priori bounds for spectral invariants of (partial differential) operators on arbitrary domains.

Content

- 1 Function spaces
- 2 Foundations of linear operator theory
- 3 Elements of the spectral theory of differential operators
- 4 Symmetric decreasing rearrangements and applications
- 5 Inequalities of spectral geometry
 - 5.1 Introduction
 - 5.2 Logarithmic potential operator
 - 5.3 Riesz potential operators
 - 5.4 Bessel potential operators
 - 5.5 Riesz transform in spherical and hyperbolic geometries
 - 5.6 Heat potential operators
 - 5.7 Cauchy-Dirichlet heat operator
 - 5.8 Cauchy-Robin heat operator
 - 5.9 Cauchy-Neumann and Cauchy-Dirichlet-Neumann heat operators

Spectral geometry

Let us consider Riesz potential operators

$$(\mathcal{R}_{\alpha, \Omega} f)(x) := \int_{\Omega} |x - y|^{\alpha-d} f(y) dy, \quad f \in L^2(\Omega), \quad 0 < \alpha < d, \quad (1)$$

where $\Omega \subset \mathbb{R}^d$ is a set with finite Lebesgue measure.

Rayleigh-Faber-Krahn inequality: The ball Ω^* is the maximiser of the first eigenvalue of the operator $\mathcal{R}_{\alpha, \Omega}$ among all domains of a given volume, i.e.

$$0 < \lambda_1(\Omega) \leq \lambda_1(\Omega^*)$$

for an arbitrary domain $\Omega \subset \mathbb{R}^d$ with $|\Omega| = |\Omega^*|$.

Hong-Krahn-Szegő inequality: The maximum of the second eigenvalue $\lambda_2(\Omega)$ of $\mathcal{R}_{\alpha, \Omega}$ among all sets $\Omega \subset \mathbb{R}^d$ with a given measure is approached by the union of two identical balls with mutual distance going to infinity.

Richard Feynman's drum



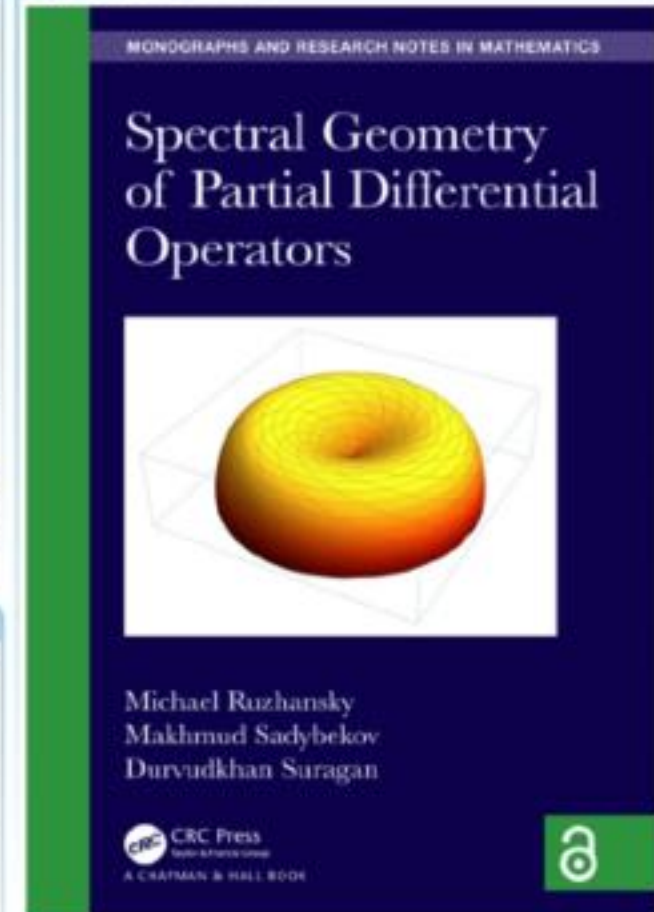
The picture from plankip.org

It is proved that the deepest bass note is produced by the circular drum among all drums of the same area (as the circular drum). Moreover, one can show that among all bodies of a given volume in the three-dimensional space with constant density, the ball has the gravitational field of the highest energy.

