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International Conference on Pseudo-differential Operators 7-8 July 2020

- Paul Alphonse (Université de Rennes 1, France)
- Duván Cardona (Ghent University, Belgium)
- Paula Cerejeiras (University of Aveiro, Portugal)
- Elena Cordero (University of Torino, Italy)
- Marianna Chatzakou (Imperial College London, UK)
- Julio Delgado (Universidad del Valle, Cali, Colombia)
- Gerd Grubb (Copenhagen University, Denmark)
- Claudia Garetto (University of Loughborough, UK)
- Arash Ghaani Farashahi (University of Leeds, UK)
- Serena Federico (Ghent University, Belgium)
- Bernard Helffer (Université de Nantes, France)
- Marius Mantoiu (University of Chile, Chile)

International Conference on Pseudo-differential Operators 7-8 July 2020

- Vishvesh Kumar (Ghent University, Belgium)
- Gerardo Mendoza (Temple University, USA)
- Rakesh Kumar Parmar (Bikaner Technical University, India)
- Alberto Parmeggiani (University of Bologna, Italy)
- Sylvie Paycha (University of Potsdam, Germany)
- Karel Pravda-Starov (Université de Rennes 1, France)
- Luigi Rodino (University of Torino, Italy)
- David Rottensteiner (Ghent University, Belgium)
- Michael Ruzhansky (Ghent University, Belgium and Queen Mary University of London, UK)

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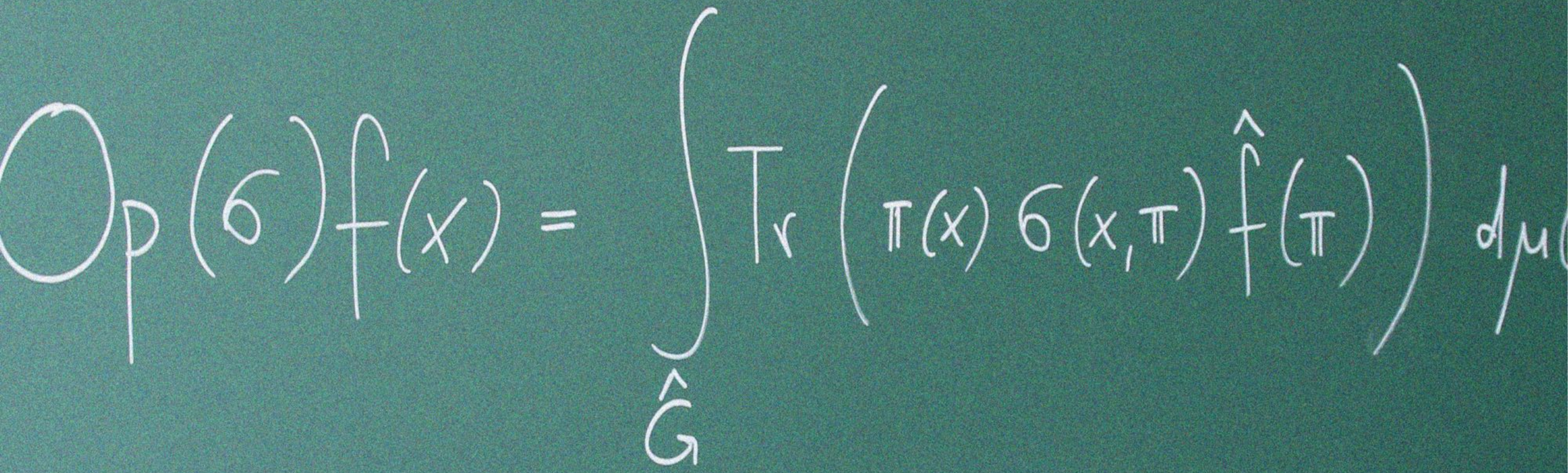
International Conference on Pseudo-differential Operators 7-8 July 2020

- Elmar Schrohe (University of Hannover, Germany)
- Johannes Sjöstrand (University of Burgundy in Dijon, France)
- Joachim Toft (Linnaeus University, Sweden)
- Ville Turunen (Aalto University, Finland)
- Gunther Uhlmann (University of Washington, USA)
- Jens Wirth (Stuttgart University, Germany)
- Man Wah Wong (York University, Canada)
- Ingo Witt (University of Göttingen)
- Maciej Zworski (University of California, Berkley, USA)



Tuesday, 7 July

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Gerd Grubb
Copenhagen University, Denmark

Fractional-order pseudodifferential operators on nonsmooth domains

Tuesday, 7 July

10:10 - 10:40

Abstract: Fractional-order pseudodifferential operators, in particular non-integer powers of the Laplacian, have recently come into focus because of their interest in financial theory and probability (they also occur in differential geometry and mathematical physics). In spite of their pseudodifferential nature, the use of ps.d.o. methods has been somewhat scarce, maybe because the ps.d.o. apparatus is technically complicated and is not part of the general curriculum in the teaching of Analysis.

For boundary value problems it was possible in 2014/15 to use ps.d.o. methods to get fine regularity results, obtained earlier only in low-order Hölder spaces by potential-theoretic methods. When lecturing on these things to applications-oriented people, I have been met by the criticism, that ps.d.o. methods require a high regularity of the set-up, C^∞coefficients and C^∞ domains.

There do exist pseudodifferential theories where symbols p(x,ξ) are only Hölder-smooth in x, but the question of how a non-smooth change of variables affects the operators has not been satisfactorily dealt with up to now.

Very recently I have completed a joint work with Helmut Abels, where we address this question, and work out conclusions for suitable fractional-order boundary problems.

Gerd Grubb
Copenhagen University, Denmark

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Johannes Sjöstrand
University of Burgundy in Dijon, France

Bergman kernels in the analytic case

Tuesday, 7 July

10:50 - 11:20

Johannes Sjöstrand

University of Burgundy in Dijon
France

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Abstract: We explain a result on the precise asymptotics of the Bergman kernel for exponentially weighted spaces and for high powers of a complex line bundle in the real analytic framework, due to Sj., Rouby, Ngoc andalso to A. Deleporte. We give some ideas from our proof, based on an adapted calculus of Fourier integral operators, and also discuss a recent simplification by A. Deleporte, M. Hitrik, Sj.

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Presentation:

Book of abstracts.

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Elena Cordero

University of Torino, Italy

Properties of Eigenfunctions of pseudodifferential operators via Gabor frames

Tuesday, 7 July

11:40 - 12:10

Elena Cordero

University of Torino, Italy

Abstract: We study the smoothness and the decay properties for eigenfunctions of compact pseudodifferential operators with symbols in modulation spaces. In particular, we focus on localization and Weyl operators. Localization operators have become popular with the papers by I. Daubechies in 1988 and from then widely investigated by several authors in different fields of mathematics and physics: in Signal Analysis, Quantum Mechanics and in the framework of pseudodifferential calculus. Localization operators can be introduced via the time-frequency representation known as short-time Fourier transform. Recall first the modulation M_ω and translation $T_x$operators of a function f on $\mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i\, t\, \omega} f(t), \qquad T_x f(t) = f(x-t), \qquad x, \omega \in \mathbb{R}^d .$$

Fix a non-zero window function ψ in the Schwartz class $S(\mathbb{R}^d)$. The short–timeFourier transform of a tempered distribution $f \in S'(\mathbb{R}^d)$, is given by

$$V_\psi f(x,\omega) = \langle f, \pi(x,\omega)\psi\rangle = F(f T_x \psi)(\omega) = \int_{\mathbb{R}^d} f(y)\, \overline{\psi(y-x)} e^{-2\pi i y \omega} dy.$$

The localization operator $A_a^{\psi_1,\psi_2}$ with symbol a and windows ψ_1, ψ_2 is formally defined by

$$A_a^{\psi_1,\psi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\psi_1} f(x,\omega) M_\omega T_x \psi_2(t)\, dx d\omega .$$

A localization operator $A_a^{\psi_1,\psi_2}$ can be represented as a Weyl pseudodifferential operator L_σ with symbol σ as

$$A_a^{\psi_1,\psi_2} = L_{a * W(\psi_2,\psi_1)}$$

Where $\sigma = a * W(\psi_2, \psi_1)$ and $W(\psi_2, \psi_1)$ is the cross-Wigner distribution:

$$W(\psi_2,\psi_1) = \int_{\mathbb{R}^d} \psi_2\left(x + \frac{t}{2}\right) \overline{\psi_1\left(x - \frac{t}{2}\right)} e^{-2\pi i y \omega} dt.$$

Considering symbols a in the wide modulation space $M^{p,\infty}$ (containing the Lebesgue space $L^p()$, $p < \infty$, and two general windows ψ_1, ψ_2 in the Schwartz class $S(\mathbb{R}^d)$, we show that $L^2(\mathbb{R}^d)$-eigenfunctions with non-zero eigenvalue are indeed highly compressed onto a few Gabor atoms.

The key idea is to study first smoothness and decay properties for eigenfunctions of compact Weyl operators. We shall show that under suitable assumptions on the Weyl symbol, anyL2-eigenfunction with non-zero eigenvalue is highly concentrated on the time-frequency space.

An important role in the proofs is played by quasi-Banach Wiener amalgam and modulation spaces. As a byproduct, new convolution relations for modulation spaces in the quasi-Banach setting are exhibited. Finally, we will show conjectures and partial results for eigenfunctions of localization and Kohn-Nirenberg operators on locally compact abelian (LCA) groups.

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Paul Alphonse
Université de Rennes 1
France

Smoothing properties of semigroups generated by accretive quadratic operators

Tuesday, 7 July

12:20 - 12:30

Abstract: Accretive quadratic operators are differential operators defined by the Weyl quantization of complex-valued quadratic forms defined on the phase space with non-negative real parts. The smoothing properties of the semigroups generated by this class of operators on $L^2(\mathbb{R}^n)$ are encoded in the structure of a vector subspace of the phase space intrinsically linked to the quadratic operator at play (called its singular space) introduced by M. Hitrik and K. Pravda-Starov. This talk is devoted to present this result in details and to give the sketch of its proof. The method is based on a polar decomposition approach, that is, on describing the decomposition of the evolution operators generated by accretive quadratic operators as the product of a selfadjoint operator and a unitary operator on $L^2(\mathbb{R}^n)$. Such a decomposition is obtained by exploiting the Fourier integral operator structure of these evolution operators given in a paper by L. Hörmander. This is a joint work with J. Bernier.

Paul Alphonse
Université de Rennes 1, France

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Lunch break: 12:30 - 14:20
Presentation of GAP

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Karel Pravda-Starov
Université de Rennes 1, France

Uncertainty principles and null-controllability of evolution equations enjoying Gelfand-Shilov smoothing effects

Tuesday, 7 July

14:20 - 14:50

Abstract: We discuss uncertainty principles for finite combinations of Hermite functions and establish some spectral inequalities for control subsets that are thick with respect to some unbounded densities growing almost linearly at infinity. These spectral inequalities allow to derive the null-controllability in any positive time for evolution equations enjoying specific regularizing effects in Gelfand-Shilov spaces.
This is a joint work with Jérémy Martin (Université de Rennes 1).

Karel Pravda-Starov

Université de Rennes 1, France

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Alberto Parmeggiani
University of Bologna, Italy

On the solvability of degenerate PDEs

Tuesday, 7 July

15:00 - 15:30

Abstract: In this talk I will survey some solvability results related to a class of degenerate second order PDEs which are generalizations of the (formal adjoint of the) Kannai operator.

Alberto Parmeggiani
University of Bologna, Italy

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Ingo Witt
University of Göttingen
Germany

Hyperbolic problems with totally characteristic boundary

Tuesday, 7 July

15:40 - 16:10

Ingo Witt

University of Göttingen, Germany

Abstract: We consider first-order $N\times N$ hyperbolic system

$$
(\mathbf{1}) \quad \begin{cases} \partial_t u + \mathcal{A}(t, x, y, xD_x, D_y)u = f(t, x, y) \ \text{ in } (0, T)\times\mathbb{R}_+^{1+d} \\ u|_{t=0} = u_0(x, y) \ \ on \ \ \mathbb{R}_+^{1+d} \end{cases}
$$

with variables $(t, x, x) \in (0, T)\times \mathbb{R}_+\times \mathbb{R}^d$. The lateral boundary at $x = 0$ is totally characteristic due to the presence of the derivative xD_x. As the lateral boundary is totally characteristic, no boundary conditions are required.

We show that the Cauchy problem **(1)** has a solution u in a class of Sobolev spaces with asymptotics, where

$$
u(t, x, y) \sim \sum_{(p,k)} x^p \log^k x \ u_{pk}(t, y) \quad \text{as } x \to +0
$$

with $(p, k) \in \mathbb{C}\times \mathbb{N}_0, \Re p \to \infty$ as $|p| \to \infty$, provided that f and u_0 have assymptotics of the same type. Furthermore, we show that the boundary traces u_{pk} solve hyperbolic Cauchy problems on $(0, T)\times\mathbb{R}^d$. A special case arises when $u(t, x, y) \sim \sum_{l\in\mathbb{N}_0} x^l u_l(t, y)$ as $x \to +0$ in which case we solve in the standard Sobolev spaces $H^s\left((0, T)\times\mathbb{R}_+^{1+d}; \mathbb{C}^N\right)$.

This is a joint work with Zhuoping Ruan (Nanjing University).

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Presentation:

Book of abstracts.

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Maciej Zworski
University of California
Berkley, USA

Analytic hypoellipticity of Keldysh operators

Tuesday, 7 July

16:30 - 17:00

Abstract: For operators modeled by $P = x_1 D_{x_1}^2 + D_{x_2}^2 + aD_{x_1}$ we show that if u is smooth and $P\,u$ is analytic then u is analytic. This is motivated by the question of analyticity of quasinormal modes of black holes across event horizons and is a consequence of a general microlocal result. Joint work with J. Galkowski.

Maciej Zworski

University of California

Berkley, USA

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Gerardo Mendoza
Temple University, USA

ℝ-actions and invariant differential operators

Tuesday, 7 July

17:10 - 17:40

Gerardo Mendoza
Temple University, USA

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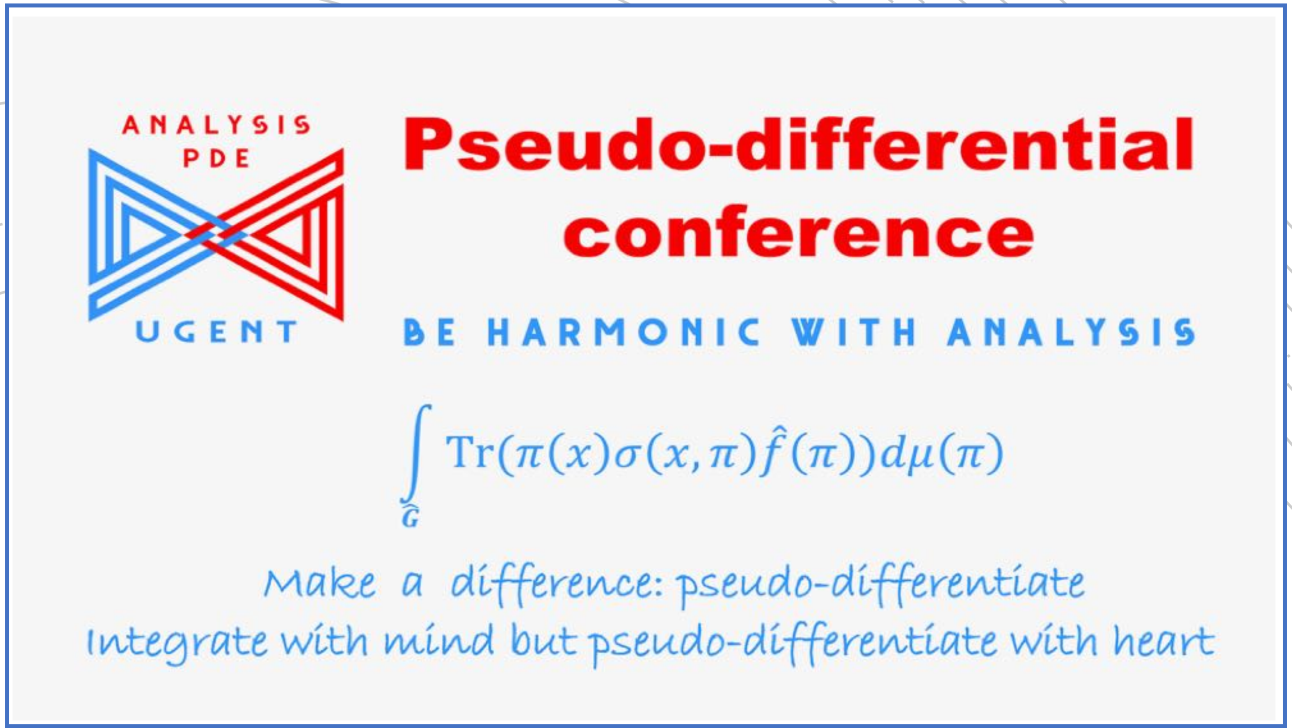
Abstract: A compact manifold together with a nowhere vanishing real vector field that leaves invariant some Riemannian metric behaves in many senses like a circle bundle over a compact base. To illustrate the point, I will discuss a theorem concerning the action of the vector field on the kernel of an invariant differential operator, the latter acting on sections of a Hermitian vector bundle and assumed to be normal with respect to the natural Hilbert space structure, the former acting as a formally self adjoint operator (all under certain hypoellipticity/ellipticity condition). As an application I will discuss how this generalizes Kodaira's vanishing theorem.



Presentation:

Book of abstracts.

Ville Turunen
Aalto University, Finland

Time-frequency analysis on locally compact groups

Tuesday, 7 July

18:00 - 18:30

Ville Turunen
Aalto University, Finland

Abstract: Time-frequency analysis can be described as Fourier analysis simultaneously both in time and in frequency. Its origins are in quantum mechanics, in signal processing in Euclidean spaces, and in pseudo-differential operators. In this presentation we show how to generalize time-frequency analysis to those locally compact groups that allow a nice-enough Fourier transform: wide families of non-commutative groups can be treated. Our results on locally compact groups shed new light also on the analysis in Euclidean spaces.

