\mathbb{R} -actions and invariant differential operators

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Virtual International Conference on Pseudo-differential Operators

Ghent, July 7, 2020

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, $[\mathcal{L}_{\mathcal{T}},P]=0$, $PP^{\star}=P^{\star}P$ acting on sections of a Hermitian vector bundle, assumed to be normal with respect to the natural Hilbert space structure. On $\ker P \subset L^2$, $\mathcal{L}_{\mathcal{T}}$ will act as a Fredholm selfadjoint operator with compact parametrix (assuming some sort of ellipticity).

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, $[\mathcal{L}_{\mathcal{T}}, P] = 0$, $PP^* = P^*P$ acting on sections of a Hermitian vector bundle, assumed to be normal with respect to the natural Hilbert space structure. On ker $P \subset L^2$, $\mathcal{L}_{\mathcal{T}}$ will act as a Fredholm selfadjoint operator with compact parametrix (assuming some sort of ellipticity).

As an application I will discuss how with certain hypoellipticity condition one gets a result resembling Kodaira's vanishing theorem.

At the end I will sketch the basic ideas of the proofs.

Set-up M is a closed manifold, compact no boundary (and connected)

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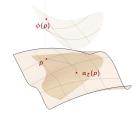
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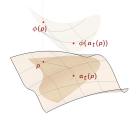
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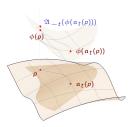
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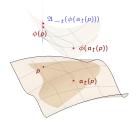
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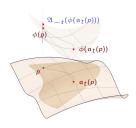
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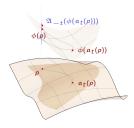
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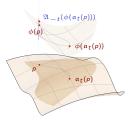
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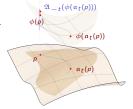
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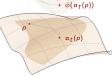
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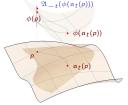
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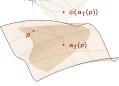
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 is invertible if $\lambda \in \Lambda$.

Let $\mathscr{D} = \{ \phi \in H : \mathcal{L}_{\mathcal{T}} \phi \in H \}.$



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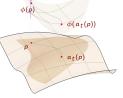
$$[\mathcal{L}_{\mathcal{T}}, P] = 0, [P, P^*] = 0$$

$$(so P + (-i\mathcal{L}_{\mathcal{T}})^m \text{ is elliptically a probable of } P)$$

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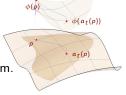
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 $H = \ker P \cap L^2$ is a Hilbert space on its own.

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· \(\alpha_t(p) \)

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P is a differential operator $C^{\infty}(M;E) \to C^{\infty}(M;E)$. We ask $[\mathcal{L}_{\mathcal{T}},P]=0$, $[P,P^*]=0$ (so $P+(-i\mathcal{L}_{\mathcal{T}})^m$ is elliptic) $\sigma(P)+\sigma(-i\mathcal{L}_{\mathcal{T}})^m-\lambda I$ is invertible if $\lambda\in\Lambda$.

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say $p \sim p'$ iff $p' \in \overline{\mathcal{O}}_p$, let $\wp : M \to B := M/\sim$. The fibers of \wp are tori.

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Reason:

The closure of a_t in Iso(M) is isomorphic to a torus \mathbb{T} .

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Let $\mathscr{D} = \{\phi \in H : \mathcal{L}_T \phi \in H\}.$ $-i\mathcal{L}_T|_{\mathscr{D}} : \mathscr{D} \subset H \to H$ is selfadjoint with compact resolvent, in particular Fredholm.

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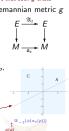
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(And there is an isomorphism $\mathcal{E}_{-m} \approx H^{0,q}(M; L^{\otimes m})$)

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5

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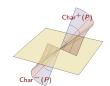
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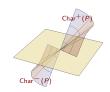
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Theorem. If P is hypoelliptic on $\operatorname{Char}^+(P)$ then $-i\mathcal{L}_{\mathcal{T}}\big|_{\mathscr{D}}$ is semibounded from above.