Rex is minimising the strain energy by circling



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Abstract

We study a class of anharmonic oscillators within the framework of the Weyl-Hörmander calculus. By associating a Hörmander metric to a given anharmonic oscillator we extend the so-called Schatten classes associated to the harmonic oscillator and the corresponding pseudo-differential calculus. Spectral properties of negative powers of anharmonic oscillators, as well as of the operator itself, are derived.

Introduction

In the study of the Schrödinger equation $\Delta u + V(X)u = E u$ the analysis of the energy levels is often reduced to the corresponding eigenvalue problem for the operator $-\Delta + V(x)$. Spectral properties of the anharmonic oscillator, on $\mathbb{R}$ or more generally on $\mathbb{R}^n$, with different potentials V have been studied in [1, 2, 3, 4, 5] (by several authors in the last 60 years). However, the exact solution of the eigenvalue problem is still unknown.

Here we consider a novel generalization on $\mathbb{R}^n$ where a prototype is of the form

$$A = (-\Delta)^j + |x|^{2\ell}, \quad \text{where } \ell, j \geq 1 \text{ integers,} \quad (1)$$

and more generally we consider operators of the form

$$T = g(D) + p(x), \quad (2)$$

where p, g are special polynomials on $\mathbb{R}^n$. In particular we write $p \in \mathcal{P}_m, g \in \mathcal{G}_k$ for some integer $k, \ell \geq 1$, where we have defined

$$\mathcal{P}_m = \left\{ p : \mathbb{R}^n \rightarrow \mathbb{R}, \text{ with } \deg(p) = m, \text{ and } \lim_{|x| \rightarrow \infty} \frac{|p(x)|}{|x|^m} = 0 \right\},$$

Therefore, for $p \in \mathcal{P}_m$ (and similarly for g), there exists $\alpha_m > 0$ such that

$$|p(x) + p_0| \geq 0, \quad \text{for every } x \in \mathbb{R}^n.$$

Weyl-Hörmander classes associated to the anharmonic oscillators

• In the case of the general anharmonic oscillator T as in (2) with bounded symbol p we have

$$T(x, \xi) = g(\xi) + p(x) \in S(M^m, g^m),$$

where the Hörmander metric g^m , and the g -weight M^m are given by

$$g^m(x, \xi) = \frac{d\xi^2}{(p_0 + p_0 + g(\xi))^2} + \frac{dx^2}{(p_0 + p_0 + p(x) + g(\xi))^2}, \quad (3)$$

and

$$M^m(x, \xi) = p_0 + g_0 + p(x) + g(\xi). \quad (4)$$

• In the case of the prototype of the anharmonic oscillator A as in (1) the metric (3) is equivalent to the metric

$$g^{\ell,j}(x, \xi) = \frac{d\xi^2}{(1 + |\xi|^2 + |\xi|^{2\ell})^2} + \frac{dx^2}{(1 + |\xi|^2 + |\xi|^{2\ell})^2}, \quad (5)$$

• In the symmetric case where $\ell = j$ in (1) the metric associated to the operator A is equivalent to

$$g^{\ell,j}(x, \xi) = \frac{d\xi^2}{(1 + |\xi|^2 + |\xi|^{2\ell})^2} + \frac{dx^2}{(1 + |\xi|^2 + |\xi|^{2\ell})^2},$$

which corresponds to the symbol (2) metric defining the Schatten classes associated to the harmonic oscillator

Associated symbol classes and operators

Let $\alpha \in \mathbb{R}$. We say that the function $a \in C^\infty(\mathbb{R}^n)$ is in the class of symbols $\mathcal{S}^\alpha_{\ell,j}(\mathbb{R}^n)$, if

$$|\partial^\alpha a(x, \xi)| \leq C_{\alpha, \ell, j} \langle \xi \rangle^{|\alpha| - \frac{\alpha}{2} - \frac{\alpha}{2}}, \quad \text{for all } \alpha \in \mathbb{N}^n,$$

where we have denoted $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, for $\ell, j \geq 1$ integers. The associated operators are denoted by $\Psi_{\ell,j}^\alpha(\mathbb{R}^n)$, and in particular

$$\Psi_{\ell,j}^0(\mathbb{R}^n) = \mathcal{B}(\mathcal{L}^2_{\ell,j}(\mathbb{R}^n)).$$

For example for prototype anharmonic oscillator A as in (1) is an operator in $\Psi_{\ell,j}^0(\mathbb{R}^n)$.