Ville Turunen
Aalto University, Finland

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$$\mathrm{Re}\,(Tf, f)_{L^2(G)} \geq -C\,\|f\|_{L^2_{(m-(\rho-\delta))/2}(G)}$$

Wednesday, 8 July

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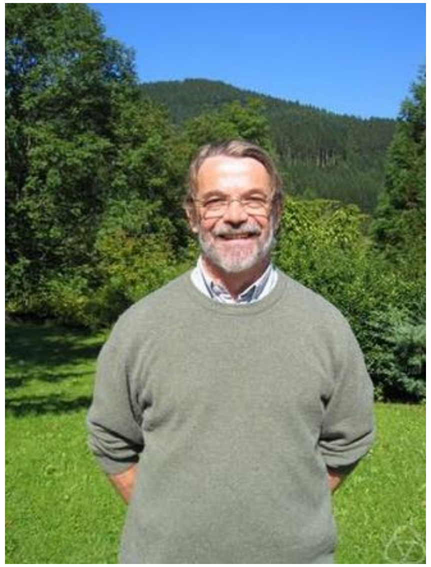

Bernard Helffer
Université de Nantes, France

Spectral analysis near a Dirac type crossing in a weak non-constant magnetic field (after H. Cornean, B. Helffer and R. Purice)

Wednesday, 8 July

10:00 - 10:30

Bernard Helffer
Université de Nantes, France

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Abstract: This is the continuation of a series of works devoted to the justification of the Peierls substitution in the case of a weak magnetic field. Here we deal with two 2d Bloch eigenvalues which have a conical crossing. It turns out that in the presence of an almost constant weak magnetic field, the spectrum near the crossing develops gaps which remind of the Landau levels of an effective mass-less magnetic Dirac operator. This involves the semi-classical analysis for the Peierls-Onsager effective Hamiltonian which is done through the combination of different pseudo-differential calculi.

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Claudia Garetto
University of Loughborough
UK

Hyperbolic Cauchy problems with multiplicities

Wednesday, 8 July

10:40 - 11:10

Claudia Garetto
University of Loughborough, UK

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Abstract: In this talk we give a survey on some recent results, obtained in collaboration with Michael Ruzhansky and Christian Jäh, on hyperbolic Cauchy problems with multiplicities. In particular, we will explain how the theory of pseudo-differential operators is employed in order to get well-posedness.

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Joachim Toft
Linnaeus University, Sweden

Liftings for ultra-modulation spaces, and one-parameter groups of Gevrey type pseudo-differential operators

Wednesday, 8 July

11:30 - 12:00

Joachim Toft

Linnaeus University, Sweden

Abstract: *Let* $\Gamma_s^{(\omega_0)}$ *be the set of all* $a \in C^\infty(R^{2d})$ *such that*

$$|\partial^\alpha a(X)| \le C_\alpha h^{|\alpha|}\alpha^s \omega_0(X), \qquad \omega_0(X+Y) \le C\omega_0(X)e^{r|Y|^{\frac{1}{s}}},$$

for some $h > 0$ *and every* $r > 0$. *We deduce one-parameter group properties for pseudo-differential operators in* $Op(\Gamma_s^{(\omega_0)})$. *We use this to show that*

$$Op(a) \circ Op(b) = Op(b) \circ Op(a) = Id \text{ for some } a \in Op(\Gamma_s^{(\omega_0)}), \quad b \in Op(\Gamma_s^{(1/\omega_0)}).$$

By combining these results with the techniques [4], we explain how to deduce lifting property for modulation spaces and construct explicit isomorphisms between them. Some ideas goes back [2,4]. Especially we prove that for ω, ω_0 *in a general class of weight functions, the Toeplitz (or localization) operator* $Tp(\omega_0)$ *is an isomorphism from* $M_\omega^{p,q}$ *onto* $M_\omega^{p,q}$ *for every* $p, q \in (0, \infty]$. We also give some examples on specific liftings, e.g. that the operators

$$(1+|x|^{2N_1})^r + (1+\Delta^{N_2})^\rho \quad \text{and} \quad Tp\left(\exp\left(\left((1+|x|^2)^r + (1+|\xi|^2)^\rho\right)\right)^t\right),$$

$t < 1/2$, are bijective mappings between suitable modulation spaces. This can be used to deduce that the Weyl operator $Op^\omega(\exp\left(\left((1+|x|^2)^r + (1+|\xi|^2)^\rho\right)\right)^t)$ has index zero as a map between these suitable modulation spaces. The talk is based on joint work with A. Abdeljawad and S. Coriasco.

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Joachim Toft

Linnaeus University, Sweden

References

[1] A. Abdeljawad, S. Coriasco, J. Toft. Liftings for ultra-modulation spaces, and one-parameter groups of Gevrey type pseudo-differential operators, Anal. Appl., appeared online 2019.

[2] P. Boggiatto, J. Toft. Embeddings and compactness for generalized Sobolev-Shubin spaces and modulation spaces, Appl. Anal. 84 (2005), 269–282.

[3] K. Gröchenig, J. Toft. Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces, J. Anal. Math. 114 (2011), 255–283.

[4] K. Gröchenig, J. Toft. The range of localization operators and lifting theorems for modulation and Bargmann-Fock spaces, Trans. Amer. Math. Soc. 365 (2013),4475–4496.

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Paula Cerejeiras
Universidad de Aveiro
Portugal

Discrete Pseudo-differential Operators and Boundary Value Problems in Hypercomplex Function Theory

Wednesday, 8 July

12:10 - 12:40

Paula Cerejeiras
Universidad de Aveiro, Portugal

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Abstract: During the last decades one can observe an increasing interest in discrete structures due to a fast growing computational power. This evolution in computational power also allows to replace the traditional approach of discretisation of partial differential equations via a variational formulation by a finite element modelation directly on the mesh. This also opens the way for discrete analogues to the usual continuous structures, in particular a discrete function theory. While there is a deep and long standing theory in the two-dimensional case, which has applications to problems in probability and statistical physics (see D. Chelkak, R. Novikov, S. Smirnov, just to name a few) the higher dimensional case is only recently being developed. One of the drawbacks of the discrete case is the lack of explicit expressions for the kernels of the corresponding discrete operators. In most cases only their Fourier symbols are known. In this talk we adapt the recent theories of discrete pseudo-differential operators by Botchway/Kibiti/Ruzhansky and Rabinovich/Roch/ Silbermann to the case of symbols with values in a complexified Clifford algebras and construct the necessary tools for a discrete hypercomplex function theory, like discrete Cauchy kernels, discrete Hilbert and Riesz-transforms, and Hardy spaces. Special emphases will be given to the study and application of discrete Riemann-Hilbert boundary value problems.

Lunch break: 12:50 - 14:20

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Poster session, Part I.

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Poster session Part II (Blitztalks)

$$Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \, \mathrm{tr}\left[\xi(x)\,\sigma(x,\xi)\,\widehat{f}(\xi)\right]$$

14:20 - 15:00

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Sylvie Paycha
University of Potsdam
Germany

Regularised traces and Getzler's rescaling

Wednesday, 8 July

15:00 - 15:30

Abstract: Inspired by Gilkey's invariance theory, Connes' deformation to the normal cone and Getzler's rescaling method, we single out a class of geometric operators among pseudodifferential operators acting on sections of a class of natural vector bundles, to which we attach a rescaling degree.
This degree is then used to express regularised traces of geometric operators in terms of a rescaled limit of Wodzicki residues. When applied to complex powers of the square of a Dirac operator, this amounts to expressing the index of a Dirac operator in terms of a local residue involving the Getzler rescaled limit of its square.

Sylvie Paycha
University of Potsdam
Germany

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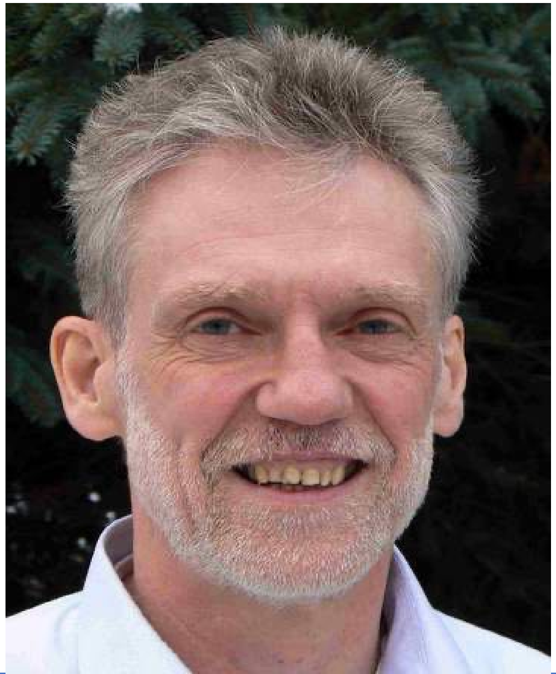
Elmar Schrohe
University of Hannover
Germany



Degenerate Elliptic Boundary Value Problems with Non-smooth Coefficients

Wednesday, 8 July

15:40 - 16:10

Elmar Schrohe
University of Hannover, Germany

Abstract: On a manifold of bounded geometry with boundary we consider a uniformly strongly elliptic second order operator A that locally is of the form

$$A = -\sum_{j,k} a_{jk}\partial_{x_j}\partial_{x_k} + \sum_j b_j \partial_{x_j} + c,$$

together with a degenerate boundary operator T of the form

$$T = \phi_0\gamma_0 + \phi_1\gamma_1,$$

where γ_0 and γ_1, denote the evaluation of a function and its exterior normal derivative, respectively, at the boundary, and ϕ_0, ϕ_1 are smooth functions on the boundary with $\phi_0 \geq 0$, $\phi_1 \geq 0$, and $\phi_0 + \phi_1 \geq c_0 > 0$. Unless either $\phi_0 \equiv 0$, or $\phi_1 \equiv 0$ this problem is not elliptic in the sense of Lopatinskij and Shapiro. We show that the realization A_T of A in $L^p(\Omega)$, has a bounded H^∞-calculus whenever the a_{jk} are Hölder continuous and b_j as well as c are L^∞. For the proof we first treat the operator with smooth coefficients on $\mathbb{R}^n_+$. Here we rely on an extension of Boutet de Monvel's calculus to operator-valued symbols of Hörmander type $(1,\delta)$. We then use H^∞-perturbation techniques in order to treat the nonsmooth case. As an application we study the porous medium equation. (Joint work with Thorben Krietenstein, Hannover).

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Presentation:

Book of abstracts.

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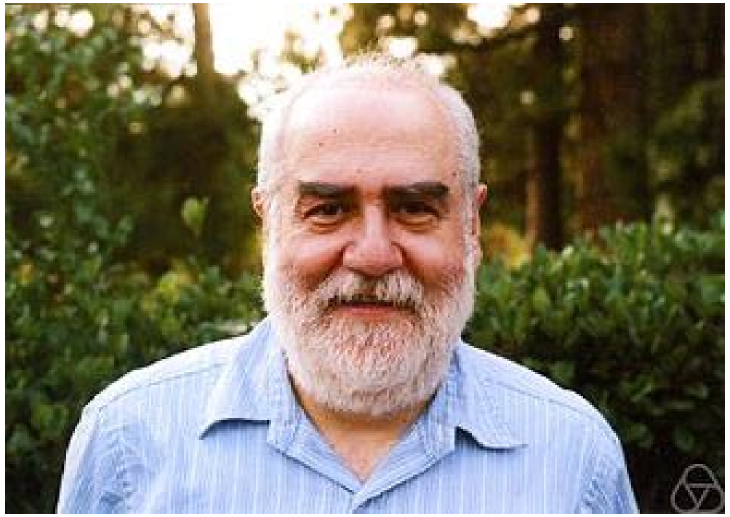
Gunther Uhlmann
University of Washington, USA

Microlocal Analysis and the X-ray Transform

Wednesday, 8 July

16:40 - 17:10



Abstract: We will consider recent applications of microlocal analysis to the study of the geodesic X-ray transform which consists of integrating a function along geodesics. This type of transforms have applications to many fields including medical imaging and geophysics.

Gunther Uhlmann

University of Washington, USA

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Man Wah Wong
York University, Canada

The Heat Semigroup of the Twisted Laplacian on the Non-Isotropic Heisenberg Group with Multi-Dimensional Center

Wednesday, 8 July

17:20 - 17:50

Man Wah Wong
York University, Canada

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Abstract: We give a formula for the one-parameter strongly continuous semigroup $e^{-\tau L^{\lambda}}, \tau > 0$, generated by the twisted Laplacian $L^{\lambda}, \lambda \in \mathbb{R}^m \backslash \{0\}$, on the non-isotropic Heisenberg group $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ in terms of Weyl transforms on theses groups, and use it to obtain an L^2-estimate for the solution of the initial value problem for the heat equation governed by L^{λ} in terms of the L^p norm of the initial data for $1 \leq p \leq \infty$. This is joint work with Ms. Shengwen Yang of York University.

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Marius Mantoiu
University of Chile

Quantizations on Nilpotent Lie Groups and Algebras Having Flat Coadjoint Orbits

Wednesday, 8 July

18:00 - 18:30

Abstract: For a connected simply connected nilpotent Lie group G with Lie algebra g and unitary dual G' one has (a) a global quantization of operator-valued symbols involving the representation theory of the group, (b) a quantization of scalar-valued symbols defined on $G \times g^*$, taking the group structure into account and (c) Weyl-type quantizations of all the coadjoint orbits. We show how these quantizations are related, by a careful analysis of the composition of two different types of Fourier transformations, interesting in itself. We also describe the concrete form of the operator-valued symbol quantization, by using *Kirillov theory* and the Euclidean version of the unitary dual and *Plancherel measure*. In the case of the Heisenberg group, this corresponds to the known picture, presenting the representation theoretical pseudo-differential operators in terms of families of Weyl operators depending on a parameter. For illustration, we work out a couple of examples and put into evidence some specific features of the case of Lie algebras with one-dimensional center. When G is also graded, we make a short presentation of the symbol classes S^m, transferred from $G \times G'$ to $G \times g^*$ by means of the connection mentioned above. Joint work with *Michael Ruzhansky*.

Marius Mantoiu

University of Chile, Chile

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Hardy Inequalities on Homogeneous Groups

Imperial College London

NAZARBAYEV UNIVERSITY

Michael Ruzhansky[1,2,3] and Durvudkhan Suragan[4]

[1]Imperial College London, [2]Ghent Analysis & PDE Center, Ghent University, [3]Queen Mary University of London, [4]Nazarbayev University

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Description

This open access book provides an extensive treatment of Hardy inequalities and closely related topics from the point of view of Folland and Stein's homogeneous (Lie) groups. The place where Hardy inequalities and homogeneous groups meet is a beautiful area of mathematics with links to many other subjects. While describing the general theory of Hardy, Rellich, Caffarelli-Kohn-Nirenberg, Sobolev, and other inequalities in the setting of general homogeneous groups, the authors pay particular attention to the special class of stratified groups. In this environment, the theory of Hardy inequalities becomes intricately intertwined with the properties of sub-Laplacians and subelliptic partial differential equations. These topics constitute the core of this book and they are complemented by additional, closely related topics such as uncertainty principles, function spaces on homogeneous groups, the potential theory for stratified groups, and the potential theory for general Hörmander's sums of squares and their fundamental solutions.

Content

- Analysis on Homogeneous Groups;
- Hardy Inequalities on Homogeneous Groups;
- Rellich, Caffarelli-Kohn-Nirenberg, and Sobolev Type Inequalities;
- Fractional Hardy Inequalities;
- Integral Hardy Inequalities on Homogeneous Groups;
- Horizontal Inequalities on Stratified Groups;
- Hardy-Rellich Inequalities and Fundamental Solutions;
- Geometric Hardy Inequalities on Stratified Groups;
- Uncertainty Relations on Homogeneous Groups;
- Function Spaces on Homogeneous Groups;
- Elements of Potential Theory on Stratified Groups;
- Hardy and Rellich Inequalities for Sums of Squares of Vector Fields.

Hardy Inequality

Classical Hardy inequality in the Euclidean space $\mathbb{R}^n$: for all $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|_E^2} dx,$$

where the constant $(n-2)^2/4$ is sharp, $|\cdot|_E$ is the standard Euclidean norm, and ∇ is the standard gradient on $\mathbb{R}^n$.

Hardy inequality on homogeneous Lie groups $\mathbb{G}$: for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, we have

$$\int_{\mathbb{G}} |\mathcal{R} f(x)|^2 dx \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^2} dx,$$

where the constant $(Q-2)^2/4$ is sharp, Q is the homogeneous dimension of $\mathbb{G}$, $|\cdot|$ is any homogeneous quasi-norm on homogeneous Lie groups $\mathbb{G}$, and $\mathcal{R}$ is the radial derivative, $\mathcal{R} f(x) := \frac{\partial f(x)}{\partial |x|}$.

100 Years of Hardy Inequalities

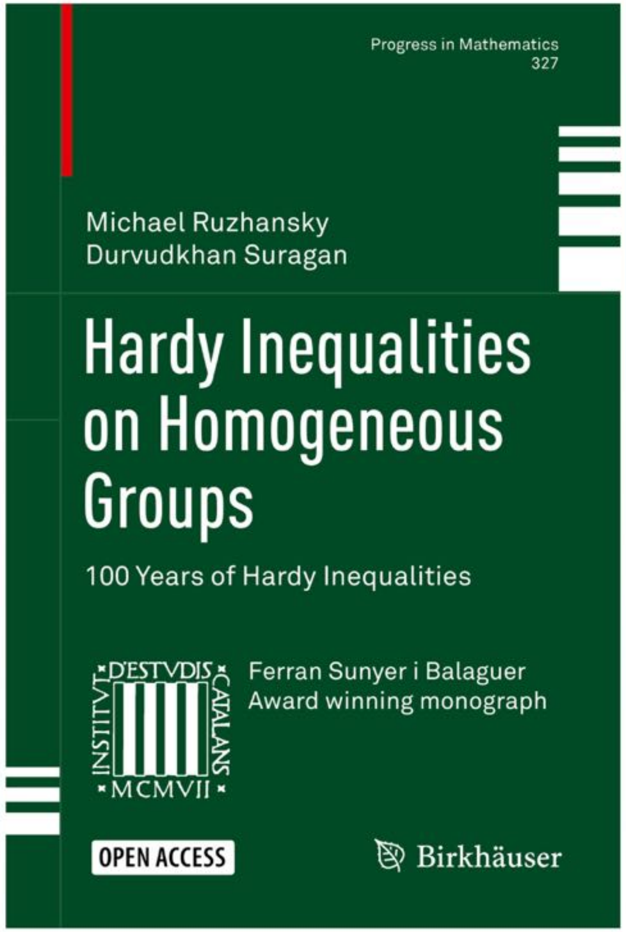
G.H. Hardy and Harald Bohr (from Wikipedia page)

G.H. Hardy reported Harald Bohr as saying 'all analysts spend half their time hunting through the literature for inequalities which they want to use but cannot prove'.