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Lie derivative: $\phi \text{ section of } E, \ p \in M. \ t \mapsto \mathfrak{A}_{-t}(\phi(\mathfrak{a}_t(p))) \text{ is a curve in } E_p, \\ \mathcal{L}_{\mathcal{T}}(\phi)(p) = \frac{d}{dt} | \mathfrak{A}_{-t}(\Phi(\mathfrak{a}_t(p))).$

 $[\mathcal{L}_{\mathcal{T}}, P] = 0, \ [P, P^*] = 0$ $[So \ P + (-i\mathcal{L}_{\mathcal{T}})^m \ is \ elliptic)$ $[So \ P + (-i\mathcal{L}_{\mathcal{T}})^m \ is \ elliptic)$ $[So \ P + (-i\mathcal{L}_{\mathcal{T}})^m \ is \ elliptic)$

Let $\mathscr{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}.$ $-i\mathcal{L}_T|_{\mathscr{D}} : \mathscr{D} \subset H \to H$

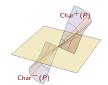
 $-i\mathcal{L}_{\mathcal{T}|_{\widehat{\mathscr{D}}}}: \mathscr{D} \subset \mathcal{H} \to \mathcal{H}$

is selfadjoint with compact resolvent, in particular Fredholm.

Theorem. If P is hypoelliptic on $\operatorname{Char}^+(P)$ then $-i\mathcal{L}_{\mathcal{T}}\Big|_{\mathscr{D}}$ above.

is semibounded from (At most finitely many posi-

(At most finitely many positive elements in spectrum.)



Observe $\sigma(-i\mathcal{L}_{\tau}) = \tau I$ for some $\tau: T^*M \to \mathbb{R}$.

Invertibility of

degree $q \neq q^{\pm}$.

$$\sigma(P) + \sigma((-i\mathcal{L}_{\mathcal{T}})^m)$$

gives $\tau \neq 0$ on Char(P):

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$$\begin{split} [\mathcal{L}_{\mathcal{T}}, P] &= 0, \ [P, P^\star] = 0 \\ \sigma(P) + \sigma(-i\mathcal{L}_{\mathcal{T}})^m &- \lambda \text{I is invertible if } \lambda \in \Lambda. \end{split}$$
Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_{\tau} \phi \in H \}.$ $-i\mathcal{L}_{\mathcal{T}}|_{\omega}: \mathcal{D} \subset H \to H$ is selfadjoint with compact resolvent, in particular Fredholm.

Theorem. If P is hypoelliptic on $\operatorname{Char}^+(P)$ then $-i\mathcal{L}_{\mathcal{T}}$ above. If M carries an invariant Levi non-degenerate CR structure then the Kohn Laplacians are hypoelliptic in

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 \overline{P} is a differential operator $C^{\infty}(M;E) \to C^{\infty}(M;E)$. We ask $[\mathcal{L}_{\mathcal{T}}, P] = 0, [P, P^*] = 0$ $[\mathcal{L}_{\mathcal{T}}, r_1] = 0$, $[r, r^n] = 0$ (so $P + (-i\mathcal{L}_{\mathcal{T}})^m$ is elliptic) $\sigma(P) + \sigma(-i\mathcal{L}_{\mathcal{T}})^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let
$$\mathscr{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}.$$

 $-i\mathcal{L}_T|_{\mathscr{D}} : \mathscr{D} \subset H \to H$

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by work of Boutet de Monvel

& Siöstrand, '70s

Theorem. If P is hypoelliptic on $\operatorname{Char}^+(P)$ then $-i\mathcal{L}_T$ above. If M carries an invariant Levi non-degenerate CR structure then the Kohn Laplacians are hypoelliptic in degree $q \neq q^{\pm}$.

Observe $\sigma(-i\mathcal{L}_{\mathcal{T}}) = \tau I$ for some $\tau : T^*M \to \mathbb{R}$.

Invertibility of

$$\sigma(P) + \sigma((-i\mathcal{L}_{\mathcal{T}})^m)$$

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 $\begin{aligned} & [\mathcal{L}_{\mathcal{T}}, P] = 0, \ [P, P^*] = 0 \\ & \sigma(P) + \sigma(-i\mathcal{L}_{\mathcal{T}})^m - \lambda I \text{ is invertible if } \lambda \in \Lambda. \end{aligned}$ Let $\mathscr{D} = \{\phi \in H : \mathcal{L}_{\mathcal{T}}\phi \in H\}.$ $-i\mathcal{L}_{\mathcal{T}}|_{\varpi} : \mathscr{D} \in H \to H$

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Theorem. If P is hypoelliptic on $\operatorname{Char}^+(P)$ then $-i\mathcal{L}_{\mathcal{T}}|_{\mathscr{D}}$ is semibounded from above. If M carries an invariant Levi non-degenerate CR structure then the Kohn Laplacians are hypoelliptic in $q^\pm = \#\operatorname{pos/neg}$ Levi eigenvalues. degree $q \neq q^\pm$. So finite spectrum in these degrees by work of Boutet de Monvel & Sjöstrand, '70s $\operatorname{Char}_{\mathcal{A}}$ Chart (P)

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Let $\mathcal{Q} = \{\phi \in H: \mathcal{L}_{\mathcal{T}}\phi \in H\}$.