Pseudo-differential calculus on $\mathcal{L}^2_{\ell,j}(\mathbb{R}^n)$

The following can be viewed as a consequence of the Weyl-Hörmander calculus

• The class of operators $\Psi_{\ell,j}^\alpha(\mathbb{R}^n)$ forms an algebra of operators that is stable under taking the adjoint.

• Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and let $k, \ell \geq 1$ integers. If $a \in \mathcal{S}^{\alpha_1}_{\ell,j}(\mathbb{R}^n)$ and $b \in \mathcal{S}^{\alpha_2}_{\ell,j}(\mathbb{R}^n)$, then there exists $c \in \mathcal{S}^{\alpha_1 + \alpha_2}_{\ell,j}(\mathbb{R}^n)$ such that $(\Psi_{\ell,j}^{\alpha_1} a)(\Psi_{\ell,j}^{\alpha_2} b) = \Psi_{\ell,j}^{\alpha_1 + \alpha_2} c$. Moreover, we have the asymptotic formula

$$c = \sum_{|\alpha| \geq 0} \frac{i^{|\alpha|} \alpha!}{\alpha!} \partial^\alpha a(x, \xi) \partial^\alpha b(x, \xi).$$

• The operators in $\Psi_{\ell,j}^\alpha(\mathbb{R}^n)$ extend boundedly to $L^2(\mathbb{R}^n)$. Furthermore, there exists $C > 0$ and $k \in \mathbb{N}$ such that if $A \in \Psi_{\ell,j}^0(\mathbb{R}^n) \in \Psi_{\ell,j}^\alpha(\mathbb{R}^n)$, then

$$\|A\|_{\mathcal{B}(\mathcal{L}^2_{\ell,j}(\mathbb{R}^n))} \leq C \|A\|_{\mathcal{B}(\mathcal{L}^2_{\ell,j}(\mathbb{R}^n))},$$

where $\|\cdot\|_{\mathcal{B}(\mathcal{L}^2_{\ell,j}(\mathbb{R}^n))}$ denotes the operator norm in the class in the class of symbols $\mathcal{S}^\alpha_{\ell,j}(\mathbb{R}^n)$ (or $\mathcal{S}^\alpha_{\ell,j}(\mathbb{R}^n)$).

Associated Sobolev spaces

Using the functional calculus on (the compact, positive spectral) A as in (1), we define the operator A^j , for $j \in \mathbb{R}$, by

$$A^j u = \sum_{k=0}^{\infty} \lambda_k^j \langle u, e_k \rangle e_k, \quad \text{for } u \in \text{Dom}(A^j),$$

where $\{\lambda_k\}_{k=0}^{\infty}$ is an increasing (and $L^2(\mathbb{R}^n)$ mode of eigenfunctions of A and $\{e_k\}_{k=0}^{\infty}$ the corresponding eigenvalues.

The Sobolev spaces related to A denoted by $\mathcal{H}^s_{\ell,j}(\mathbb{R}^n)$, for $s \in \mathbb{R}$, $\ell, j \geq 1$ integers, is the subspace $\mathcal{H}^s_{\ell,j}(\mathbb{R}^n)$ that is the completion of $\text{Dom}(A^j)$ for the norm

$$\|u\|_{\mathcal{H}^s_{\ell,j}(\mathbb{R}^n)} = \|A^{j/2} u\|_{L^2(\mathbb{R}^n)}.$$

Continuity Properties

The characterization of the Sobolev spaces $\mathcal{H}^s_{\ell,j}(\mathbb{R}^n)$ with suitable Sobolev spaces $\mathcal{B}(\mathcal{H}^s_{\ell,j}(\mathbb{R}^n))$ in the Weyl-Hörmander setting, and the general theory yield

• Let $\alpha \in \mathbb{R}$ and $k, \ell \geq 1$ integers. If $a \in \mathcal{S}^\alpha_{\ell,j}(\mathbb{R}^n)$, then

$$(\Psi_{\ell,j}^\alpha a)(\mathcal{H}^s_{\ell,j}(\mathbb{R}^n)) \xrightarrow{\text{weak}} \mathcal{H}^s_{\ell,j}(\mathbb{R}^n), \quad \forall s \in \mathbb{R}.$$