Ferran Sunyer i Balaguer Prize

This monograph is **the winner of the 2018 Ferran Sunyer i Balaguer Prize**, a prestigious award for books of expository nature presenting the latest developments in an active area of research in mathematics. As can be attested as the winner of such an award, it is a vital contribution to literature of analysis not only because it presents a detailed account of the recent developments in the field, but also because the book is accessible to anyone with a basic level of understanding of analysis. Undergraduate and graduate students as well as researchers from any field of mathematical and physical sciences related to analysis involving functional inequalities or analysis of homogeneous groups will find the text beneficial to deepen their understanding.

Open Access Book



Information

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L^p-bounds for pseudo-differential operators on graded Lie groups

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Abstract

We present the sharp L^p-estimates for pseudo-differential operators on arbitrary graded Lie groups proved by the authors in [1]. The results are presented within the setting of the global symbolic calculus on graded Lie groups by using the Fourier analysis associated to every graded Lie group which extends the usual one due to Hörmander on $\mathbb{R}^n$. The main result extends the classical Fefferman's sharp theorem on the L^p-boundedness of pseudo-differential operators for Hörmander classes on $\mathbb{R}^n$ to general graded Lie groups, also adding the borderline $\rho = \delta$.

Introduction

- The investigation of the L^p boundedness of pseudo-differential operators is a crucial task for a large variety of problems in mathematical analysis and its applications, mainly due to its consequences for the regularity, approximation and existence of solutions on L^p-Sobolev spaces.
- There is an extensive literature on the subject, in particular, devoted to operators associated with symbols belonging to the Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, (see for instance, J.J. Kohn and L. Nirenberg [5], L. Hörmander [4] and C. Fefferman [2]).
- Our main goal is to extend a classical and sharp result by C. Fefferman [2] and to provide a critical order for the L^p-boundendess of pseudo-differential operators on graded Lie groups based on the quantization procedure developed by the third author and V. Fischer in [3].
- Our main estimate recover the sharp Fefferman theorem on $\mathbb{R}^n$, adding the critical case $\rho = \delta$.

Fourier analysis on nilpotent Lie groups

- Let G be a simply connected nilpotent Lie group and let $\widehat{G}$ be its unitary dual.
- Let $\exp_G : \mathfrak{g} \to G$, be the exponential mapping on G. The Schwartz class on G, is defined by those $f \in C^\infty(G)$, such that $f \circ \exp_G \in \mathscr{S}(\mathfrak{g})$, with $\mathfrak{g} \simeq \mathbb{R}^{\dim(G)}$.

The Fourier transform of $f \in \mathscr{S}(G)$, at $\pi \in \widehat{G}$, is defined by:

$$\widehat{f}(\pi) = \int_G f(x)\pi(x)^* dx$$

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Pseudo-differential operators on graded Lie groups

- Roughly speaking, a pseudo-differential operator is a continuous linear operator on $\mathscr{S}(G)$, defined by the (quantization) formula:

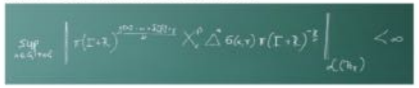
$$\mathrm{Op}(\sigma)f(x) = \int_{\widehat{G}} \mathrm{Tr}\left(\pi(x)\sigma(x,\pi)\widehat{f}(\pi)\right) d\mu(\pi)$$

In such a case, we say that σ is the symbol associated with $\mathrm{Op}(\sigma)$. We have denoted by $d\mu(\pi)$ the Plancherel measure on $\widehat{G}$.

- A Rockland operator is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree $\nu = \nu_{\mathcal{R}}$ and such that, for every unitary irreducible non-trivial representation $\pi \in \widehat{G}$, $\pi(\mathcal{R})$ is injective on $\mathcal{H}^\infty_\pi$; $\sigma_{\mathcal{R}}(\pi) = \pi(\mathcal{R})$ is the symbol associated to $\mathcal{R}$.
- It can be shown that a Lie group G is graded if and only if there exists a differential Rockland operator on G.
- The basic example of graded Lie group is the Heisenberg group $\mathbb{H}^n$.

Hörmander classes on graded Lie groups

Hörmander classes on the phase space $G \times \widehat{G}$ can be defined by using Rockland operators. Indeed, the Hörmander class of order m, and of type (ρ, δ), $S^m_{\rho,\delta}(G \times \widehat{G})$, is defined by those *symbols* σ satisfying symbol inequalities of the kind:



L^p-L^p, H^1-L^1 and L^∞-BMO-boundedness of pseudo-differential operators

Theorem (Fefferman type estimates on Graded Lie groups)

Let G be a graded Lie group of homogeneous dimension Q. Let $A \equiv Op(\sigma) : C^\infty(G) \to \mathscr{D}'(G)$ be a pseudo-differential operator with symbol $\sigma \in S^{-m}_{\rho,\delta}(G \times \widehat{G})$, $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$. Then,

- *(a) if $m = \frac{Q(1-\rho)}{2}$, then A extends to a bounded operator from $L^\infty(G)$ to $BMO(G)$, from the Hardy space $H^1(G)$ to $L^1(G)$, and from $L^p(G)$ to $L^p(G)$ for all $1 < p < \infty$.*
- *(b) If $m \geq m_p := Q(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$, $1 < p < \infty$, then A extends to a bounded operator from $L^p(G)$ into $L^p(G)$.*

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L^p-L^q boundedness of pseudo-differential operators on smooth manifolds and its applications to nonlinear equations

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Abstract

We present the boundedness of global pseudo-differential operators on smooth manifolds obtained in [3]. By using the notion of global symbol [6, 7] we extend a classical condition of Hörmander type to guarantee the L^p-L^q-boundedness of global operators. First we investigate L^p-boundedness of pseudo-multipliers in view of the Hörmander-Mihlin condition. Later, we concentrate to settle L^p-L^q boundedness of the Fourier multipliers and pseudo-differential operators for the range $1 < p \leq 2 \leq q < \infty$. Finally, we present applications of our boundedness theorems to the well-posedness properties of different types of nonlinear partial differential equations.

Introduction

- The boundedness results for pseudo-differential operators in their own right are interesting and important but also these serve as crucial tools to tackle several important problems of mathematics, in particular, of nonlinear PDEs. We refer [8, 7] and references therein for available extensive literature on these topics.
- Our main purpose is to extend seminal and classical result of L. Hörmander on L^p-L^q boundedness for pseudo-differential operators [5] to manifolds using the nonharmonic Fourier analysis and global quantization developed the last two authors [6] (see also [4]). Recently, L^p-L^q boundedness results for Fourier multipliers have been established in [1, 2] in the context of unimodular locally compact groups.
- To prove L^p-L^q boundedness of global operators we establish and apply the Paley-inequality and Hausdorff-Young-Paley inequality in the setting of nonharmonic analysis.

Fourier analysis associated to a model operator L on M

- Let L be a pseudo-differential operator (need not be elliptic or self-adjoint) of order m on the interior M of a smooth manifold with boundary $\overline{M}$ in the sense of Hörmander.
- Assume that some boundary conditions (BC) are fixed and lead to a discrete spectrum with a family of eigenfunctions yielding a Riesz basis in $L^2(M)$.
- The discrete spectrum is $\{\lambda_\xi \in \mathbb{C} : \xi \in \mathcal{I}\}$ of L with corresponding eigenfunctions in $L^2(M)$ denoted by u_ξ which satisfy the boundary conditions (BC).
- The conjugate spectral problem is
$$L^* v_\xi = \overline{\lambda_\xi} v_\xi \text{ in } M, \quad \text{for all } \xi \in \mathcal{I},$$
which we equip with the conjugate boundary conditions $(BC)^*$. We further assume that the functions u_ξ, v_ξ are normalised and the systems $\{u_\xi\}_{\xi\in\mathcal{I}}$ and $\{v_\xi\}_{\xi\in\mathcal{I}}$ are bi-orthogonal.
- The space $C_L^\infty(M) := \cap_{k=1}^\infty \mathrm{Dom}(L^k)$, where $\mathrm{Dom}(L^k) := \{f \in L^2(M) \,|\, L^j f \in \mathrm{Dom}(L),\ j = 0, 1, \cdots, k-1\}$, so that the boundary condition (BC) are satisfied by the operators L^j.
- The L-Fourier transform of $f \in C_L^\infty(M)$ is defined by
$$(\mathcal{F}_L f)(\xi) := \widehat{f}(\xi) := \int_M f(x)\overline{v_\xi(x)}\,dx.$$
- We refer to [6, 4] for more details.

L-pseudo-differential operators on manifolds with boundary

- An L-pseudo-differential operator is a continuous linear operator $A : C_L^\infty(M) \to C_L^\infty(M)$, defined by
$$Af(x) = \sum_{\xi\in\mathcal{I}} u_\xi(x) m(x,\xi)\widehat{f}(\xi), \quad f \in \mathrm{Dom}(A).$$
In this case, the function $m : M \times \mathcal{I} \to \mathbb{C}$, is called the L-symbol associated with A.
- Denote by L° the densely defined operator given by $L^\circ u_\xi = \lambda_\xi u_\xi$, $\quad \xi \in \mathcal{I}$.
- An L-pseudo-differential operator A is called L-pseudo-multiplier if there exists a continuous function $\tau_m : M \times \mathbb{R} \to \mathbb{C}$, such that for every $\xi \in \mathcal{I}$, and $x \in M$, we have $m(x,\xi) = \tau_m(x, |\lambda_\xi|)$. So, A is given by
$$Af(x) = \tau_m(x, \sqrt{L^\circ L})f(x) := \sum_{\xi\in\mathcal{I}} u_\xi(x)\tau_m(x,|\lambda_\xi|)\widehat{f}(\xi)$$

Assumptions for H-M theorem

- There exist $-\infty < \gamma_p^{(1)}, \gamma_p^{(2)} < \infty$, satisfying
$$\|u_\xi\|_{L^p(M)} \lesssim |\lambda_\xi|^{\gamma_p^{(1)}}, \quad \|v_\xi\|_{L^p(M)} \lesssim |\lambda_\xi|^{\gamma_p^{(2)}},$$
for $1 \leq p \leq \infty$.
- The operator $\sqrt{L^\circ L}$ satisfies the Weyl-eigenvalue counting formula
$$N(\lambda) := \sum_{\xi\in\mathcal{I}: |\lambda_\xi|\leq\lambda} = O(\lambda^Q), \ \lambda \to \infty,$$
where $Q > 0$. If $Q' > Q$, then $N(\lambda) = O(\lambda^{Q'})$, $\lambda \to \infty$, so that we assume that Q is the smallest real number with this property.

L^p-boundedness of L-pseudo-multipliers operators on M

Theorem (Hörmander-Mihlin (H-M) theorem for pseudo-multipliers)

Let M be a smooth manifold with boundary and let $A : C_L^\infty(M) \to C_L^\infty(M)$ be an L-pseudo-multiplier. Let us assume that τ_m satisfies the following Hörmander condition,
$$\|\tau_m\|_{l.u,\mathcal{H}^s} = \sup_{r>0,x\in M} r^{(s-\frac{Q_m}{2})}\|\langle\cdot\rangle^s \mathscr{F}[\tau_m(x,\cdot)\psi(r^{-1}\cdot)]\|_{L^2(\mathbb{R})} < \infty,$$
for $s > \max\{1/2, \gamma_p + Q + (Q_m/2)\}$. Then $A \equiv T_m : L^p(M) \to L^p(M)$ extends to a bounded linear operator for all $1 < p < \infty$.

L^p-L^q-boundedness of L-Fourier multipliers operators on M

Theorem (Hörmander theorem for L-Fourier multipliers)

Let $1 < p \leq 2 \leq q < \infty$ and assume that
$$\sup_{\xi\in\mathcal{I}}\left|\frac{\|v_\xi\|_{L^\infty(M)}}{\|u_\xi\|_{L^\infty(M)}}\right| < \infty \quad and \quad \sup_{\xi\in\mathcal{I}}\left|\frac{\|u_\xi\|_{L^\infty(M)}}{\|v_\xi\|_{L^\infty(M)}}\right| < \infty.$$
Suppose that $A : C_L^\infty(M) \to C_L^\infty(M)$ is an L-Fourier multiplier with L-symbol $\sigma_{A,L}$ on M, that is, A satisfies
$$\mathcal{F}_L(Af)(\xi) = \sigma_{A,L}(\xi)\mathcal{F}_L f(\xi), \quad for\ all\ \xi \in \mathcal{I},$$
where $\sigma_{A,L} : \mathcal{I} \to \mathbb{C}$ is a function. Then we have
$$\|A\|_{\mathscr{B}(L^p(M),L^q(M))} \lesssim \sup_{s>0} s\left(\sum_{\substack{\xi\in\mathcal{I}\\ |\sigma_{A,L}(\xi)|\geq s}} \max\{\|u_\xi\|^2_{L^\infty(M)}, \|v_\xi\|^2_{L^\infty(M)}\}\right)^{\frac{1}{p}-\frac{1}{q}}$$

L^p-L^q-boundedness of L-pseudo-differential operators on M

Theorem (Hörmander theorem for L-pseudo-differential operators)

Let $1 < p \leq 2 \leq q < \infty$ and assume that
$$\sup_{\xi\in\mathcal{I}}\left|\frac{\|v_\xi\|_{L^\infty(M)}}{\|u_\xi\|_{L^\infty(M)}}\right| < \infty \quad and \quad \sup_{\xi\in\mathcal{I}}\left|\frac{\|u_\xi\|_{L^\infty(M)}}{\|v_\xi\|_{L^\infty(M)}}\right| < \infty.$$
Suppose that $A : C_L^\infty(M) \to C_L^\infty(M)$ is a continuous linear operators with L-symbol $\sigma_{A,L} : M \times \mathcal{I} \to \mathbb{C}$, where M is a compact manifold (with or without boundary), satisfying
$$\|\sigma_{A,L}\|_{(\beta)} := \sup_{s>0,y\in M} s\left(\sum_{\substack{\xi\in\mathcal{I}\\ |\partial_y^\beta\sigma_{A,L}(y,\xi)|\geq s}} \max\{\|u_\xi\|^2_{L^\infty(M)}, \|v_\xi\|^2_{L^\infty(M)}\}\right)^{\frac{1}{p}-\frac{1}{q}} < \infty,$$
for all $|\beta| \leq \left[\frac{\dim(M)}{p}\right] + 1$, where ∂_y denotes the local partial derivative. If $\partial M \neq \emptyset$, let us assume additionally that $\mathrm{supp}(\sigma_{A,L}) \subset \{(y,\xi) \in M \times \mathcal{I} : y \in M\setminus V\}$ where $V \subset M$ is an open neighbourhood of the boundary ∂M. Then A admits a bounded extension from $L^p(M)$ into $L^q(M)$.

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Applications to non-linear equations

We apply our L^p-L^q boundedness to establish the well-posedness properties the solutions of nonlinear equations in the space $L^\infty(0,T; L^2(M))$.

- In the nonlinear stationary problem case, we consider the following equation in $L^2(M)$
$$Au = |Bu|^p + f,$$
where $A, B : L^2(M) \to L^2(M)$ are linear operators and $1 \leq p < \infty$.
- In the case of the nonlinear heat equation, we study the Cauchy problem in the space $L^\infty(0,T; L^2(M))$
$$u_t(t) - |Bu(t)|^p = 0, u(0) = u_0,$$
where B is a linear operator in $L^2(M)$ and $1 \leq p < \infty$.
- In the non-linear wave equation case, we study the following initial value problem (IVP)
$$u_{tt}(t) - b(t)|Bu(t)|^p = 0,$$
$$u(0) = u_0, \ u_t(0) = u_1,$$
where b is a positive bounded function depending only on time, B is a linear operator in $L^2(M)$ and $1 \leq p < \infty$.

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The Harmonic Oscillator on The Heisenberg Group

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Abstract

We present a notion of harmonic oscillator on the Heisenberg group $\mathbf{H}_n$. This operator forms a natural analogue of the harmonic oscillator on $\mathbb{R}^n$.

Introduction

Our ansatz is based on a few reasonable assumptions: the harmonic oscillator on $\mathbf{H}_n$ should be a negative sum of squares of operators related to the sub-Laplacian on $\mathbf{H}_n$, essentially self-adjoint with purely discrete spectrum, and its eigenvectors should be smooth functions and form an orthonormal basis of $L^2(\mathbf{H}_n)$. This leads to a differential operator on $\mathbf{H}_n$ which is determined by the Dynin-Folland Lie algebra, a stratified 3-step nilpotent Lie algebra.

Ansatz

Our approach is motivated by the following three realizations of the classical harmonic oscillator $\mathcal{Q}_{\mathbb{R}^n}$ on $\mathbb{R}^n$:

- (R1) the negative sum of squares $-\Delta + |x|^2$ of partial derivatives of order 1 and coordinate multiplication operators;
- (R2) the Weyl and Kohn-Nirenberg quantizations on $\mathbb{R}^n$ of the symbol $\sigma(x,\xi) := |\xi|^2 + |x|^2$ with $x, \xi \in \mathbb{R}^n$;
- (R3) the image $d\rho_1(-\mathcal{L}_{\mathbf{H}_n})$ of the negative sub-Laplacian $-\mathcal{L}_{\mathbf{H}_n}$ on $\mathbf{H}_n$ under the infinitesimal Schrödinger representation $d\rho_1$ (of Planck's constant equal to 1) of the Heisenberg Lie algebra $\mathfrak{h}_n$.

The Schrödinger representation ρ_1 of $\mathbf{H}_n$ acting on $L^2(\mathbb{R}^n)$ and the associated Lie algebra representation, naturally acting on $\mathcal{S}(\mathbb{R}^n)$, clearly relate each of the realisations (R1) - (R3) to the others. One can expect that similar realisations should be available for the canonical harmonic oscillator on $\mathbf{H}_n$.

Main Result

The harmonic oscillator $\mathcal{Q}_{\mathbf{H}_n}$ on the Heisenberg group $\mathbf{H}_n$ has a purely discrete spectrum $\mathrm{spec}(\mathcal{Q}_{\mathbf{H}_n}) \subset (0,\infty)$. The number of its eigenvalues, counted with multiplicities, which are less or equal to $\lambda > 0$ is asymptotically (as $\lambda \to \infty$) given by

$$N(\lambda) \sim \lambda^{\frac{6n+3}{2}},$$

and the magnitude of the eigenvalues is asymptotically equal to

$$\lambda_k \sim k^{\frac{2}{6n+3}} \text{ for } k = 1, 2, \ldots.$$

Moreover, the eigenvectors of $\mathcal{Q}_{\mathbf{H}_n}$ are in $\mathcal{S}(\mathbf{H}_n)$ and form an orthonormal basis of $L^2(\mathbf{H}_n)$.