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(0, q)-forms on M

Theorem. If P is hypoelliptic on $\operatorname{Char}^+(P)$ then $-i\mathcal{L}_{\mathcal{T}}|_{\mathcal{L}}$ above. If M carries an invariant Levi non-degenerate CR structure then the Kohn Laplacians are hypoelliptic in

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degree $q \neq q^{\pm}$. So finite spectrum in these degrees by work of Boutet de Monvel & Sjöstrand, '70s

Example. Let B be a compact complex manifold, $\wp: M \to B$ the circle bundle

of a holomorphic line bundle $L \to \mathcal{B}$, \mathcal{T} the generator of $t \mapsto e^{it} p$, $p \in M$, \mathcal{V} be the CR structure of $M \subset L$, P be the Kohn Laplacian acting on $C^{\infty}(\bigwedge^q \overline{\mathcal{V}}^*)$

 $P + (-i\mathcal{L}_T)^2$ is elliptic, $\ker P = \bigoplus_{m \in \mathbb{Z}} \mathcal{E}_{m_1} - i\mathcal{L}_T \phi = m\phi$.

(And there is an isomorphism $\mathcal{E}_{-m} \approx H^{0,q}(M; L^{\otimes m})$)



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Parametrix:

Suppose Q is a \mathcal{T} -invariant parametrix for $P + (-i\mathcal{L}_T)^m$.

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Symmetry:

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$$d\big(h(\phi,\psi)\mathcal{T}\rfloor\mathfrak{m}\big)\underbrace{\int_{M}d(h(\phi,\psi)\mathcal{T}\rfloor\mathfrak{m})}_{=0}$$

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Selfadjointness uses $\sigma(P) + \sigma((-i\mathcal{L}_T)^m) - \lambda I$ invertible for large λ (Λ is a ray of minimal growth) and formal normality of P.

Proofs, details in (1)

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$$\begin{split} [\mathcal{L}_{\mathcal{T}},P] &= 0, \ [P,P^{\star}] = 0 \\ &\text{(so } P + (-i\mathcal{L}_{\mathcal{T}})^m \text{ is elliptic)} \\ &\sigma(P) + \sigma(-i\mathcal{L}_{\mathcal{T}})^m - \lambda \text{I is invertible if } \lambda \in \Lambda. \end{split}$$
Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_{\tau} \phi \in H \}.$ $-i\mathcal{L}_{\mathcal{T}}|_{\mathcal{Q}}: \mathcal{D} \subset H \to H$ is selfadjoint with compact resolvent, in particular Fredholm

Parametrix:

Suppose
$$Q$$
 is a \mathcal{T} -invariant parametrix for $P+(-i\mathcal{L}_{\mathcal{T}})^m.$ Then

$$\phi - R\phi = Q(P + (-i\mathcal{L}_T)^m)\phi = Q(-i\mathcal{L}_T)^m\phi = [Q(-i\mathcal{L}_T)^{m-1}](-i\mathcal{L}_T)\phi$$
if $P\phi = 0$

Symmetry:

Selfadjointness uses $\sigma(P) + \sigma((-i\mathcal{L}_T)^m) - \lambda I$ invertible for large λ (Λ is a ray of minimal growth) and formal normality of P.

(1) —, Hypoellipticity and vanishing theorems, Bull. Inst. Math. Acad. Sin. (N.S.) 8 (2013), 231–258.

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Lie derivative:

P is a differential operator $C^{\infty}(M; E) \to C^{\infty}(M; E)$. We ask $[\mathcal{L}_{\mathcal{T}},P]=0,\,[P,P^{\star}]=0$ $\frac{[\mathcal{L}_{\mathcal{T}}, r] = 0, [r, r'] = 0}{\sigma(P) + \sigma(-i\mathcal{L}_{\mathcal{T}})^m - \lambda I} \text{ is invertible if } \lambda \in \Lambda.$

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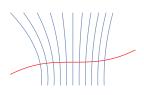
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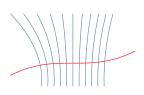
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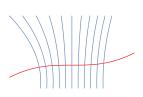
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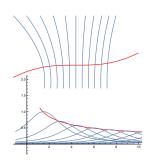
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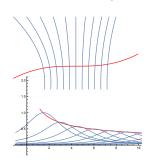
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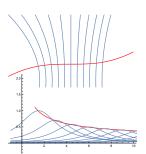
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