More generally, equivalence of quantizations in our particular case yield

• For $\alpha \in \mathcal{S}^\alpha_{\ell,j}(\mathbb{R}^n)$ (or $\mathcal{S}^\alpha_{\ell,j}(\mathbb{R}^n)$) we have

$$(\Psi_{\ell,j}^\alpha a)(\mathcal{H}^s_{\ell,j}(\mathbb{R}^n)) \xrightarrow{\text{weak}} \mathcal{H}^s_{\ell,j}(\mathbb{R}^n), \quad \forall s \in \mathbb{R}, \forall \alpha \in \mathbb{R}.$$

Anharmonic oscillators and Schatten-classes of operators

For $1 \leq p < \infty$, we denote by $\mathcal{S}^p_{\ell,j}(\mathbb{R}^n)$ the $\mathcal{S}^p$ Schatten class of operators.

• For $p = p^* \in [1, \infty]$, and for $\alpha \in \mathcal{S}^\alpha_{\ell,j}(\mathbb{R}^n)$, $\alpha > \frac{n}{p}$, we have

$$(\Psi_{\ell,j}^\alpha a) \in \mathcal{S}^p_{\ell,j}(\mathbb{R}^n), \quad \text{for all } \alpha \in \mathbb{R}.$$

• For the operator A as in (1), or more generally, for the operator T as in (2), and for $\mu > \frac{n}{2\ell}$ (or ℓ, j as in (1) or according to the choice of μ, ℓ as in (2)) we have

$$T^{-\mu}, A^{-\mu} \in \mathcal{S}^p_{\ell,j}(\mathbb{R}^n).$$

Eigenvalue asymptotics for negative powers of operators

The general theory on the Schatten-classes $\mathcal{S}^p_{\ell,j}(\mathbb{R}^n)$ and the Weyl inequality (see [6])

$$\sum_{k=1}^N \frac{\lambda_k(T)^p}{\sum_{k=1}^N \lambda_k(T)^p} \leq \sum_{k=1}^N \frac{\lambda_k(T)^p}{\sum_{k=1}^N \lambda_k(T)^p}, \quad \text{for } p > 0,$$

imply

• For $p = p^* \in [1, \infty]$, and for $\alpha \in \mathcal{S}^\alpha_{\ell,j}(\mathbb{R}^n)$, $\alpha > \frac{n}{p}$, and for any $\beta \in \mathbb{R}$ we have

$$\lambda_j(\Psi_{\ell,j}^\alpha a) = o(\lambda_j^\beta), \quad \text{as } j \rightarrow \infty.$$

• For the operator A as in (1), or more generally, for the operator T as in (2), and for $\mu > \frac{n}{2\ell}$ (or ℓ, j as in (1) or according to the choice of μ, ℓ as in (2)) we have

$$\lambda_j(A^{-\mu}) = o(\lambda_j^{-\mu}), \quad \text{as } j \rightarrow \infty.$$

Rate of growth of the eigenvalues of the anharmonic oscillator A

Let λ_j be as in (1) and $\mu > \frac{n}{2\ell}$. Then for every $N \in \mathbb{N}$ there exists $N_0 \in \mathbb{N}$ such that

$$\lambda_j^2 \leq \lambda_{j+N_0}^2, \quad \text{for } j \geq N_0.$$

Thus, the eigenvalues $\lambda_j(A)$ are of fast growth

$$j^2, \quad \text{as } j \rightarrow \infty.$$

Asymptotic expansions of $\lambda_j(A)$ have also been studied in [1] and [3].