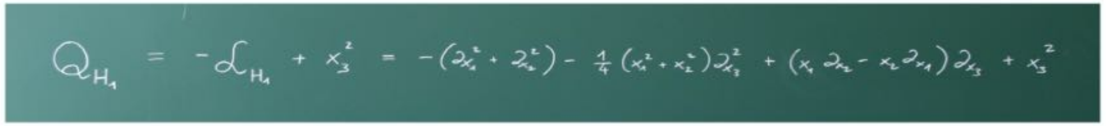

Figure 1: The Harmonic Oscillator on $\mathbf{H}_1$.

Definition

The Dynin-Folland Lie group $\mathbf{H}_{n,2} = \mathbb{R}^{2n+2} \rtimes \mathbf{H}_n$ acts on $f \in L^2(\mathbf{H}_n)$ by the unitary irreducible representation

$$(\pi(z,y,x)f)(t) = e^{iz} e^{i\langle t \cdot \frac{1}{2}x, y\rangle} f(t \cdot x),$$

where $t \cdot \frac{1}{2}x$ and $t \cdot x$ denote the $\mathbf{H}_n$-group products of the corresponding coordinate vectors.
For the basis $\{X_1, \ldots, Y_{2n+1}, Z\}$ of its Lie algebra $\mathfrak{h}_{n,2}$ we define the **harmonic oscillator** on $\mathbf{H}_n$ by

$$\mathcal{Q}_{\mathbf{H}_n} := d\pi(-\mathcal{L}_{\mathbf{H}_{n,2}}) = -d\pi(X_1)^2 - \ldots - d\pi(X_{2n})^2 - d\pi(Y_{2n+1})^2,$$

where $-\mathcal{L}_{\mathbf{H}_{n,2}}$ is the sub-Laplacian on $\mathbf{H}_{n,2}$. Its natural domain includes the smooth vectors $\mathcal{H}_\pi^\infty \cong \mathcal{S}(\mathbf{H}_n)$.

Interpretation

The essentially self-adjoint differential operator $\mathcal{Q}_{\mathbf{H}_n}$ on $\mathbf{H}_n$ admits analogues of (R1) – (R3):

- (R1') the differential operator $-\mathcal{L}_{\mathbf{H}_n} + x_{2n+1}^2$;
- (R2') i) the Dynin-Weyl quantization on $\mathbf{H}_n$ of the symbol $\sigma(x,\xi) := \xi_1^2 + \cdots + \xi_{2n}^2 + x_{2n+1}^2$ with $(x,\xi) \in \mathbb{R}^{2n+1} \times \widehat{\mathbb{R}}^{2n+1}$;
  ii) the Kohn-Nirenberg quantization, in the sense of [3], of the operator-valued symbol on $\mathbf{H}_n \times \widehat{\mathbf{H}}_n$ $\sigma(x,\rho_\lambda) := -\rho_\lambda(X_1)^2 - \ldots - \rho_\lambda(X_{2n})^2 + x_{2n+1}^2$;
- (R3') the image $d\pi(-\mathcal{L}_{\mathbf{H}_{n,2}})$ of the sub-Laplacian $\mathcal{L}_{\mathbf{H}_{n,2}}$ on $\mathbf{H}_{n,2}$, here by definition.

Methods

$\mathcal{Q}_{\mathbf{H}_n}$ has purely discrete spectrum in $(0,\infty)$ by [5]. The asymptotic growth rate of its eigenvalues is obtained via a powerful method developed in [8]. The number of eigenvalues (with multiplicities), asymptotically behaves like the volumes of certain subsets of the coadjoint orbit $\mathcal{O}_\pi \subset \mathfrak{h}_{n,2}^*$ corresponding to representation $\pi \in \widehat{\mathbf{H}}_n$. The subsets in question are determined (up to a multiplicative constant) by a (any) homogeneous quasi-norm on $\mathfrak{g}^*$. The flatness of $\mathcal{O}_\pi$ and a convenient choice of quasi-norm facilitate the computations substantially.

Remark

The power $\frac{6n+3}{2}$ intimately related to the homogeneous structure of $\mathfrak{h}_{n,2}$: the nominator $6n+3$ is the homogenous dimension of the first two strata $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \subseteq \mathfrak{h}_{n,2}$, while the denominator 2 is the homogeneous degree of $-\mathcal{L}_{\mathbf{H}_{n,2}}$.

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Smoothing and Strichartz estimates for Degenerate Schrödinger Operators

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Abstract

In what follows we shall present some recent results about the validity of smoothing and Strichartz-type estimates for time-degenerate Schrödinger operators. These results have important applications in the study of the local well-posedness of the initial value problem (IVP) associated with the operators under consideration.

Time degenerate Schrödinger-type Operators

We shall consider the following classes of degenerate Schrödinger-type operators

$$\mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x + b(t,x)\cdot\nabla_x, \tag{1}$$

$$\mathcal{L}_c = i\partial_t + c'(t)\Delta_x, \tag{2}$$

where $\alpha > 0$, $b(t,x) = (b_1(t,x), \dots, b_n(t,x))$ is such that, for all $j = 1, \dots, n$, $b_j \in C([0,T], C_b^\infty(\mathbb{R}^n))$, while $c \in C^1(\mathbb{R})$.

The class (1) was considered in [1] where both homogeneous and inhomogeneous weighted local smoothing estimates are derived. These estimates are also employed to obtain local well-posedness results for the associated nonlinear IVP (see [1]).

The class (2) was studied in [2] where global weighted homogeneous smoothing estimates are proved by means of comparison principles. For the class (2) weighted Strichartz estimates are proved in [2] where the application to the local well-posedness of the semilinear IVP is given.

The main difference between the operators of the form (1) and (2) and the other Schrödinger operators studied so far **is the presence of degeneracies**. Specifically, the degeneracies are given by the coefficient t^α and $c'(t)$ in (1) and (2) respectively.

Why to study smoothing and Strichartz estimates?

These estimates give us important information about the regularity properties of the solution of the IVP. In particular:

- **The homogeneous smoothing** effect describes a gain of smoothness of the homogeneous solution of the IVP with respect to the smoothness of the initial data.
- **The inhomogeneous smoothing** effect describes a gain of smoothness of the solution of the inhomogeneous IVP with respect to the regularity of the inhomogeneous data.
- **Strichartz estimates** describe a gain of integrability instead of a gain of smoothness of the solution of the IVP.

Additionally, **these estimates are fundamental in order to prove the local well-posedness of the corresponding semilinear and nonlinear IVP** through the standard fixed point argument.

What are comparison principles? (Following [3])

Question: Given two operators $P_a(t,x,D_t,D_x)$ and $P_b(t,x,D_t,D_x)$ depending on two functions a and b respectively, is it possible to compare (in a suitable sense) the solutions of the HIVP (homogeneous IVP) for P_a and P_b if a and b are comparable (in a suitable sense)?

This is essentially what the **comparison principles** we refer to do, that is, **they translate a relation between a and b in a relation between the solutions of the HIVP for P_a and P_b.**

Example (see [3]). Let $a, \tilde{a} \in C^1(\mathbb{R})$ be real valued and strictly monotone on the support of a measurable function χ, and let $\sigma, \tau \in C^0(\mathbb{R})$. Then, if $\forall \xi \in \operatorname{supp}\chi$ we have

$$\frac{|\sigma(\xi)|}{|a'(\xi)|^{1/2}} \le C\frac{|\tau(\xi)|}{|\tilde{a}'(\xi)|^{1/2}}, \quad \text{then} \quad \|\chi(D_x)\sigma(D_x)e^{ita(D)}\varphi\|_{L^2(\mathbb{R}^2)} \le C\|\chi(D_x)\tau(D_x)e^{it\tilde{a}(D)}\varphi\|_{L^2(\mathbb{R}^2)}$$

for all $\varphi = \varphi(x)$ smooth enough.

Weighted local smoothing effect for $\mathcal{L}_\alpha$

We consider the IVP

$$\begin{cases} \partial_t u = it^\alpha \Delta_x u + ib(t,x)\cdot\nabla_x u + f(t,x) \\ u(0,x) = u_0(x). \end{cases} \tag{3}$$

When $b = 0$ one can proceed by using Fourier analysis. However, in the general case $b \neq 0$, the use of pseudo-differential calculus is needed.

Theorem (F.-Staffilani)

Let $u_0 \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Assume that, for all $j = 1, \dots, n$, b_j is such that $b_j \in C([0,T], C_b^\infty(\mathbb{R}^n))$ and there exists $\sigma > 1$ such that

$$|\operatorname{Im}\partial_x^\gamma b_j(t,x)|, |\operatorname{Re}\partial_x^\gamma b_j(t,x)| \lesssim t^\alpha \langle x\rangle^{-\sigma-|\gamma|}, \quad x \in \mathbb{R}^n, \tag{4}$$

and denote by $\lambda(|x|) := \langle x\rangle^{-\sigma}$ and by $\Lambda := \langle\xi\rangle$.
Then

(i) If $f \in L^1([0,T]; H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0\le t\le T}\|u(t)\|_s \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)}\left(\|u_0\|_s + \int_0^T \|f(t)\|_s dt\right);$$

(ii) If $f \in L^2([0,T]; H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist two positive constants C_1, C_2 such that

$$\sup_{0\le t\le T}\|u(t)\|_s^2 + \int_0^T\int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)\,dx\,dt \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)}\left(\|u_0\|_s^2 + \int_0^T \|f(t)\|_s^2 dt\right);$$

(iii) If $\Lambda^{s-1/2}f \in L^2([0,T]\times\mathbb{R}^n; t^{-\alpha}\lambda(|x|)^{-1}dt\,dx)$ then the IVP (3) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0\le t\le T}\|u(t)\|_s^2 + \int_0^T\int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)\,dx\,dt \le C_1 e^{C_2\frac{T^{\alpha+1}}{\alpha+1}}\left(\|u_0\|_s^2 + \int_0^T\int_{\mathbb{R}^n} t^{-\alpha}\lambda(|x|)^{-1}\left|\Lambda^{s-1/2}f\right|^2 dx\,dt\right).$$

Local well-posedness results for $\mathcal{L}_\alpha$

We consider the nonlinear IVPs

$$\text{IVP1} = \begin{cases} \mathcal{L}_\alpha u = \pm u|u|^{2k} \\ u(0,x) = u_0(x), \end{cases} \qquad \text{IVP2} = \begin{cases} \mathcal{L}_\alpha u = \pm t^\beta \nabla u \cdot u^{2k}, \; \beta \ge \alpha > 0, \\ u(0,x) = u_0(x). \end{cases}$$

Theorem (F.-Staffilani). Let $\mathcal{L}_\alpha$ be such that condition (4) is satisfied. Then the IVP1 is locally well posed in H^s for $s > n/2$.

Theorem (F.-Staffilani). Let $\mathcal{L}_\alpha$ be such that condition (4) is satisfied with $\sigma = 2N$ (thus $\lambda(|x|) = \langle x\rangle^{-2N}$) for some $N \ge 1$, and $s > n + 4N + 3$ such that $s - 1/2 \in 2\mathbb{N}$. Let $H_\lambda^s := \{u_0 \in H^s(\mathbb{R}^n); \lambda(|x|)u_0 \in H^s(\mathbb{R}^n)\}$, then the IVP2 with $\beta \ge \alpha > 0$, is locally well posed in H_λ^s.

Global weighted smoothing and Strichartz estimates for $\mathcal{L}_c$

For operators of the form $\mathcal{L}_c$ several comparison principles are proved in [2]. These are used to obtain global smoothing estimates. We state below only one of them, namely the one corresponding to the suitable generalization of the standard (corresponding to the case $c'(t) = 1$) global homogeneous smoothing estimate. For more global smoothing estimates see [2].

Theorem (F.-Ruzhansky)

Let $n \ge 1$, $c \in C^1(\mathbb{R})$ be such that it vanishes at 0. Then, $\forall x \in \mathbb{R}^n$,

(i) If c is such that $\{t \in \mathbb{R}: c'(t) = 0\}$ is finite, then there exist a constant $C > 0$ such that, for all j,

$$\sup_{x_j}\||c'(t)|^{1/2}|D_{x_j}|^{1/2}e^{ic(t)\Delta_x}\varphi\|_{L^2(R^{n-1}_{x'}\times\mathbb{R}_t)} \le C\|\varphi\|_{L^2_x(\mathbb{R}^n)}, \quad \forall\varphi \in L^2_x(\mathbb{R}^n);$$

(ii) If c is such that the set $\{t \in \mathbb{R}; c'(t) = 0\}$ is countable, then there exists a function $\tilde{c} \in C(\mathbb{R})$, and a positive constant C such that, for all j,

$$\sup_{x_j}\||\tilde{c}(t)c'(t)|^{1/2}|D_{x_j}|^{1/2}e^{ic(t)\Delta_x}\varphi\|_{L^2(R^{n-1}_{x'}\times\mathbb{R}_t)} \le C\|\varphi\|_{L^2_x(\mathbb{R}^n)}, \quad \forall\varphi \in L^2_x(\mathbb{R}^n);$$

where $x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n)$.

Both global and local Strichartz estimates are satisfied by $\mathcal{L}_c$. Below we give the statement of the local ones which are those employed to prove the local well-posedness of the semilinear IVP.

Theorem (F.-Ruzhansky)

Let $c \in C^1([0,T])$ be vanishing at 0 and such that $\sharp\{t \in [0,T]; c'(t) = 0\} = k \ge 1$. Then, on denoting by $L_t^q L_x^p := L^q([0,T]; L^p(\mathbb{R}^n))$, we have that for any (q,p) admissible pair $\left(\frac{2}{q} + \frac{n}{p} = \frac{n}{2}\right)$ such that $2 < q, p < \infty$, the following estimates hold

$$\||c'(t)|^{1/q}e^{ic(t)\Delta}\varphi\|_{L_t^q L_x^p} \le C(n,q,p,k)\|\varphi\|_{L^2_x(\mathbb{R}^n)},$$

$$\|e^{ic(t)\Delta}\varphi\|_{L_t^\infty L_x^2} \le \|\varphi\|_{L^2_x(\mathbb{R}^n)},$$

$$\||c'(t)|^{1/q}\int_0^t |c'(s)|e^{i(c(t)-c(s))\Delta}g(s)ds\|_{L_t^q L_x^p} \le C(n,q,p,k)\||c'|^{1/q'}g\|_{L_t^{q'}L_x^{p'}},$$

$$\|\int_0^t |c'(s)|e^{i(c(t)-c(s))\Delta}g(s)ds\|_{L_t^\infty L_x^2} \le C(n,q,p,k)\||c'|^{1/q'}g\|_{L_t^{q'}L_x^{p'}}.$$

Local well-posedness of the semilinear IVP for $\mathcal{L}_c$

We can now apply the previous results to obtain the local well-posedness of the semilinear IVP

$$\begin{cases} \partial_t u + ic'(t)\Delta u = \mu|c'(t)||u|^{p-1}u, \quad \mu \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \tag{5}$$

Theorem (F.-Ruzhansky). Let $1 < p < \frac{4}{n} + 1$ and $c \in C^1([0,+\infty))$ be vanishing at 0 and it is either strictly monotone or such that $\sharp\{t \in [0,T]; c'(t) = 0\}$ is finite for any $T < \infty$. Then for all $u_0 \in L^2(\mathbb{R}^n)$ there exists $T = T(\|u_0\|_2, n, \mu, p) > 0$ such that there exists a unique solution u of the IVP (5) in the time interval $[0,T]$ with $u \in C([0,T]; L^2(\mathbb{R}^n)) \bigcap L_t^q([0,T]; L_x^{p+1}(\mathbb{R}^n))$ and $q = \frac{4(p+1)}{n(p-1)}$. Moreover the map $u_0 \mapsto u(\cdot,t)$, locally defined from $L^2(\mathbb{R}^n)$ to $C([0,T); L^2(\mathbb{R}^n))$, is continuous.

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Hardy-Littlewood inequality and L^p-L^q Fourier multipliers on compact hypergroups

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Aim

- To prove Hardy-Littlewood inequality and Paley inequality for compact hypergroups [8].
- To establish Hörmander's L^p-L^q Fourier multiplier theorem on compact hypergroups for the range $1 < p \leq 2 \leq q < \infty$ [8].

Hypergroups: What & why?

- Roughly, a hypergroup K is a locally compact Hausdorff space with a convolution on the space $M_b(K)$ of regular bounded Borel measures on K with properties similar to those of group convolution.
- In non commutative setting, the analysis on hypergroups provides a natural extension of analysis on locally compact groups. While in commutative setting, they extend the theory of spherical functions and Gelfand pairs.
- Some of important examples are double coset spaces, the space of conjugacy classes on (Lie) groups and the space of group orbits.
- In particular, the results presented here are true for several interesting examples including Jacobi hypergroups with Jacobi polynomials as characters, compact hypergroup structure on the fundamental alcove with Heckman-Opdam polynomials as characters and multivariant disk hypergroups.
- A compact hypergroup can be countable infinite also ([6]). This property distinguishes them from compact groups.
- Unlike locally compact abelian groups, the support of the Plancherel measure on the dual space may not be full space in the case of commutative hypergroups.
- For more details on analysis on hypergroups and several interesting examples, see [4, 10, 6].

Fourier analysis on compact hypergroups

- Let K be compact hypergroup with normalised Haar measure λ. Denote by $\widehat{K}$ the dual space consisting of irreducible inequivalent continuous representations of K equipped with the discrete topology.
- Every $\pi \in \widehat{K}$ is finite dimensional but may not be unitary in contrast to compact groups case.
- In commutative setting also, the dual space $\widehat{K}$ may not have a hypergroup structure, in contrary to abelian groups.
- Denote the dimension and hyperdimension of $\pi \in \widehat{K}$ by d_π and k_π.
- For each $\pi \in \widehat{K}$, the Fourier transform $\widehat{f}$ of $f \in L^1(K)$ is defined as

$$\widehat{f}(\pi) = \int_K f(x)\bar{\pi}(x)\, d\lambda(x),$$

where $\bar{\pi}$ is the conjugate representation of π.
- We refer to [10, 4, 9] for more details on Fourier analysis and representation theory of compact hypergroups.