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Hyperbolic systems with non-diagonalisable principal part

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Introduction

We consider

$$\begin{cases} \partial_t u + \partial_x(A(x, \lambda)u) + B(x, \lambda)u = f(x, t), & (x, t) \in (0, T) \times \mathbb{R}^n, \\ u(x, 0) = u_0, & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where $n \geq 2$, $\lambda \in \mathbb{C}$ and $B(x, \lambda) = -\partial_x(A(x, \lambda))$. We assume that $A(x, \lambda) \in \mathbb{R}^{n \times n}$, $B(x, \lambda) \in \mathbb{R}^{n \times n}$, $f(x, t) \in C([0, T]; L^2(\mathbb{R}^n))$ and that $B(x, \lambda) = B_0(x) + B_1(x, \lambda)$, $B_0(x) \in C([0, T]; L^2(\mathbb{R}^n))$ and that $B_1(x, \lambda) \in C([0, T]; L^2(\mathbb{R}^n))$. We also assume that the matrix A is upper triangular and hyperbolic, i.e.,

$$A(x, \lambda) = \begin{pmatrix} a_{11}(x, \lambda) & & 0 \\ & \ddots & \\ 0 & & a_{nn}(x, \lambda) \end{pmatrix}, \quad a_{ii}(x, \lambda) = a_{ii}(x) + b_{ii}(x, \lambda), \quad a_{ii}(x) \in C([0, T]; L^2(\mathbb{R}^n))$$

with real eigenvalues $\lambda_1(x, \lambda), \lambda_2(x, \lambda), \dots, \lambda_n(x, \lambda)$ of $A(x, \lambda)$ and

$$B(x, \lambda) = \begin{pmatrix} b_{11}(x, \lambda) & b_{12}(x, \lambda) & \dots & b_{1n}(x, \lambda) \\ 0 & b_{22}(x, \lambda) & \dots & b_{2n}(x, \lambda) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{nn}(x, \lambda) \end{pmatrix}, \quad b_{ij}(x, \lambda) \in C([0, T]; L^2(\mathbb{R}^n))$$

Furthermore, we introduce the following two hypotheses:

(H1) For the coefficients of the lower order term $B(x, \lambda)$

the lower order term $b_{ij}(x, \lambda)$ belong to $C([0, T]; W^{1,\infty}(\mathbb{R}^n))$.

(H2) For some constants, we assume that A does not depend on λ , i.e. $A = A(x, \lambda)$ and similarly there exists $B = B(x, \lambda)$ such that $B(x, \lambda) = B_0(x) + B_1(x, \lambda)$ for some $B_0(x) \in C([0, T]; L^2(\mathbb{R}^n))$ and $B_1(x, \lambda) \in C([0, T]; L^2(\mathbb{R}^n))$ and that $B_1(x, \lambda)$ can not identically equal zero on $\mathbb{R}^n$ for some values $\lambda \in \mathbb{C}$. If

$$A(x, \lambda) = A_0(x) + A_1(x, \lambda) \quad \text{with } A_0(x) = \begin{pmatrix} a_{11}(x) & & 0 \\ & \ddots & \\ 0 & & a_{nn}(x) \end{pmatrix}, \quad A_1(x, \lambda) = \begin{pmatrix} b_{11}(x, \lambda) & & 0 \\ & \ddots & \\ 0 & & b_{nn}(x, \lambda) \end{pmatrix},$$

where the matrix $A_0(x)$ is invertible for all $x \in \mathbb{R}^n$.

Well-posedness

In Garetto et al. (2018), we prove a well-posedness result for (1) under hypothesis (H1).

Theorem 1. Consider the Cauchy problem (1), where $A(x, \lambda)$ and $B(x, \lambda)$ are as described in the introduction and (H1) is satisfied. If $u_0 \in L^2(\mathbb{R}^n)$ and $f \in C([0, T]; L^2(\mathbb{R}^n))$, then (1) has a unique anisotropic Sobolev solution u , i.e., $u \in C([0, T]; W^{1,\infty}(\mathbb{R}^n))$ for all $x \in \mathbb{R}^n$.