Methods

- To establish Hausdorff-Young-Paley inequality we first prove Paley inequality [2, 11] for compact hypergroups [8] and then we use weighted interpolation with Hausdorff-Young inequality [9].
- An application of Paley inequality gives Hardy-Littlewood inequality for compact hypergroups.
- We obtain Hörmander L^p-L^q boundedness of Fourier multiplier [7] in context of compact hypergroup with the help of the Hausdorff-Young-Paley inequality.

Literature

- Hardy-Littlewood inequality [5] was recently estalished for compact homogeneous spaces [2] and for compact quantum groups [1, 11].
- L^p-L^q boundedness of Fourier multipliers on locally compact unimodular groups was proved in [3] using von-Neumann algebra techniques.

Hausdorff-Young-Paley inequality on compact hypergroups

Theorem (Hausdorff-Young-Paley (Pitt) inequality)

Let K be a compact hypergroup and let $1 < p \leq b \leq p' < \infty$. If a positive sequence $\varphi(\pi), \pi \in \widehat{K}$, satisfies the condition

$$M_\varphi := \sup_{y>0} y \sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \geq y}} k_\pi^2 < \infty,$$

then we have

$$\left(\sum_{\pi \in \widehat{K}} k_\pi^2 \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \varphi(\pi)^{\frac{1}{b}-\frac{1}{p'}} \right)^b \right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b}-\frac{1}{p'}} \|f\|_{L^p(K)}.$$

Non-commutative version of Hardy-Littlewood inequality

Theorem (Hardy-Littlewood inequality for compact hypergroups)

Let $1 < p \leq 2$ and let K be a compact hypergroup. Assume that a sequence $\{\mu_\pi\}_{\pi \in \widehat{K}}$ grows sufficiently fast, that is,

$$\sum_{\pi \in \widehat{K}} \frac{k_\pi^2}{|\mu_\pi|^\beta} < \infty \quad \textit{for some } \beta \geq 0.$$

Then we have

$$\sum_{\pi \in \widehat{K}} k_\pi^2 |\mu_\pi|^{\beta(p-2)} \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \right)^p \lesssim \|f\|_{L^p(K)}.$$

L^p-L^q-boundedness of Fourier multipliers on compact hypergroups

Theorem (Hörmander theorem for Fourier multipliers)

Let K be a compact hypergroup and let $1 < p \leq 2 \leq q < \infty$. Let A be a left Fourier multiplier with symbol σ_A, that is, A satisfies

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi), \quad \pi \in \widehat{K}.$$

Then we have

$$\|A\|_{L^p(K) \to L^q(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi \in \widehat{K} \\ \|\sigma_A(\pi)\|_{op} \geq y}} k_\pi^2 \right)^{\frac{1}{p}-\frac{1}{q}}.$$

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H-L inequality for Conj(SU(2))

If $1 < p \leq 2$ and $f \in L^p(\mathrm{Conj}(\mathrm{SU})(2))$, then we have

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p \leq C \|f\|_{L^p(\mathrm{Conj}(\mathrm{SU})(2))}$$

H-L for D-R hypergroups [6]

If $1 < p \leq 2$ then there exists a constant $C = C(p)$ such that

$$f(0) + \sum_{n \in \mathbb{N}} ((1-a)a^{-n})^{p(\frac{2}{p}-1)} |\widehat{f}(n)|^p \leq C \|f\|_{L^p(H_a)}$$

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Van der Corput lemmas for Mittag-Leffler functions

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Introduction

In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals. This estimate was first obtained by the Dutch mathematician Johannes Gaultherus van der Corput [1] and named in his honour. While the paper [1] was published in Mathematische Annalen in 1921, he submitted it there on 17 December 1920 (from Utrecht). Therefore, it seems appropriate to us to dedicate this paper to the 100^{th} anniversary of this lemma.

Let us state the classical van der Corput lemmas as follows:

- **van der Corput lemma.** Suppose ϕ is a real-valued and smooth function in $[a,b]$. If ψ is a smooth function and $|\phi^{(k)}(x)| \geq 1$, $k \geq 1$, for all $x \in (a,b)$, then
$$\left|\int_a^b \exp(i\lambda\phi(x))\psi(x)dx\right| \leq \frac{C}{\lambda^{1/k}}, \quad \lambda \to \infty, \qquad (1)$$
for $k=1$ and ϕ' is monotonic, or $k \geq 2$. Here C does not depend on λ.

Formulation of problem

The main goal of the present paper is to study van der Corput lemmas for the oscillatory integrals defined by (see [2], [3]) respectively:
$$I_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx, \qquad (2)$$
and
$$\mathcal{I}_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i^\alpha\lambda\phi(x))\,\psi(x)dx, \qquad (3)$$
where $\alpha > 0$, $\beta > 0$, ϕ is a phase and ψ is an amplitude, and λ is a positive real number that can vary. Here $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined as
$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R},$$
with the property that
$$E_{1,1}(z) = e^z. \qquad (4)$$

Main results for $I_{\alpha,\beta}$

van der Corput lemmas on $\mathbb{R}$: consider $I_{\alpha,\beta}$ defined by (2).
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha < 1$, $\beta > 0$, and $m = \operatorname{ess\,inf}_{x\in\mathbb{R}} |\phi(x)| > 0$, then
$$|I_{\alpha,\beta}(\lambda)| \leq \frac{M}{1+\lambda m}\|\psi\|_{L^1(\mathbb{R})}, \ \lambda \geq 1,$$
where M does not depend on ϕ, ψ and λ.

van der Corput lemmas on $I = [a,b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$.

- Let $0 < \alpha < 1$, $\beta > 0$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then
$$|I_{\alpha,\beta}(\lambda)| \leq M_k\lambda^{-\frac{1}{k}}\log^{\frac{1}{k}}(1+\lambda), \quad \lambda \geq 1,$$
where M_k does not depend on λ.
- Let $-\infty < a < b < +\infty$ and $I = [a,b] \subset \mathbb{R}$. Let $0 < \alpha \leq 1$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$. Then
$$|I_{\alpha,\alpha}(\lambda)| \leq M_k\lambda^{-1/k}, \quad \lambda \geq 1, \qquad (5)$$
for $k=1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ.

Main results for $\mathcal{I}_{\alpha,\beta}$

van der Corput lemmas on $\mathbb{R}$: consider $\mathcal{I}_{\alpha,\beta}$ defined by (3).
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha \leq 2$, $\beta > 1$, and $m = \operatorname{ess\,inf}_{x\in\mathbb{R}} |\phi(x)| > 0$, then
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq \frac{M}{1+\lambda m}\|\psi\|_{L^1(\mathbb{R})}, \ \lambda \geq 1, \ 0 < \alpha < 2, \ \beta \geq \alpha+1,$$
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq \frac{M}{(1+\lambda m)^{\frac{\beta-1}{\alpha}}}\|\psi\|_{L^1(\mathbb{R})}, \ \lambda \geq 1, \ 0 < \alpha \leq 2, \ 1 < \beta < \alpha+1,$$
where M does not depend on ϕ, ψ and λ.

van der Corput lemmas on $I = [a,b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$.

- Let $0 < \alpha < 2$, $\beta > 1$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq M_k\lambda^{-\frac{1}{k}}\log^{\frac{1}{k}}(1+\lambda), \quad \lambda \geq 1, \ 0 < \alpha < 2, \ \beta \geq \alpha+1,$$
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq M_k\lambda^{-\frac{1}{k}}(1+\lambda)^{\frac{\alpha+1-\beta}{\alpha k}}, \quad \lambda \geq 1, \ 0 < \alpha < 2, \ 1 < \beta < \alpha+1,$$
where M_k does not depend on λ.
- Let $-\infty < a < b < +\infty$ and $I = [a,b] \subset \mathbb{R}$. Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$. Then
$$|\mathcal{I}_{\alpha,\alpha}(\lambda)| \leq M_k\lambda^{-1/k}, \quad \lambda \geq 1, \qquad (6)$$
for $k=1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ.

Remark

The case of $\alpha = 1$ in (5) and (6) corresponds to the classical van der Corput lemma (1).

Conclusion

The main goal of the paper was to study van der Corput lemmas for the integrals defined by (2) and (3). Van der Corput type lemmas were obtained, for the different cases of parameters α and β.

As an immediate application of the obtained results, time-estimates of the solutions of time-fractional Klein-Gordon and Schrödinger equations and generalisations of the Riemann-Lebesgue lemma were also considered in [2] and [3].

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Gelfand triples for the Kohn-Nirenberg quantization on a homogeneous Lie group

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Abstract

A Gelfand triple is a triple of spaces (E, H, E') such that H is a Hilbert space and there are continuous dense embeddings $E \hookrightarrow H \hookrightarrow E'$. A classical example is $(\mathcal{S}(G), L^2(G), \mathcal{S}'(G))$ in which $\mathcal{S}(G)$ is the space of rapidly decreasing functions on a nilpotent Lie group G. But characterizing the image of $\mathcal{S}(G)$ under the group Fourier transform $\mathcal{F}_G$ is difficult and unwieldy. For this purpose we define and study new Gelfand triples for the Fourier transform and the Kohn-Nirenberg quantization.
The new Gelfand triples contain distribution spaces for which we may define multiplication on the Fourier side for a large class of smooth operator valued functions.
In this regime we may also show that the formula $a = \rho^* \cdot (A \otimes 1)(\rho)$ for the Kohn-Nirenberg symbol a holds for any operator $A \in \mathcal{L}(\mathcal{O}_M(G))$.

Gelfand triples for the Fourier transform

Homogeneous Lie groups

Let G be a homogeneous Lie group, i.e. G is a nilpotent Lie group together with an invertible derivation A on the Lie algebra $\mathfrak{g}$ with only positive eigenvalues. The group of automorphisms

$$\mathbb{R}^+ \to \mathrm{Hom}(G) : \lambda \mapsto \delta_\lambda \quad \text{where} \quad \delta_\lambda x = e^{A \log \lambda} \log_G(x)$$

are the dilations on G. The trace $Q = \operatorname{tr}(A)$ is the *homogeneous dimension* of G. Let $\widehat{G}$ be the dual of G, i.e. the set of unitary equivalence classes of irreducible unitary representations. Since G is nilpotent we may use Kirillov's correspondence between elements of $\widehat{G}$ and coadjoint orbits.

The group Fourier transform for $\dim Z(G) = 1$ and $\mathrm{SI}/\mathrm{Z} \neq \emptyset$

Let π be a fixed irreducible unitary representation that is square integrable modulo the center (in symbols $\pi \in \mathrm{SI}/\mathrm{Z}$). Furthermore we choose a real structure C_π on $(H_\pi^\infty, H_\pi, H_\pi^{-\infty})$, where H_π is the representation space of π, H_π^∞ is the space of smooth vectors of π and $H_\pi^{-\infty} = (H_\pi^\infty)'$. We will write

$$\pi_\lambda(x) := \pi(\delta_\lambda x),\ \lambda > 0 \quad \text{and} \quad \pi_\lambda(x) := C_\pi \pi_{-\lambda}(x) C_\pi,\ \lambda \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}.$$

Using this definition, we may define the following isomorphism of measure spaces

$$(\mathbb{R}^\times, c_\pi |\lambda|^{Q-1} \mathrm{d}\lambda) \to (\widehat{G}_{gen}, \widehat{\mu}) : \lambda \mapsto [\pi_\lambda], \tag{1}$$

in which $\widehat{G}_{gen} \subset \widehat{G}$ are the equivalence classes corresponding to generic orbits and $\widehat{\mu}$ is the Plancherel measure. Since $\widehat{\mu}(\widehat{G} \setminus \widehat{G}_{gen}) = 0$, we may use (1) and Pedersen's machinery [3] to construct a unitary isomorphism

$$j_\pi : L^2(\mathbb{R}^\times, c_\pi |\lambda|^Q \mathrm{d}\lambda) \widehat{\otimes}_{\mathrm{HS}} \mathcal{HS}(H_\pi) \to L^2(\widehat{G}) = \mathcal{F}_G L^2(G).$$

We will use j_π to define an adjusted Fourier transform as described in Figure 1.

Defining new Gelfand triples

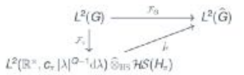

Figure 1: Commutative diagram describing $\mathcal{F}_\pi$

But it is rather inconvenient that the Fourier image of the space of Schwartz functions $\mathcal{S}(G)$ does not decompose into a tensor product of a functions space and an operator space. In order to bypass the problematic behaviour of $\mathcal{F}_\pi \varphi(\lambda)$ in $\lambda \to 0$ for $\varphi \in \mathcal{S}(G)$, we will use operator valued functions that vanish in $\lambda = 0$. Denote by $\mathcal{S}_*(\mathbb{R})$ the space of Schwartz functions on $\mathbb{R}$ that are orthogonal to all polynomials. We will also denote by $\mathcal{S}(\mathbb{R}^\times)$ the set of Schwartz functions on $\mathbb{R}^\times$ that vanish of arbitrarily high order in $\lambda \to 0$. We split the Lie algebra into vector subspaces $\mathfrak{g} = Z(\mathfrak{g}) \oplus \omega$ and define

$$\mathcal{S}_*(G) := \{\varphi \in \mathcal{S}(G) \mid [(\lambda, x) \mapsto \varphi \circ \exp_G(\lambda z + x)] \in \mathcal{S}_*(\mathbb{R}) \widehat{\otimes}_\varepsilon \mathcal{S}(\omega)\}$$

for any $z \in Z(\mathfrak{g}) \setminus \{0\}$. Using this definition we may formulate the following theorem.

Theorem
The map $\mathcal{F}_\pi$ is an isomorphism in each row of

$$\mathcal{F}_\pi : \begin{pmatrix} \mathcal{S}_*(G) \\ L^2(G) \\ \mathcal{S}'_*(G) \end{pmatrix} \to \begin{pmatrix} \mathcal{S}(\mathbb{R}^\times) \widehat{\otimes}_\varepsilon \mathcal{L}(H_\pi^{-\infty}, H_\pi^\infty) \\ L^2(\mathbb{R}^\times, c_\pi |\lambda|^{Q-1} \mathrm{d}\lambda) \widehat{\otimes}_{\mathrm{HS}} \mathcal{HS}(H_\pi) \\ \mathcal{S}'(\mathbb{R}^\times) \widehat{\otimes}_\varepsilon \mathcal{L}(H_\pi^\infty, H_\pi^{-\infty}) \end{pmatrix} =: \begin{pmatrix} \mathcal{S}(\mathbb{R}^\times; \pi) \\ L^2(\mathbb{R}^\times; \pi) \\ \mathcal{S}'(\mathbb{R}^\times; \pi) \end{pmatrix}.$$

Multiplication operators on the Fourier side

Due to [1] we may identify a large class of operator valued functions that act as multiplication operators on $\mathcal{S}(\mathbb{R}^\times; \pi)$ and $\mathcal{S}'(\mathbb{R}^\times; \pi)$. In the following $\mathcal{O}_M(\cdot)$ denotes the space of multiplication operators on $\mathcal{S}(\cdot)$.

Theorem
The multiplication $(f, \sigma) \mapsto f \cdot \sigma$, defined by pointwise composition, is a hypocontinuous bilinear map

$$\mathcal{O}_M(\mathbb{R}^\times) \widehat{\otimes}_\varepsilon \mathcal{L}(H_\pi^\infty) \times \mathcal{S}(\mathbb{R}^\times; \pi) \to \mathcal{S}(\mathbb{R}^\times; \pi); \tag{2}$$

and by transposition there is a hypocontinuous multiplication

$$\mathcal{O}_M(\mathbb{R}^\times) \widehat{\otimes}_\varepsilon \mathcal{L}(H_\pi^{-\infty}) \times \mathcal{S}'(\mathbb{R}^\times; \pi) \to \mathcal{S}'(\mathbb{R}^\times; \pi). \tag{3}$$

Gelfand triples for the Kohn-Nirenberg quantization

We may see the Kohn-Nirenberg quantization as the map

$$\mathrm{Op} : L^2(G) \widehat{\otimes}_{\mathrm{HS}} L^2(\widehat{G}) \to \mathcal{HS}(L^2(G)), \quad \mathrm{Op}(a) := \mathcal{K}^{-1} \mathcal{T}^{-1} (1 \otimes \mathcal{F}_G^{-1}), \tag{4}$$

in which $\mathcal{T} f(x, y) = f(x, xy^{-1})$ and $\mathcal{K} : \mathcal{HS}(L^2(G)) \to L^2(G \times G)$ is the canonical isomorphism with $\langle A\varphi, \psi \rangle = \langle \mathcal{K}(A), \psi \otimes \varphi \rangle$. Let us call Op_π the map we obtain by substituting $\mathcal{F}_G$ by $\mathcal{F}_\pi$. Similar to Figure 1 we get the commutative diagram

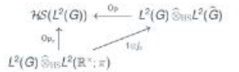

Figure 2: Commutative diagram describing Op_π

As before, we want to construct Gelfand triples such that Op_π becomes an easy to describe isomorphism between Gelfand triples. For this purpose it is enough to realize, that $\mathcal{T}$ restricts to an isomorphism on $\mathcal{S}(G) \widehat{\otimes}_\varepsilon \mathcal{S}_*(G)$ such that $\langle \mathcal{T} f, g \rangle = \langle f, \mathcal{T}^{-1} g \rangle$. By (4) we may formulate the next theorem.

Theorem
The Kohn-Nirenberg quantization, Op_π, is an isomorphism in each row of

$$\mathrm{Op}_\pi : \begin{pmatrix} \mathcal{S}(G) \widehat{\otimes}_\varepsilon \mathcal{S}(\mathbb{R}^\times; \pi) \\ L^2(G) \widehat{\otimes}_{\mathrm{HS}} L^2(\mathbb{R}^\times; \pi) \\ \mathcal{S}'(G) \widehat{\otimes}_\varepsilon \mathcal{S}'(\mathbb{R}^\times; \pi) \end{pmatrix} \to \begin{pmatrix} \mathcal{L}(\mathcal{S}'_*(G), \mathcal{S}(G)) \\ \mathcal{HS}(L^2(G)) \\ \mathcal{L}(\mathcal{S}_*(G), \mathcal{S}'(G)) \end{pmatrix}.$$

On the integral formula for Kohn-Nirenberg operators

We may realize that $\rho : (x, \lambda) \to \pi(x, \lambda)$ is an operator valued function in $\mathcal{O}_M(\mathbb{R}^\times \times G) \widehat{\otimes}_\varepsilon \mathcal{L}(H_\pi^\infty)$. By using the continuity and invertibility of Op_π, the continuous dense embedding $\mathcal{S}_*(G) \hookrightarrow \mathcal{O}_M(G)$ and the hypocontinuous multiplication on $\mathcal{O}_M(G \times \mathbb{R}^\times) \widehat{\otimes} \mathcal{L}(H_\pi^\infty)$, we are able to prove the following theorem.