This theorem is proved by making use of the triangular form, solving the first equation and then iteratively finding the solution of the system from the solutions to scalar equations. For each characteristic λ_i of A , we denote by U_i and V_i the respective vector solutions to

$$\begin{cases} \partial_t U_i + \partial_x(A(x, \lambda_i)U_i) + B(x, \lambda_i)U_i = f(x, t), \\ U_i(x, 0) = u_0, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t V_i + \partial_x(A(x, \lambda_i)V_i) + B(x, \lambda_i)V_i = g(x, t), \\ V_i(x, 0) = 0. \end{cases}$$

The operators U_i and V_i can be iteratively represented by Fourier integral operators

$$U_i(x, t) = \int e^{i\phi(x, \xi, t)} u_i(\xi, t) d\xi$$

and

$$V_i(x, t) = \int e^{i\phi(x, \xi, t)} v_i(\xi, t) d\xi + \int e^{i\phi(x, \xi, t)} w_i(\xi, t) d\xi$$

where

$$U_i(x, t) = \int e^{i\phi(x, \xi, t)} u_i(\xi, t) d\xi$$

with $u_i(\xi, t) \in \mathbb{C}$ satisfying the ordinary equation

$$\begin{cases} \partial_t u_i = A_i(x, t)u_i + B_i(x, t)u_i, \\ u_i(x, 0) = u_0(x). \end{cases}$$

The symbols $A_i(x, t) \in \mathbb{C}$ and $B_i(x, t) \in \mathbb{C}$ are given by $A_i(x, t) = A(x, \lambda_i)$ and then satisfy the usual transport equations with initial data $u_0(x) = u_0(x)$ and $u_0(x) = u_0(x)$.

The components of the solution $u(x, t)$ is given by a composition of the operators described above together with principal part coefficients and lower order coefficients. That is where hypothesis (H2) comes into play. For example in the case $n = 2$, we get

$$\begin{cases} u_1 = U_1 + V_1, \\ u_2 = U_2 + V_2. \end{cases} \quad \text{with } U_1 = A_1(x, t)U_1 + B_1(x, t)U_1, \quad U_2 = A_2(x, t)U_2 + B_2(x, t)U_2$$

Then this gives

$$\begin{cases} u_1 = U_1 + V_1, \\ u_2 = U_2 + V_2. \end{cases} \quad \text{with } U_1 = A_1(x, t)U_1 + B_1(x, t)U_1, \quad U_2 = A_2(x, t)U_2 + B_2(x, t)U_2$$

One then gets to the final result by getting an initial point problem to which Duhamel's point theorem can be applied. A general time interval $[0, T]$ can be iteratively covered since the estimates involved for the U_i are only depend on the coefficients against the initial data.

Solution representations and regularity results

In the case of $A = A(x, \lambda)$, adding to addition to (H1) as the lower order term due to (H2) for the principal part, the solutions of (1) can be represented explicitly including some smoothing operators. Here we state the principal results and refer to Garetto et al. (2018) for the details and proofs.

Theorem 2. Consider (1) with $A = A(x, \lambda)$ and $B(x, \lambda)$ satisfying properties described above and if it satisfies (H1) and (H2) described. Let u_0 and f have components u_0^i and f_i respectively with $u_0^i \in W^{1,\infty}(\mathbb{R}^n)$ and $f_i \in C([0, T]; L^2(\mathbb{R}^n))$ for $i = 1, \dots, n$. Then, for any $t \in [0, T]$, the components $u_i(x, t) \in C([0, T]; L^2(\mathbb{R}^n))$ are given by

$$u_i(x, t) = \sum_{j=1}^n \left(u_0^j(x) + b_{ij}(x, t) \right) e^{i\phi(x, \xi, t)} + \int e^{i\phi(x, \xi, t)} f_j(x, t) d\xi$$

where $b_{ij}(x, t) \in C([0, T]; W^{1,\infty}(\mathbb{R}^n))$ and the operators $U_i^j(x, t) \in C([0, T]; W^{1,\infty}(\mathbb{R}^n))$ are integral Fourier integral operators of order $1 - \frac{1}{2}$.

Using the explicit solution representations, we get

Theorem 3. Let $A = A(x, \lambda)$ and $B = B(x, \lambda)$ satisfy (H1) and (H2). Then, for any compactly supported $u_0 \in L^2(\mathbb{R}^n)$, the solution $u = u(x, t)$ of the Cauchy problem (1) satisfies $u(x, t) \in C([0, T]; L^2(\mathbb{R}^n))$ for all $x \in \mathbb{R}^n$. Moreover, there is a positive constant C_1 such that

$$\|u\|_{C([0, T]; L^2(\mathbb{R}^n))} \leq C_1 \|u_0\|_{L^2(\mathbb{R}^n)}$$

Local estimates can be obtained in other spaces as well for $x \in \mathbb{R}^n$ and $t \in [0, T]$. In spirit, assuming u_0 to be compactly supported, we have that $u(x, t) \in C([0, T]; L^2(\mathbb{R}^n))$ for all $x \in \mathbb{R}^n$ and $t \in [0, T]$. This implies $u(x, t) \in C([0, T]; L^2(\mathbb{R}^n))$ for all $x \in \mathbb{R}^n$ and $t \in [0, T]$. This implies $u(x, t) \in C([0, T]; L^2(\mathbb{R}^n))$ for all $x \in \mathbb{R}^n$ and $t \in [0, T]$.

Propagation of singularities

Operators of the form $\partial_t U_i + \partial_x(A(x, \lambda_i)U_i) + B(x, \lambda_i)U_i$, which appear in the solution representation, are in the operators $U_i^j(x, t)$ and the general form

$$U_i = \int e^{i\phi(x, \xi, t)} u_i(\xi, t) d\xi + \int e^{i\phi(x, \xi, t)} v_i(\xi, t) d\xi + \int e^{i\phi(x, \xi, t)} w_i(\xi, t) d\xi$$

For these operators, we have $U_i \in C([0, T]; W^{1,\infty}(\mathbb{R}^n))$ where $U_i(x, t) \in C([0, T]; W^{1,\infty}(\mathbb{R}^n))$. The singularity propagation along bicharacteristic flows.

Let $\lambda = \lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $\lambda_i \in \mathbb{C}$ for $i = 1, \dots, n$. Then the definition of (H1) we have that the characteristic $\lambda_i \in \mathbb{C}$ for $i = 1, \dots, n$ is given by

$$\lambda_i = \begin{pmatrix} a_{11}(x, \lambda_i) & \dots & a_{nn}(x, \lambda_i) \end{pmatrix}, \quad \lambda_i = \begin{pmatrix} a_{11}(x, \lambda_i) & \dots & a_{nn}(x, \lambda_i) \end{pmatrix}$$

and the U_i are the singularities corresponding to λ_i by solving the equations of the characteristic flow defined by the λ_i .

Let $U_i(x, t) \in C([0, T]; W^{1,\infty}(\mathbb{R}^n))$ be the corresponding bicharacteristic flow. It means that points follow bicharacteristics of λ_i and moving the characteristic of λ_i and then continue along the bicharacteristic of λ_i .

Operators of the type U_i can be written as standard Fourier integral operators where the Ansatz of composition is that the whole space has a symbol. For these operators, following arguments of Hörmander, we get $U_i^j(x, t) \in C([0, T]; W^{1,\infty}(\mathbb{R}^n))$. For details see Kato and Ruzhansky (2007), Garetto et al. (2018).

Corollary 4. Let $A = A(x, \lambda)$ and consider (1) with $A = A(x, \lambda)$ under hypothesis (H1) and (H2). Then, we obtain an explicit representation of the solution u continuously, up to any Sobolev order depending on T . The representation of u is given by

$$U_i^j(x, t) \in C([0, T]; W^{1,\infty}(\mathbb{R}^n)) \cup C([0, T]; W^{1,\infty}(\mathbb{R}^n)), \quad (2)$$

with each of the terms that are the terms in the right hand side of (2) given by the propagation along the bicharacteristic flow as described above.

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Abstract

This work [3] is devoted to present the geometric Hardy and Hardy-Sobolev inequalities for the sub-Laplacian in the half-spaces of the Heisenberg group with a sharp constant. This result answers a conjecture posed by S. Larson in [2]. As a consequence, a geometric Hardy-Sobolev Maz'ya inequality is recovered.

Preliminaries on the Heisenberg group:

Let $\mathbb{H}^n$ be the Heisenberg group, that is, the set $\mathbb{R}^{2n+1}$ equipped with the group law

$$\xi \circ \tilde{\xi} := (x + \tilde{x}, y + \tilde{y}, t + \tilde{t} + 2 \sum_{i=1}^n (\tilde{x}_i y_i - x_i \tilde{y}_i)),$$

where $\xi := (x, y, t) \in \mathbb{H}^n$, $x := (x_1, \dots, x_n)$, $y := (y_1, \dots, y_n)$, and $\xi^{-1} = -\xi$ is the inverse element of ξ with respect to the group law. The dilation operation of the Heisenberg group with respect to the group law has the following form $\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t)$ for $\lambda > 0$.