Theorem
For any $A \in \mathcal{L}(\mathcal{O}_M(G), E)$, $E \in \{\mathcal{S}(G), \mathcal{O}_M(G)\}$, the equality $a := \rho^* \cdot (A \otimes 1)(\rho) = \mathrm{Op}_\pi^{-1}(A)$ is valid and we may evaluate A by

$$A\varphi = \int_{\mathbb{R}^\times} \operatorname{tr}[\pi_\lambda a(\cdot, \lambda) \mathcal{F}_\pi \varphi(\lambda)]\, c_\pi |\lambda|^{Q-1} \mathrm{d}\lambda, \quad \text{for} \quad \varphi \in \mathcal{S}(G),$$

where the integral exists weakly in E. Furthermore

$$\mathrm{Op}_\pi^{-1} : \mathcal{L}(\mathcal{O}_M(G)) \to \mathcal{O}_M(G \times \mathbb{R}^\times) \widehat{\otimes}_\varepsilon \mathcal{L}(H_\pi^\infty)$$

is continuous.

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Noncommutative Fourier Series on $\Gamma\backslash SE(2)$

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Pseudo-differential conference

Abstract

This work begins with a systematic study of abstract noncommutative Fourier series on $\Gamma\backslash SE(2)$, where Γ is a discrete and co-compact subgroup of $SE(2)$, the group of handedness-preserving Euclidean isometries of the Euclidean plane.

Introduction

The 2D special Euclidean group, usually denoted as $SE(2)$, is one of the simplest examples of a noncommutative, noncompact real finite dimensional Lie group. The right coset space of discrete and co-compact subgroups in $SE(2)$, such as $\mathbb{Z}^2\backslash SE(2)$, appears as the configuration space in many recent applications in computational science and engineering including computer vision, robotics, mathematical crystallography, computational biology, and material science [2, 3, 4].

Preliminaries and Notation

The 2D special Euclidean group, $SE(2)$, is the semidirect product of $\mathbb{R}^2$ with the 2D special orthogonal group $SO(2)$. We denote elements $g \in SE(2)$ as $g = (\mathbf{x}, \mathbf{R})$ where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{R} \in SO(2)$.
The classical Fourier Plancherel formula on the group $SE(2)$ is

$$\int_{SE(2)} |f(g)|^2 dg = \int_0^\infty \|\hat{f}(p)\|_{\mathrm{HS}}^2 p\,dp,$$

and also the noncommutative Fourier reconstruction formula is

$$f(g) = \int_0^\infty \mathrm{tr}\left[\hat{f}(p) U_p(g)\right] p\,dp, \tag{1}$$

for $f \in L^1 \cap L^2(SE(2))$ and $g \in SE(2)$, where for each $p > 0$, the irreducible representation $U_p : SE(2) \to \mathcal{U}(L^2(\mathbb{S}^1))$ is

$$[U_p(g)\varphi](\mathbf{u}) := e^{-ip\langle \mathbf{u}, \mathbf{t}\rangle} \varphi(\mathbf{R}^T \mathbf{u}), \tag{2}$$

for all $g = (\mathbf{t}, \mathbf{R}) \in SE(2)$, $\varphi \in L^2(\mathbb{S}^1)$, $\mathbf{u} \in \mathbb{S}^1$, and the Fourier transform of each $f \in L^1(SE(2))$ at $p > 0$ is given by

$$\hat{f}(p) := \int_{SE(2)} f(g) U_p(g^{-1}) dg. \tag{3}$$

The standard orthonormal basis for the Hilbert function space $L^2(\mathbb{S}^1)$ is $\mathcal{B} := \{\mathbf{e}_k : k \in \mathbb{Z}\}$, where for each $k \in \mathbb{Z}$, $\mathbf{e}_k : \mathbb{S}^1 \to \mathbb{C}$ is given by

$$\mathbf{e}_k(\mathbf{u}_\alpha) = \mathbf{e}_k(\cos\alpha, \sin\alpha) := e^{ik\alpha}.$$

For each $g \in SE(2)$ and $p > 0$, the matrix elements of the operator $U_p(g)$ with respect to the basis $\mathcal{B}$, are expressed as

$$\mathrm{u}_{mn}(g, p) = \langle U_p(g)\mathbf{e}_m, \mathbf{e}_n\rangle_{L^2(\mathbb{S}^1)}, \tag{4}$$

for each $m, n \in \mathbb{Z}$, see [1].

State of the problem

- Let Γ be a discrete co-compact subgroup of $SE(2)$.
- Suppose μ is the finite $SE(2)$-invariant measure on the right coset space $\Gamma\backslash SE(2)$ which is normalized with respect to Weil's formula given by

$$\int_{\Gamma\backslash SE(2)} \tilde{f}(\Gamma g) d\mu(\Gamma g) = \int_{SE(2)} f(g) dg, \tag{5}$$

for $f \in L^1(SE(2))$, where

$$\tilde{f}(\Gamma g) := \sum_{\gamma\in\Gamma} f(\gamma \circ g), \tag{6}$$

for $g \in SE(2)$.
- Let $\varphi = \tilde{f} \in L^1(\Gamma\backslash SE(2), \mu)$ with $f \in L^1(SE(2))$ be given.
- How to expand φ using coefficients of Fourier matrix elements of f on $SE(2)$ given by (3)?

Theorem (GF-Chirikjian)

Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$. Let $f \in L^1 \cap L^2(SE(2))$ such that $\tilde{f} \in L^2(\Gamma\backslash SE(2), \mu)$. We then have

$$\langle \tilde{f}, \psi_\ell\rangle = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \int_0^\infty \hat{f}(p)_{nm} Q^\ell(p)_{mn} p\,dp, \tag{7}$$

where

$$Q^\ell(p)_{mn} := \int_{SE(2)} \mathrm{u}_{mn}(g, p) \overline{\psi_\ell(\Gamma g)} dg,$$

and

$$\hat{f}(p)_{nm} = \langle \hat{f}(p)\mathbf{e}_n, \mathbf{e}_m\rangle,$$

for $p > 0$, $\ell \in \mathbb{I}$, and $m, n \in \mathbb{Z}$.

Proposition (GF-Chirikjian)

Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$. Let $f \in L^1(SE(2))$ such that $\widetilde{|f|} \in L^2(\Gamma\backslash SE(2), \mu)$. We then have

$$\langle \tilde{f}, \psi_\ell\rangle = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \int_0^\infty \hat{f}(p)_{nm} Q^\ell(p)_{mn} p\,dp, \tag{8}$$

Let $f \in L^1(SE(2))$ and $\psi \in L^1(\Gamma\backslash SE(2), \mu)$. We then define the convolution of f with ψ as the function $\psi \oslash f : \Gamma\backslash SE(2) \to \mathbb{C}$ via

$$(\psi \oslash f)(\Gamma g) := \int_{SE(2)} \psi(\Gamma h) f(h^{-1} \circ g) dh, \tag{9}$$

for $g \in SE(2)$.

Theorem (GF-Chirikjian)

Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$. Let $f_k \in L^1(SE(2))$ with $k \in \{1, 2\}$ such that $\widetilde{|f_1|} \in L^2(\Gamma\backslash SE(2), \mu)$. We then have

$$\langle \widetilde{f_1} \oslash f_2, \psi_\ell\rangle = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \int_0^\infty \widehat{f_1}(p)_{nk} \widehat{f_2}(p)_{km} Q^\ell(p)_{mn} p\,dp. \tag{10}$$

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PROPAGATION OF SINGULARITIES ON HYPO-ANALYTIC STRUCTURES

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GEVREY VECTORS

Let Ω be a smooth manifold of dimension $m > 0$ and consider $\{L_1, \dots, L_n\}$, pairwise commuting complex vector fields defined on Ω, where $n \leq m$. Let $s \geq 1$, we say that a distribution $u \in \mathcal{D}'(\Omega)$ is a **Gevrey-s vector** for $\{L_1, \dots, L_n\}$ if $u \in \mathcal{C}^\infty(\Omega)$ and for every $K \subset \Omega$, compact set, there is a positive constant $C = C_K$ such that

$$\sup_{x \in K} |L^\alpha u(x)| \leq C^{|\alpha|+1} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^m,$$

and we write $u \in G^s(\Omega; L_1, \dots, L_n)$. If $s = 1$ we write $u \in \mathcal{C}^\omega(\Omega; L_1, \dots, L_n)$, and we say that u is an **analytic vector** for $\{L_1, \dots, L_n\}$.

HYPO-ANALYTIC STRUCTURES

Let Ω be a smooth manifold of dimension $n + m$. A hypo-analytic structure on Ω of corank m is a family $\{U_\alpha, (Z_{\alpha,1}, \cdots, Z_{\alpha,m})\}_{\alpha \in \Lambda}$ satisfying the following conditions:

- $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open covering of Ω;
- $Z_{\alpha,j} : U_\alpha \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function for every $j = 1, \cdots, m$ and $\alpha \in \Lambda$;
- $dZ_{\alpha,1}, \cdots, dZ_{\alpha,m}$ are $\mathbb{C}$-linearly independent at each point of U_α, for every $\alpha \in \Lambda$
- If $U_\alpha \cap U_\beta \neq \emptyset$ then for every $p \in U_\alpha \cap U_\beta$ there exists a biholomorphic map $F^p_{\alpha,\beta}$ such that $Z_\beta = F^p_{\alpha,\beta} \circ Z_\alpha$ in some neighborhood of p in $U_\alpha \cap U_\beta$.

We call a pair (U, Z) a hypo-analytic chart. We can associate it a locally integrable structure $\mathcal{V}$ setting its orthogonal, T', as the complex bundle defined, locally, by the differentials $dZ_1, \dots, dZ_m$.

EXAMPLE

Let $M \subset \mathbb{C}^N$ be a **generic CR** submanifold of codimension d, so the CR dimension is equal to $N - d$. We can associate a hypo-analytic structure on M setting $Z = z|_M$, for every holomorphic coordinate system (U, z). In this case the locally integrable structure $\mathcal{V}$ is equal to $T^{0,1}M$, *i.e.*, the collection of all anti-holomorphic vector fields that are tangent to M.

THE LOCAL SET UP AND THE REAL STRUCTURE BUNDLE

For simplicity assume that $0 \in \Omega$. There is a hypo-analytic chart $(U, Z_1(x,t), \dots, Z_m(x,t))$ centered at 0 with $U = V \times W$, where $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$ are open balls centered at the origin, and the map $Z = (Z_1, \dots, z_m)$ is given by

$$Z_j(x,t) = x_j + i\phi_j(x,t), \qquad j = 1, \dots, m,$$

with $(x,t) \in V \times W$, where the map $\phi(x,t) = (\phi_1(x,t), \dots, \phi_m(x,t))$ is smooth, real-valued, $\phi(0) = 0$ and $d_x\phi(0) = 0$. We associate the complex vector fields $\{M_1, \dots, M_m, L_1, \dots, L_n\}$ with the following properties:

$$\begin{matrix} L_j Z_k = 0 & M_j Z_k = \delta_{j,k} \\ L_j t_k = \delta_{j,k} & M_j t_k = 0. \end{matrix}$$

For every $t \in W$ the set $\Sigma_t = \{Z(x,t) \; : \; x \in V\}$ is a maximally real submanifold of $\mathbb{C}^m$. The **real structure bundle** of Σ_t, $\mathbb{R}T'_{\Sigma_t}$, is given by

$$\mathbb{R}T'_{\Sigma_t} = \{(Z(x,t), {}^tZ_x(x,t)^{-1}\xi) \; : \; x \in V, \xi \in \mathbb{R}^m\}.$$

AN FBI CHARACTERIZATION OF GEVREY VECTORS

Let $\mathcal{H} \subset \mathbb{C}^m$ be a maximally real submanifold and $u \in \mathcal{E}'(\mathcal{H})$. We define the FBI transform of u by

$$\mathfrak{F}[u](z,\zeta) \doteq \left\langle u(z'), e^{i\zeta\cdot(z-z') - \langle\zeta\rangle(z-z')^2} \Delta(z - z', \zeta) \right\rangle,$$

for $z \in \mathbb{C}^m$ and $\zeta \in \mathfrak{C}_1 = \{\eta \in \mathbb{C}^m \; : \; |\mathrm{Im}\,\eta| < |\mathrm{Re}\,\eta|\}$, where $\Delta(z,\zeta) = \det(\mathrm{Id} + i(z \odot \zeta)/\langle\zeta\rangle)$. Now if u is a distribution on U (the same as before) such that $L_j u \in G^s(U; L_1, \dots, L_n, M_1, \dots, M_m)$, then actually $u \in \mathcal{C}^\infty(W; \mathcal{D}'(W))$. So we define the FBI transform of $u \in \mathcal{C}^\infty(W; \mathcal{E}'(V))$ by

$$\mathfrak{F}[u](t; z, \zeta) = \mathfrak{F}[u_t^*](z, \zeta),$$

where u_t^* is the compactly supported distribution on Σ_t defined by $u_t^*(z) = u(x,t)$, for $z = Z(x,t)$.

Theorem 1 (N. Braun Rodrigues, 2020) *Let $u \in \mathcal{C}^\infty(W; \mathcal{D}'(V))$ be a solution of*

$$\begin{cases} L_1 u = f_1, \\ \quad \vdots \\ L_n u = f_n, \end{cases}$$

where $f_j \in G^s(U; L_1, \cdots, L_n, M_1, \dots, M_m)$, $j = 1, \dots, n$. Then are equivalent:

1. *$u|_{U_0} \in G^s(U; L_1, \cdots, L_n, M_1, \dots, M_m)$, for some open neighborhood of the origin U_0;*
2. *For every $\chi \in \mathcal{C}_c^\infty(V)$, with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in some open neighborhood of the origin, there exist $\widetilde{V} \subset V$, $\widetilde{W} \subset W$, open balls centered at the origin, constants $C, \epsilon > 0$ such that*

$$|\mathfrak{F}[\chi u](t; z, \zeta)| \leq C e^{-\epsilon|\zeta|^{\frac{1}{s}}}, \qquad \forall t \in \widetilde{W}, (z,\zeta) \in \mathbb{R}T'_{\Sigma_t}|_{\widetilde{V}} \setminus 0,$$

where $(z,\zeta) \in \mathbb{R}T'_{\Sigma_t}|_{\widetilde{V}}$ means that $z = Z(x,t)$, $\zeta = {}^tZ_x(x,t)^{-1}\xi$, $\xi \in \mathbb{R}^m \setminus 0$ and $x \in \widetilde{V}$.

ACKNOWLEDGEMENTS

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"PROPAGATORS"

We shall consider only analytic tube structures, *i.e.*, locally the hypo-analytic structure is given by $Z(x,t) = x + i\phi(t)$, defined on $U = V \times W$, and $\phi(t)$ is analytic. One of the reasons we are only dealing with tube structures is that the real structure bundle is trivial, *i.e.*,

$$\mathbb{R}T'_{\Sigma_t} = \{(Z(x,t), \xi) \; : \; x \in V, \xi \in \mathbb{R}^m\},$$

for every $t \in W$. Let $\Sigma \subset \Omega$ be a connected submanifold of Ω, satisfying the following properties:

1. For every $p \in \Sigma$ there is (U, Z), a hypo-analytic chart, with $p \in U$, such that $\Sigma \cap U \subset Z^{-1}(0)$;
2. In the same situation as above, for every $q \in \Sigma \cap U$, and $\widetilde{U}_1 \Subset U$, an open neighborhood of p, there is $\widetilde{U}_2 \Subset U$, an open neighborhood of q, such that the connected component of the fiber $Z^{-1}(Z(q'))$ that contains q' intersects $\widetilde{U}_1$, for every $q' \in \widetilde{U}_2$;
3. the map $\Sigma \ni p \mapsto \sup\{r > 0 \; : \; B_r(p) \subset U\}$ is continuous.

PROPAGATION OF SINGULARITIES

Let $u \in \mathcal{D}'(\Omega)$. We say that $\mathbb{L}u \in G^s$ if for every (U, Z) described in the local setup section we have that

$$L_j u \in G^s(U; L_1, \cdots, L_n, M_1, \dots, M_m),$$

for $j = 1, \dots, n$. Note that since the structure is real-analytic, this is equivalent to $L_j u \in G^s(U)$.

Theorem 2 (N. Braun Rodrigues, 2020)
In the same situation as above, if $u \in \mathcal{D}'(\Omega)$ is such that $\mathbb{L}u \in G^s(\Omega)$, then $\mathrm{singsupp}_s\, u \cap \Sigma = \emptyset$ or $\Sigma \subset \mathrm{singsupp}_s\, u$.

REFERENCES

[1] Nicholas Braun Rodrigues. An FBI characterization for Gevrey vectors on hypo-analytic structures and propagation of Gevrey singularities. *arXiv:2006.04590*, 2020.

Feature Extraction for License Plate Location Based on L_0-norm Smoothing

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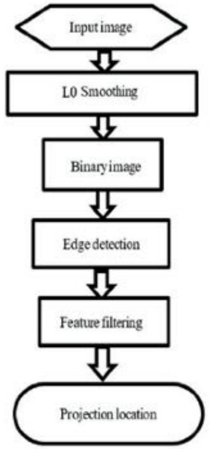
Overview

Abstact: The license plate location plays an important role in license plate recognition systems. As a premise of character recognition, the accuracy and robustness of license location directly determine the performance of the entire license plate recognition system. In many practical applications, license plate recognition systems mostly work outdoors, and the captured images inevitably suffer from different kinds of degradations caused by lighting, weather, and complex backgrounds, and so on. In general, it is a challenging problem under such a diverse, uncertain and complex environment and has also attracted considerable attention in both academic and industrial fields.

Main Idea: In regard to the complex background in license images, we first use an edge-aware filter, L^0-norm smoothing to remove the majority background textures but keep the license plate characters. Then, we take a series of feature filtering steps based on the geometrical textures and structures to furtherly to reduce the interference of pseudo-licenses. Finally, a simple projection location method is used to extract the position and size of the license plates. The whole location procedure is:



Contributions:

- Propose a practical license plate location system based on a series of feature extraction and filter steps.
- Use L_0 image smoothing algorithm to remove the background noise.
- Use a binarized image for fast multiscale resolution analysis.
- Take full use of textural information and projection location method to extract license plates.