The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}^n$ is spanned by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \text{ and } Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

and with their (non-zero) commutator $[X_i, Y_i] = -4 \frac{\partial}{\partial t}$. The horizontal gradient of $\mathbb{H}^n$ is $\nabla_H := (X_1, \dots, X_n, Y_1, \dots, Y_n)$.

Let us define the half-space of the Heisenberg group by

$$\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu \rangle > d\},$$

where $\nu := (\nu_x, \nu_y, \nu_t)$ with $\nu_x, \nu_y \in \mathbb{R}^n$ and $\nu_t \in \mathbb{R}$ is the Riemannian outer unit normal to $\partial\mathbb{H}^+$ (see [1]) and $d \in \mathbb{R}$. The Euclidean distance to the boundary $\partial\mathbb{H}^+$ is defined by

$$\text{dist}(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d.$$

Let us define

$$X_i(\xi) = (\underbrace{0, \dots, 1}_{i\text{-th}}, \dots, 0, \underbrace{0, \dots, 0}_n, 2y_i),$$

$$Y_i(\xi) = (\underbrace{0, \dots, 0}_n, \underbrace{0, \dots, 1}_{i\text{-th}}, \dots, 0, -2x_i).$$

Then we have

$$\langle X_i(\xi), \nu \rangle = \nu_{x,i} + 2y_i \nu_t, \quad \langle Y_i(\xi), \nu \rangle = \nu_{y,i} - 2x_i \nu_t,$$

where $\xi := (x, y, t)$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\nu := (\nu_x, \nu_y, \nu_t)$.

Introduction

The Hardy inequality in the half-space on the Heisenberg group was shown by Luan and Young in [4] as follows

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi. \quad (1)$$

An alternative proof of this inequality was given by Larson in [2], where the author generalised it to any half-space of the Heisenberg group,

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4} \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2}{\text{dist}(\xi, \partial\mathbb{H}^+)^2} |u|^2 d\xi,$$

where X_i and Y_i (for $i = 1, \dots, n$) are left-invariant vector fields on the Heisenberg group, ν is the Riemannian outer unit normal to the boundary. Also, there is the L^p -generalisation of the above inequality

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^p + \langle Y_i(\xi), \nu \rangle^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi. \quad (2)$$

Conjecture posed by S. Larson

A more natural weight in the right-hand side of (2) would be

$$\frac{(\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2)^{p/2}}{\text{dist}(\xi, \partial\mathbb{H}^+)^p}.$$

L^p -Hardy inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all functions $u \in C_0^\infty(\mathbb{H}^+)$ and $p > 1$ we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq C \int_{\mathbb{H}^+} \frac{(\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2)^{p/2}}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi, \quad (3)$$

where the constant $C := \left(\frac{p-1}{p}\right)^p$ is sharp.

L^2 -Hardy inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all functions $u \in C_0^\infty(\mathbb{H}^+)$ we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi, \quad (4)$$

where the constant is sharp.

This corollary can be proved by considering $p = 2$, $\text{dist}(\xi, \partial\mathbb{H}^+) = t$ and $\nu = (0, 0, 1)$.

Hardy-Sobolev inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$ and $2 \leq p < Q$ with $Q = 2n + 1$, there exists some $C_1 > 0$ such that we have

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - C \int_{\mathbb{H}^+} \frac{(\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2)^{p/2}}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \right)^{\frac{1}{p}} \geq C_1 \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{1}{p^*}},$$

where $p^* := Qp/(Q - p)$ and the constant $C := \left(\frac{p-1}{p}\right)^p$.

Hardy-Sobolev-Maz'ya inequality

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$, there exists some $C > 0$ such that we have

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi \right)^{\frac{1}{2}} \geq C \left(\int_{\mathbb{H}^+} |u|^{2^*} d\xi \right)^{\frac{1}{2^*}},$$

where $2^* := 2Q/(Q - 2)$ with the homogeneous dimension $Q = 2n + 1$.

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