Algorithm

L^0 Smoothing: In license plate images, license characters have high contrast textures, while the image background contains abundant low contrast details. We use the L_0-norm smoothing filters to suppress the details, which can be phased as,

$$\arg\min_f E(f) = \|f-g\|_2^2 + \lambda\|\nabla f\|_0, \qquad (1)$$

where g and f are the source and target images in $\mathbb{R}^N$, ∇ is gradient operator, $\|\cdot\|_2$ is L^2-norm, and $\|\cdot\|_0$ is so-called L^0-norm, counting the number of non-zero elements of an vector, which leads to a sparse regularization, and λ is a weight scalar.

L^0-norm minimization: The Eq.(1) is NP-hard to solve, we instead introduce two auxiliary variables h_i and v_i for the 2D discrete image, and rewrite it as,

$$\min_{f,h,v} E(f) = \sum_i (f_i - g_i)^2 + \lambda C(h,v) + \beta((\partial_x f_i - h_i)^2 + (\partial_y f_i - v_i)^2), \qquad (2)$$

where $C(h,v) = 1$ if $|h_i| + |v_i| \neq 0$, else 0. It is clear that the solution of Eq. (2) approximates that of Eq. (1) when $\beta \to +\infty$. We solve the Eq. (2) minimizing (h,v) and f alternatively.

1. Computing f: By fixing h_i and v_i and minimizing the function,

$$\min_f E(f) = \sum_i (f_i - g_i)^2 + \beta((\partial_x f_i - h_i)^2 + (\partial_y f_i - v_i)^2). \qquad (3)$$

The above function is quadratic and thus has a global minimum. Here we use fast Fourier transform (FFT) to accelerate the solver,

$$f = \mathscr{F}^{-1}\left\{\frac{\mathscr{F}(g) + \beta(\mathscr{F}(\partial_x) * \mathscr{F}(h) + \mathscr{F}(\partial_y) * \mathscr{F}(v))}{\mathscr{F}(1) + \beta(\mathscr{F}(\partial_x) * \mathscr{F}(\partial_x) + \mathscr{F}(\partial_y) * \mathscr{F}(\partial_y))}\right\}, \qquad (4)$$

where $*$ is a component-wise multiplication, $\mathscr{F}$ and $\mathscr{F}^{-1}$ denotes FFT and its inverse operators, and $\mathscr{F}(1)$ is the Fourier Transform of δ function.

2. computing (h,v): We solve (h,v) by minimizing

$$\min_{h,v} E(f) = \lambda C(h,v) + \beta((\partial_x f_i - h_i)^2 + (\partial_y f_i - v_i)^2), \qquad (5)$$

where $C(h,v)$ returns the number of non-zero elements of $|\partial_x f_i| + |\partial_y f_i|$. Eq. (5) can be spatially decomposed and solved fastly, because each element h_p and v_p can be estimated individually. It reaches its minimum E^* under the condition,

$$h_i, v_i = \begin{cases} (0,0), (\partial_y f_i + \partial_y f_i)^2 \leq \frac{\lambda}{\beta}, \\ 0, \text{ otherwise.} \end{cases} \qquad (6)$$

Post-processing: To extract and locate the license more accurately, we also introduce the following post-processing steps:

Binarized image: Let I be an image, and I_s is resized from I with a scale s, where $s \in S = \{s_{m,n}\} = \{2^{-m}, 2^{-n}\}_{m,n \in Z}$ are row and column scales. If we rearrange the images $\{I_s\}_{s \in S}$ on a 2-D plane in descending scales, a new image $B = \{I_s\}_{s \in S}$, called binarized image, is obtained, which provides a multiscale analysis of license image.

Feature Filtering: By using the L^0-norm smoothing, a mass of local textures can be removed and an optimal scale of the license is also available with binarized image. We further propose a feature filtering, that is, edge extraction, removing long lines, structure analysis, and texture density analysis to further remove the pseudo-licenses.

Projection Location: After the above steps, the license is easy to locate, we use a very simple projection location method to extract the position and size information.

Experiments & Results

1. L^0-norm smoothing

- Left: input image, right: L^0-norm smoothing result. It is clear that the low-contrast details are significantly suppressed with L^0-norm smoothing.

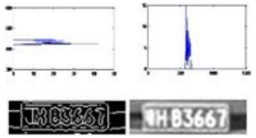
2. Binarized image

- The binarized image provides a multiscale analysis for licenses.

3. Feature filtering

- Edge extraction, removing lines, structural and textural analysis.

4. Projection results



- The horizontal and vertical projection histograms, and the output license plate.

Quantitative evaluation:

Method	Precision(%)	Recall (%)	F_1-score(%)
Top-hat	77.50	84.50	88.61
MSER&SIFT	83.73	90.47	86.97
Wavelet-based	90.30	94.03	95.00
CNN-based	97.80	95.30	96.01
Ours	96.10	95.40	95.30

- Quantitative evaluation on natural image datasets. We collected 1000 images with both simple and complex backgrounds.

References

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Spectral Geometry of Partial Differential Operators

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GHENT ANALYSIS PDE

Queen Mary University of London

Description

The aim of Spectral Geometry of Partial Differential Operators is to provide a basic and self-contained introduction to the ideas underpinning spectral geometric inequalities arising in the theory of partial differential equations. Historically, one of the first inequalities of the spectral geometry was the minimisation problem of the first eigenvalue of the Dirichlet Laplacian. Nowadays, this type of inequalities of spectral geometry have expanded to many other cases with numerous applications in physics and other sciences. The main reason why the results are useful, beyond the intrinsic interest of geometric extremum problems, is that they produce a priori bounds for spectral invariants of (partial differential) operators on arbitrary domains.

Content

Spectral geometry

Let us consider **Riesz potential operators**

$$(\mathcal{R}_{\alpha,\Omega} f)(x) := \int_\Omega |x-y|^{\alpha-d} f(y)dy, \;\; f \in L^2(\Omega), \;\; 0 < \alpha < d, \qquad (1)$$

where $\Omega \subset \mathbb{R}^d$ is a set with finite Lebesgue measure.

Rayleigh-Faber-Krahn inequality: The ball Ω^* is the maximiser of the first eigenvalue of the operator $\mathcal{R}_{\alpha,\Omega}$ among all domains of a given volume, i.e.

$$0 < \lambda_1(\Omega) \leq \lambda_1(\Omega^*)$$

for an arbitrary domain $\Omega \subset \mathbb{R}^d$ with $|\Omega| = |\Omega^*|$.

Hong-Krahn-Szegö inequity: The maximum of the second eigenvalue $\lambda_2(\Omega)$ of $\mathcal{R}_{\alpha,\Omega}$ among all sets $\Omega \subset \mathbb{R}^d$ with a given measure is approached by the union of two identical balls with mutual distance going to infinity.

Richard Feynman's drum

The picture from planksip.org

It is proved that the deepest bass note is produced by the circular drum among all drums of the same area (as the circular drum). Moreover, one can show that among all bodies of a given volume in the three-dimensional space with constant density, the ball has the gravitational field of the highest energy.

Rex is minimising the strain energy by circling

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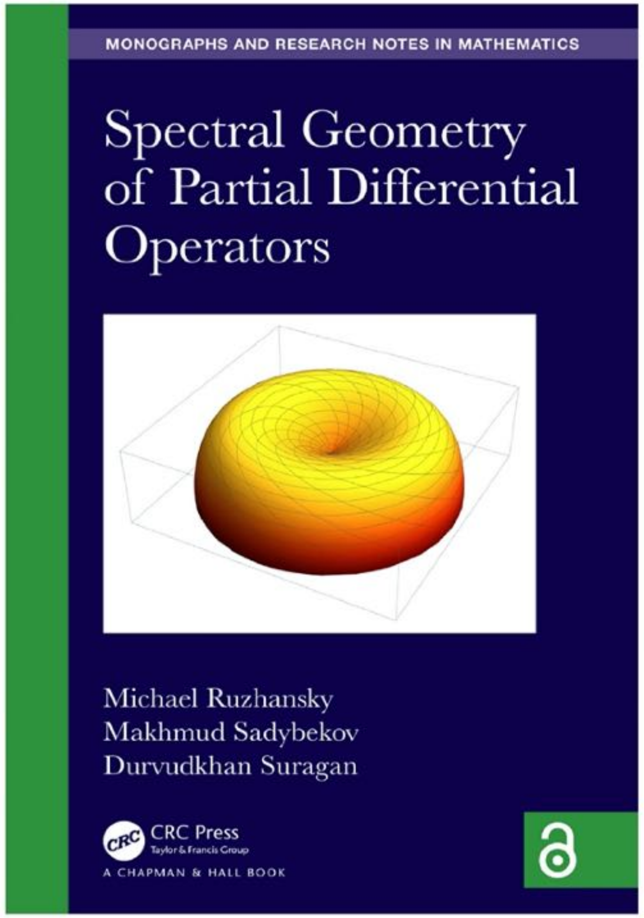

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On a class of anharmonic oscillators

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Abstact

We study a class of anharmonic oscillators within the framework of the Weyl-Hörmander calculus. By associating a Hörmander metric to a given anharmonic oscillator we extend the so-called *Shubin classes* associated to the harmonic oscillator and the corresponding pseudo-differential calculus. Spectral properties of negative powers of anharmonic oscillators, as well as of the operator itself, are derived.

Introduction

In the study of the **Schrödinger equation** $i\partial_t\psi = -\Delta\psi + V(X)\psi$ the analysis of the energy levels is often reduced to the corresponding eigenvalue problem for the operator $-\Delta + V(x)$.
Spectral properties of the anharmonic oscillator, on $\mathbb{R}$ or more generally on $\mathbb{R}^n$, with different potentials V have been studied (c.f. [1],[2],[4]) by several authors in the last 40 years. However, the exact solution of the eigenvalue problem is still unknown.
Here we consider a more general case on $\mathbb{R}^n$ where a **prototype** is of the form

$$\mathcal{A} = (-\Delta)^l + |x|^{2k}, \quad \text{where} \quad k,l \geq 1 \quad \text{integers}, \tag{1}$$

and **more generally** we consider operators of the form

$$T = q(D) + p(x), \tag{2}$$

where p, q are special polynomials on $\mathbb{R}^n$. In particular we write $p \in \mathcal{P}_{2k}, q \in \mathcal{P}_{2l}$, for some integers $k,l \geq 1$, where we have defined

$$\mathcal{P}_{2k} = \left\{ p : \mathbb{R}^n \to \mathbb{R}, \text{with } \deg(p) = 2k, \text{and } \lim \inf_{|x|\to\infty} \frac{p(x)}{|x|^{2k}} > 0 \right\}.$$

Therefore, for $p \in \mathcal{P}_{2k}$ (and similarly for q), there exists $p_0 > 0$ such that

$$p(x) + p_0 > 0, \quad \text{for every} \quad x \in \mathbb{R}^n.$$

Weyl-Hörmander classes associated to the anharmonic oscillators

- **In the case of the general anharmonic oscillator** T as in (2) with (rescalled) symbol τ we have

$$\tau(x,\xi) = p(x) + q(\xi) \in S(M^{p,q}, g^{p,q}),$$

where the Hörmander metric $g^{p,q}$, and the g-weight $M^{p,q}$ are given by

$$g^{p,q}_{x,\xi}(dx, d\xi) = \frac{dx^2}{\underbrace{(p_0 + q_0 + p(x) + q(\xi))^{\frac{1}{k}}}_{>0}} + \frac{d\xi^2}{(p_0 + q_0 + p(x) + q(\xi))^{\frac{1}{l}}}, \tag{3}$$

and

$$M^{p,q}(x,\xi) = p_0 + q_0 + p(x) + q(\xi) \tag{4}$$

- **In the case of the prototype of the anharmonic oscillator** $\mathcal{A}$ as in (1) the metric (3) is equivalent to the metric

$$g^{k,l}_{x,\xi}(dx, d\xi) = \frac{dx^2}{(1 + |x|^{2k} + |\xi|^{2l})^{\frac{1}{k}}} + \frac{d\xi^2}{(1 + |x|^{2k} + |\xi|^{2l})^{\frac{1}{l}}}. \tag{5}$$

- **In the symmetric case where** $k = l$ in (1) the metric associated to the operator $\mathcal{A}$ is equivalent to

$$g_{x,\xi}(dx, d\xi) = \frac{dx^2}{1 + |x|^2 + |\xi|^2} + \frac{d\xi^2}{1 + |x|^2 + |\xi|^2},$$

which corresponds to the symplectic metric defining the *Shubin classes* associated to the *harmonic oscillator*.

Associated symbol classes and operators

Let $m \in \mathbb{R}$. We say that the function $a \in C^\infty(\mathbb{R}^{2n})$ is in the **class of symbols** $\Sigma^m_{k,l}(\mathbb{R}^n)$, if

$$|\partial^\alpha_x \partial^\beta_\xi a(x,\xi)| \leq C_{\alpha,\beta} \Lambda(x,\xi)^{m - \frac{|\alpha|}{k} - \frac{|\beta|}{l}}, \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}^n,$$

where we have denoted $\Lambda(x,\xi) := (1 + |x|^{2k} + |\xi|^{2l})^{\frac{1}{2}}$, for $k,l \geq 1$ integers. The **associated operators** are denoted by $\Psi\Sigma^m_{k,l}(\mathbb{R}^n)$, and in particular

$$\Psi\Sigma^m_{k,l}(\mathbb{R}^n) = Op^w(\Sigma^m_{k,l}(\mathbb{R}^n)).$$

For example the **prototype anharmonic oscillator** $\mathcal{A}$ as in (1) is an operator in $\Psi\Sigma^2_{k,l}(\mathbb{R}^n)$.

Pseudo-differential calculus on $\Sigma^m_{k,l}(\mathbb{R}^n)$

The following can be viewed as a consequence of the Weyl-Hörmander calculus:

- The class of operators $\cup_{m\in\mathbb{R}} \Psi\Sigma^m_{k,l}(\mathbb{R}^n)$ forms an algebra of operators that is stable under taking the adjoint.
- Let $m_1, m_2 \in \mathbb{R}$ and let $k,l \geq 1$ integers. If $a \in \Sigma^{m_1}_{k,l}$ and $b \in \Sigma^{m_2}_{k,l}$, then there exists $c \in \Sigma^{m_1+m_2}_{k,l}$ such that $Op^w(a) \circ Op^w(b) = Op^w(c)$. Moreover, we have the asymptotic formula

$$c \sim \sum_\alpha \frac{(2\pi i)^{-|\alpha|}}{\alpha!} (\partial^\alpha_\xi a)(\partial^\alpha_x b).$$

- The operators in $\Psi\Sigma^0_{k,l}$ extend boundedly to $L^2(\mathbb{R}^n)$. Furthermore, there exists $C > 0$ and $N \in \mathbb{N}$ such that if $A = Op^w(a) \in \Psi\Sigma^0_{k,l}$, then

$$\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C\|a\|_{\Sigma^0_{k,l},N},$$

where $\|\cdot\|_{\Sigma^0_{k,l},N}$ denotes the inherited seminorm in the class in the class of symbols $\Sigma^0_{k,l}(\mathbb{R}^n)(\equiv S(1, g^{k,l}))$.

Associated Sobolev spaces

Using the functional calculus on (the compact, positive operator) $\mathcal{A}$ as in (1), we define the operator $\mathcal{A}^{\frac{m}{2}}$, for $m \in \mathbb{R}$, by

$$\mathcal{A}^{\frac{m}{2}} u = \sum_{j=1}^\infty \lambda_j^{\frac{m}{2}} (\phi_j, u)_{L^2} \phi_j, \quad \text{for} \quad u \in Dom\left(\mathcal{A}^{\frac{m}{2}}\right),$$

where $(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ made of eigenfunctions of $\mathcal{A}$ and $(\lambda_j)_{j\in\mathbb{N}}$ the corresponding eigenvalues.

The **Sobolev spaces related to** $\mathcal{A}$, denoted by $\mathcal{A}^m_{k,l}(\mathbb{R}^n)$, for $m \in \mathbb{R}$, $k,l \geq 1$, integers, is the subspace $\mathcal{S}'(\mathbb{R}^n)$ that is the completion of $Dom\,(\mathcal{A}^{\frac{m}{2}})$ for the norm

$$\|u\|_{\mathcal{A}_m} := \|\mathcal{A}^{\frac{m}{2}} u\|_{L^2(\mathbb{R}^n)}.$$

Continuity Properties

The identification of the Sobolev spaces $A^m_{k,l}(\mathbb{R}^n)$ with suitable Sobolev spaces $H(M, g)$ in the Weyl-Hörmander setting, and the general theory yield:

- Let $m \in \mathbb{R}$ and $k,l \geq 1$ integers. If $a \in \Sigma^m_{k,l}(\mathbb{R}^n)$, then

$$Op^w(a) : \mathcal{A}^s_{k,l}(\mathbb{R}^n) \xrightarrow{\text{cont.}} \mathcal{A}^{s-m}_{k,l}(\mathbb{R}^n), \quad \forall s \in \mathbb{R}.$$

More generally, equivalence of quantizations in our particular case yield:

- For $a \in \Sigma^m_{k,l}(\mathbb{R}^n)$ $(m \in \mathbb{R}, k,l \geq 1)$ we have

$$Op^\tau(a) : \mathcal{A}^s_{k,l}(\mathbb{R}^n) \xrightarrow{\text{cont.}} \mathcal{A}^{s-m}_{k,l}(\mathbb{R}^n), \quad \forall s \in \mathbb{R}, \forall \tau \in \mathbb{R}.$$

Anarmonic oscillators and Schatten-classes of operators

For $1 \leq r < \infty$, we denote by $S_r(L^2(\mathbb{R}^n))$ the r^{th}-Shatten class of operators.

- For $g = g^{p,q}$ as in (3), and for $a \in S(\lambda_g^{-\mu}, g)$, $\mu > \frac{n}{r}$, we have

$$Op^\tau(a) \in S_r(L^2(\mathbb{R}^n)), \quad \text{for all} \quad \tau \in \mathbb{R}.$$

- For the operator $\mathcal{A}$ as in (1), or more generally, for the operator T as in (2), and for $\mu > \frac{n(k+l)}{2klr}$ (for k,l as in (1), or accordingly to the choices of p,q as in (2)) we have

$$T^{-\mu}, \mathcal{A}^{-\mu} \in S_r(L^2(\mathbb{R}^n)).$$

Eigenvalue asymptotics for negative powers of operators

The general theory on the Schatten-classes $S_r(L^2(\mathbb{R}^n))$ and the Weyl-inequality (see [5])

$$\sum_{j=1}^\infty |\underbrace{\lambda_j(T)^r}_{\text{eigen. of } T}| \leq \sum_{j=1}^\infty \underbrace{s_j(T)^r}_{\text{sing. val. of } T}, \quad r > 0,$$

imply:

- For $g = g^{p,q}$ as in (3), and for $a \in S(\lambda_g^{-\mu}, g)$, $\mu > \frac{n}{r}$, and for any $\tau \in \mathbb{R}$ we have

$$\lambda_j(Op^\tau(a)) = o(j^{-\frac{1}{r}}), \quad \text{as} \quad j \to \infty.$$

- For the operator $\mathcal{A}$ as in (1), or more generally, for the operator T as in (2), and for $\mu > \frac{n(k+l)}{2klr}$ (for k,l as in (1), or accordingly to the choices of p,q as in (2)) we have

$$\lambda_j(\mathcal{A}^{-\mu}) = o(j^{-r}), \quad \text{as} \quad j \to \infty.$$

Rate of growth of the eigenvalues of the anharmonic oscillator $\mathcal{A}$

Let k,l be as in (1) and $r > \frac{n(k+l)}{2kl}$. Then for every $N \in \mathbb{N}$ there exists $N_0 \in \mathbb{N}$ such that

$$N j^{\frac{1}{r}} \leq \lambda_j(\mathcal{A}), \quad \text{for} \quad j \geq N_0.$$

Thus, the eigenvalues $\lambda_j(\mathcal{A})$ are at least of growth

$$j^{\frac{1}{r}}, \quad \text{as} \quad j \to \infty.$$

Asymptotic expansions of $\lambda_j(\mathcal{A})$ have also been studied in [1] and [2].

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Hyperbolic systems with non-diagonalisable principal part

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Introduction

We consider

$$\begin{cases} D_t u = A(t,x,D_x)u + B(t,x,D_x)u + f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where $n \geq 1$, $m \geq 2$ and $D_t = -i\partial_t$, $D_x = -i\partial_x$. We assume that $A(t,x,D_x) = (a_{ij}(t,x,D_x))_{i,j=1}^m$ is an $m \times m$ matrix of continuously time dependent pseudo-differential operators of order 1, i.e., $a_{ij} \in C([0,T], \Psi^1_{1,0}(\mathbb{R}^n))$ and that $B(t,x,D_x) = (b_{ij}(t,x,D_x))_{i,j=1}^m$ is an $m \times m$ matrix of pseudo-differential operators of order 0, i.e., $b_{ij} \in C([0,T], \Psi^0_{1,0}(\mathbb{R}^n))$. We also assume that the matrix A is upper triangular and hyperbolic, i.e.,

$$\begin{aligned} A(t,x,D_x) &= \Lambda(t,x,D_x) + N(t,x,D_x) \\ &= \mathrm{diag}(\lambda_1(t,x,D_x), \lambda_2(t,x,D_x), \ldots, \lambda_m(t,x,D_x)) + N(t,x,D_x) \end{aligned}$$

with real eigenvalues $\lambda_1(t,x,\xi), \lambda_2(t,x,\xi), \ldots, \lambda_m(t,x,\xi)$ of $A(t,x,\xi)$ and

$$N(t,x,D_x) = \begin{pmatrix} 0 & a_{12}(t,x,D_x) & a_{13}(t,x,D_x) & \cdots & a_{1m}(t,x,D_x) \\ 0 & 0 & a_{23}(t,x,D_x) & \cdots & a_{2m}(t,x,D_x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{m-1m}(t,x,D_x) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Furthermore, we introduce the following two hypotheses:

(H1) For the coefficients of the lower order term $B(t,x,D_x)$:

the lower order terms b_{ij} belong to $C([0,T], \Psi^{j-i})$ for $i > j$.

(H2) For some theorems, we assume that A does not depend on t, i.e. $A = A(x,D_x)$ and satisfies: there exists $M \in \mathbb{N}$ such that if $\lambda_j(x,\xi) = \lambda_k(x,\xi)$ for some $j,k \in \{1,\ldots,m\}$ and $\lambda_j(x,\xi)$ and $\lambda_k(x,\xi)$ are not identically equal near (x,ξ) then there exists some $N \leq M$ such that

$$\lambda_j(x,\xi) = \lambda_k(x,\xi) \Rightarrow H^N_{\lambda_j}(\lambda_k) := \{\lambda_j, \{\lambda_j, \ldots, \{\lambda_j, \lambda_k\}\} \ldots\}(x,\xi) \neq 0, \quad \text{(H2)}$$

where the Poisson bracket $\{\cdot,\cdot\}$ in $H^N_{\lambda_j}$ is iterated N times.

Well-posedness

In Garetto et al. [2018], we prove a well-posedness result for (1) under hypothesis (H1):

Theorem 1. *Consider the Cauchy problem (1), where $A(t,x,D_x)$ and $B(t,x,D_x)$ are as described in the introduction and $B(t,x,D_x)$ satisfies (H1). If now $u^0_k \in H^{s+k-1}(\mathbb{R}^n)$ and $f_k \in C([0,T], H^{s+k-1})$ for $k = 1,\ldots,m$, then (1) has a unique anisotropic Sobolev solution u, i.e., $u_k \in C([0,T], H^{s+k-1})$ for $k = 1,\ldots,m$.*

This theorem is proved by making use of the triangular form, solving the last equation and then iteratively building the solution of the system from the solutions to scalar equations. For each characteristic λ_j of A, we denote by $G^0_j\theta$ and $G_j g$ the respective solution solution to

$$\begin{cases} D_t w = \lambda_j(t,x,D_x)w + b_{jj}(t,x,D_x)w, \\ w(0,x) = \theta(x), \end{cases} \text{ and } \begin{cases} D_t w = \lambda_j(t,x,D_x)w + b_{jj}(t,x,D_x)w + g(t,x), \\ w(0,x) = 0. \end{cases}$$

The operators G^0_j and G_j can be microlocally represented by Fourier integral operators

$$G^0_j\theta(t,x) = \int e^{i\varphi_j(t,x,\xi)} a_j(t,x,\xi)\hat{\theta}(\xi)d\xi$$

and

$$G_j g(t,x) = \int_0^t \int e^{i\varphi_j(t,s,x,\xi)} A_j(t,s,x,\xi)\hat{g}(s,\xi)d\xi ds = \int_0^t \mathcal{E}_j(t,s)g(s,x)ds,$$

where

$$\mathcal{E}_j(t,s)g(s,x) = \int e^{i\varphi_j(t,s,x,\xi)} A_j(s,x,\xi)\hat{g}(s,\xi)d\xi$$

with $\varphi_j(t,s,x,\xi)$ solving the eikonal equation

$$\begin{cases} \partial_t\varphi_j = \lambda_j(t,x,\nabla_x\varphi_j), \\ \varphi_j(s,s,x,\xi) = x\cdot\xi, \end{cases} \quad \varphi_j(t,x,\xi) := \varphi_j(t,0,x,\xi).$$

The amplitudes $A_{j,-k}(t,s,x,\xi)$ of order $-k$, $k \in \mathbb{N}$, giving $A_j \sim \sum_{k=0}^\infty A_{j,-k}$, and they satisfy the usual transport equations with initial data at $t = s$, and we have $a_j(t,x,\xi) = A_j(t,0,x,\xi)$.

The components of the solution u of (1) is given by a composition of the operators described above together with principal part coefficients and lower order coefficients. That is where hypothesis (H1) comes into play. For example in the case $m = 2$, we get

$$\begin{cases} u_1 = U^0_1 + G_1((a_{12} + b_{12})u_2), \\ u_2 = U^0_2 + G_2(b_{21}u_1), \end{cases} \quad U^0_j = G^0_j u^0_j + G_j(f_j), \quad j = 1,2.$$

That then gives

$$\begin{aligned} u_1 &= U^0_1 + G_1(a_{12}G_2(b_{21}u_1)) + G_1(b_{12}G_2(b_{21}u_1)) \\ U^0_1 &= G^0_1 u^0_1 + G_1(f_1) + G_1((a_{12} + b_{12})U^0_2). \end{aligned}$$

One then gets to the final result by setting up a fixed point problem to which Banach's fixed point theorem can be applied. A general time interval $[0,T]$ can be iteratively covered since the estimates involved for the G's do only depend on the coefficients and not the initial data.

Solution representations and regularity results

In the case of $A = A(x,D_x)$, asking in addition to (H1) on the lower order terms also (H2) for the principal part, the solutions of (1) can be represented explicitly modulo some smoothing operators. Here, we state the principal results and refer to Garetto et al. [2020] for the details and proofs.

Theorem 2. *Consider (1) with $A = A(x,D_x)$ and $B(t,x,D_x)$ satisfying properties described above and let in addition (H1) and (H2) be satisfied. Let u_0 and f have components u^0_j and f_j, respectively, with $u^0_j \in H^{s+j-1}(\mathbb{R}^n)$ and $f_j \in C([0,T], H^{s+j-1})$ for $j = 1,\ldots,m$. Then, for any $N \in \mathbb{N}$, the components u_j, $j = 1,\ldots,m$, of the solution u are given by*

$$u_j(t,x) = \sum_{l=1}^m \left(\mathcal{H}^{l-j}_{jl}(t) + R_{jl}(t)\right)u^0_l + \left(\mathcal{K}^{l-j}_{jl}(t) + S_{jl}(t)\right)f_l,$$

where R_{jl}, $S_{jl} \in \mathcal{L}(H^s, C([0,T], H^{s+N-l+j}))$ and the operators $\mathcal{H}^{l-j}_{jl}$, $\mathcal{K}^{l-j}_{jl} \in \mathcal{L}(C([0,T], H^s), C([0,T], H^{s-l+j}))$ are integrated Fourier Integral Operators of order $l - j$.

Using the explicit solution representations, we get

Theorem 3. *Let $p \in (1,\infty)$ and $\alpha = (n-1)|\frac{1}{p} - \frac{1}{2}|$. Consider (1) under (H1) and (H2). Then, for any compactly supported $u_0 \in L^p_\alpha \cap L^2_{comp}$, the solution $u = u(t,x)$ of the Cauchy problem (1) satisfies $u(t,\cdot) \in L^p_{loc}$, for all $t \in [0,T]$. Moreover, there is a positive constant C_T such that*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^p_{loc}} \leq C_T\|u_0\|_{L^p_\alpha}.$$

Local estimates can be obtained in other spaces as well, for $s \in \mathbb{R}$ and α as above. In detail, assuming u_0 below is compactly supported, we have that $u_0 \in L^p_{s+\alpha}$ implies $u(t,\cdot) \in L^p_s$; that $u_0 \in C^{s+\frac{n-1}{2}}$ implies $u(t,\cdot) \in C^s$; and, for $1 < p \leq q \leq 2$, that $u_0 \in L^p_{s-\frac{1}{q}+\frac{n}{p}-\frac{n-1}{2}}$ implies $u(t,\cdot) \in L^q_s$.

Propagation of singularities

Operators of the form $(\prod_\sigma G_{\sigma(i)})^k$, which appear in the solution representation above in the operators $\mathcal{H}^{l-j}_{jl}$, are of the general form

$$Q_l = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} D(\bar{t})H(\bar{t})dt_l \ldots dt_1, \quad H(\bar{t}) = e^{i\lambda_{j_1}t_1}e^{i\lambda_{j_2}(t_2-t_1)} \cdot \ldots \cdot e^{i\lambda_{j_l}(t_l-t_{l-1})}e^{-i\lambda_{j_l}t_l},$$

For these operators, we have $Q_l \in \mathcal{L}(H^s, H^{s+N(l)})$, where $N(l) \to +\infty$ as $l \to +\infty$. The singularities propagate along broken Hamiltonian flows.

Let $J = \{j_1,\ldots,j_{l+1}\}$, $1 \leq j_k \leq m$, $j_k \neq j_{k+1}$. From the definition of $H(\bar{t})$, we have that its canonical relation $\Lambda^{\bar{t}} \subseteq T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ is given by

$$\Lambda^{\bar{t}} = \left\{(x,p,y,\xi) : (x,p) = \Psi^{\bar{t}}(y,\xi)\right\}, \quad \Psi^{\bar{t}} = \Phi^{t_1}_{j_1} \circ \cdots \circ \Phi^{t_l - t_{l-1}}_{j_l} \circ \Phi^{-t_l}_{j_{l+1}}$$

and the Φ^t_j are the transformations corresponding to a shift by t along the trajectories of the Hamiltonian flow defined by the λ_j.

Let $\Phi_J(t,x,\xi)$ be the corresponding broken Hamiltonian flow. It means that points follow bicharacteristics of λ_{j_1} until meeting the characteristic of λ_{j_2}, and then continue along the bicharacteristic of λ_{j_2}, etc.

Operators of the type Q_l can be rewritten as standard Fourier integral operators where the domain of integration is not the whole space but a simplex. For those operators, following arguments of Hörmander, we get $WF(Iu) \subset \bigcup_j \Lambda_j(WF(u))$. For details see Kamotski and Ruzhansky [2007], Garetto et al. [2020].

Corollary 1. *Let $n \geq 1$, $m \geq 2$, and consider (1) with $A = A(x,D_x)$ under hypotheses (H1) and (H2). Then, see above, we have an explicit representation of the solution u. Consequently, up to any Sobolev order (depending on N), the wave front set of u_j is given by*

$$WF(u_j(t,\cdot)) \subset \left(\bigcup_{l=1}^m WF(\mathcal{H}^{l-j}_{jl}(t)u^0_l)\right) \bigcup \left(\bigcup_{l=1}^m WF(\mathcal{K}^{l-j}_{jl}(t)f_l)\right), \quad (2)$$

with each of the wave front sets for terms in the right hand side of (2) given by the propagation along the broken Hamiltonian flow as described above.

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Hardy and Hardy-Sobolev inequalities on Heisenberg groups

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Abstract

This work [3] is devoted to present the geometric Hardy and Hardy-Sobolev inequalities for the sub-Laplacian in the half-spaces of the Heisenberg group with a sharp constant. This result answers a conjecture posed by S. Larson in [2]. As a consequence, a geometric Hardy-Sobolev-Maz'ya inequality is recovered.

Preliminaries on the Heisenberg group:

Let $\mathbb{H}^n$ be the Heisenberg group, that is, the set $\mathbb{R}^{2n+1}$ equipped with the group law

$$\xi \circ \tilde{\xi} := (x + \tilde{x}, y + \tilde{y}, t + \tilde{t} + 2\sum_{i=1}^{n}(\tilde{x}_i y_i - x_i \tilde{y}_i)),$$

where $\xi := (x, y, t) \in \mathbb{H}^n$, $x := (x_1, \ldots, x_n)$, $y := (y_1, \ldots, y_n)$, and $\xi^{-1} = -\xi$ is the inverse element of ξ with respect to the group law. The dilation operation of the Heisenberg group with respect to the group law has the following form $\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t)$ for $\lambda > 0$.
The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}^n$ is spanned by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \text{ and } Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

and with their (non-zero) commutator $[X_i, Y_i] = -4\frac{\partial}{\partial t}$. The horizontal gradient of $\mathbb{H}^n$ is $\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$
Let us define the half-space of the Heisenberg group by

$$\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu \rangle > d\},$$

where $\nu := (\nu_x, \nu_y, \nu_t)$ with $\nu_x, \nu_y \in \mathbb{R}^n$ and $\nu_t \in \mathbb{R}$ is the Riemannian outer unit normal to $\partial\mathbb{H}^+$ (see [1]) and $d \in \mathbb{R}$. The Euclidean distance to the boundary $\partial\mathbb{H}^+$ is defined by

$$dist(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d.$$

Let us define

$$X_i(\xi) = (\underbrace{0, \ldots, \overbrace{1}^{i}, \ldots, 0}_{n}, \underbrace{0, \ldots, 0}_{n}, 2y_i),$$

$$Y_i(\xi) = (\underbrace{0, \ldots, 0}_{n}, \underbrace{0, \ldots, \overbrace{1}^{i}, \ldots, 0}_{n}, -2x_i).$$

Then we have

$$\langle X_i(\xi), \nu \rangle = \nu_{x,i} + 2y_i\nu_t, \qquad \langle Y_i(\xi), \nu \rangle = \nu_{y,i} - 2x_i\nu_t,$$

where $\xi := (x, y, t)$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\nu := (\nu_x, \nu_y, \nu_t)$.

Introduction

The Hardy inequality in the half-space on the Heisenberg group was shown by Luan and Young in [4] as follows

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi. \qquad (1)$$

An alternative proof of this inequality was given by Larson in [2], where the author generalised it to any half-space of the Heisenberg group,

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4} \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2}{dist(\xi, \partial\mathbb{H}^+)^2} |u|^2 d\xi,$$

where X_i and Y_i (for $i = 1, \ldots, n$) are left-invariant vector fields on the Heisenberg group, ν is the Riemannian outer unit normal to the boundary. Also, there is the L^p-generalisation of the above inequality

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n |\langle X_i(\xi), \nu \rangle|^p + |\langle Y_i(\xi), \nu \rangle|^p}{dist(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi. \qquad (2)$$

Conjecture posed by S. Larson

A more natural weight in the right-hand side of (2) would be

$$\frac{\left(\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2\right)^{p/2}}{dist(\xi, \partial\mathbb{H}^+)^p}.$$

L^p-Hardy inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all functions $u \in C_0^\infty(\mathbb{H}^+)$ and $p > 1$ we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq C \int_{\mathbb{H}^+} \frac{\left(\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2\right)^{p/2}}{dist(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi, \qquad (3)$$

where the constant $C := \left(\frac{p-1}{p}\right)^p$ is sharp.

L^2-Hardy inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all functions $u \in C_0^\infty(\mathbb{H}^+)$ we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi, \qquad (4)$$

where the constant is sharp.
This corollary can be proved by considering $p = 2$, $dist(\xi, \partial\mathbb{H}^+) = t$ and $\nu = (0, 0, 1)$.

Hardy-Sobolev inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$ and $2 \leq p < Q$ with $Q = 2n + 1$, there exists some $C_1 > 0$ such that we have

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - C \int_{\mathbb{H}^+} \frac{\left(\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2\right)^{p/2}}{dist(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi\right)^{\frac{1}{p}} \geq C_1 \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{\frac{1}{p^*}},$$

where $p^* := Qp/(Q-p)$ and the constant $C := \left(\frac{p-1}{p}\right)^p$.

Hardy-Sobolev-Maz'ya inequality

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$, there exists some $C > 0$ such that we have

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi\right)^{\frac{1}{2}} \geq C \left(\int_{\mathbb{H}^+} |u|^{2^*} d\xi\right)^{\frac{1}{2^*}},$$

where $2^* := 2Q/(Q-2)$ with the homogeneous dimension $Q = 2n + 1$.

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