$\mathbb{R}$-actions and invariant differential operators

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Temple University

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As an application I will discuss how with certain hypoellipticity condition one gets a result resembling Kodaira's vanishing theorem.

At the end I will sketch the basic ideas of the proofs.

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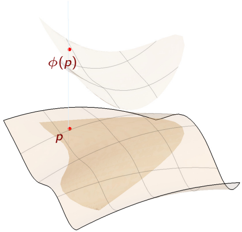

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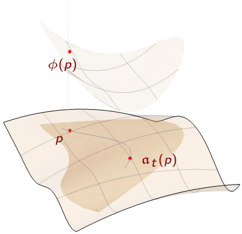

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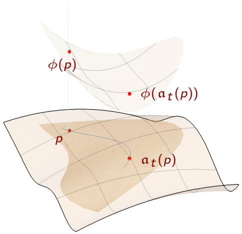

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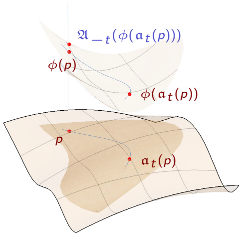

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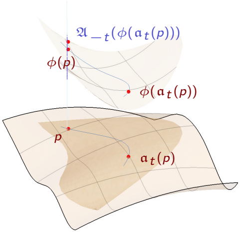

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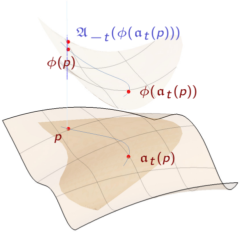

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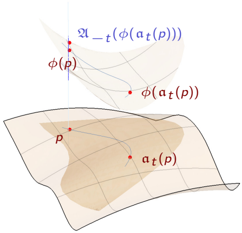

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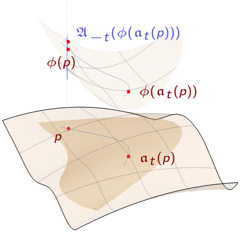

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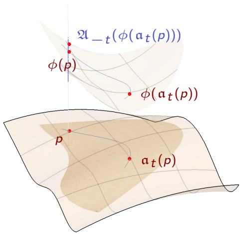

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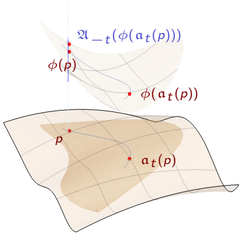

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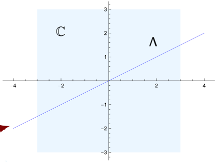

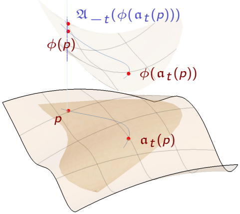

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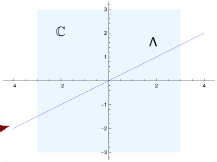

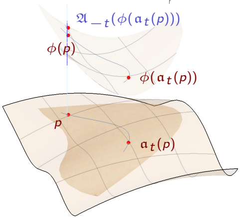

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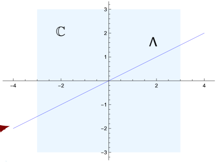

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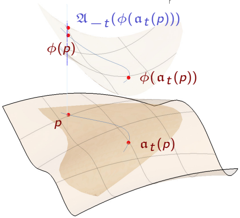

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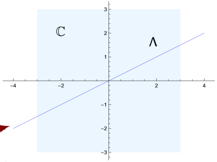

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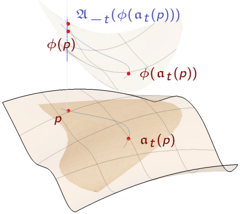

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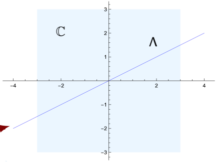

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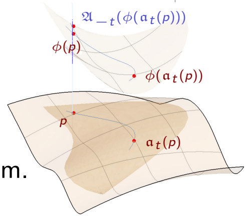

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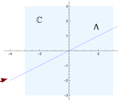

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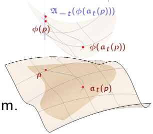

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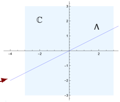

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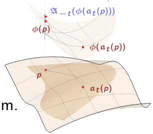

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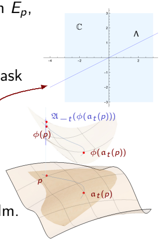

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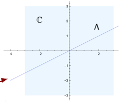

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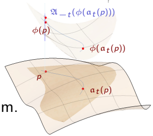

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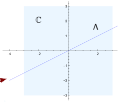

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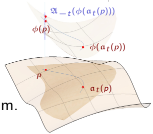

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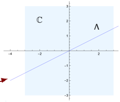

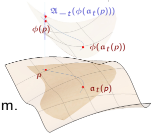

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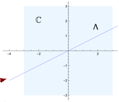

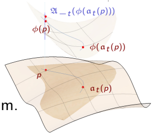

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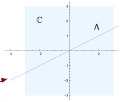

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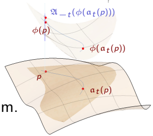

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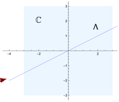

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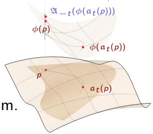

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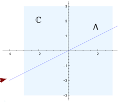

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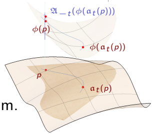

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Example. Let B be a compact complex manifold, $\wp : M \to B$ the circle bundle of a holomorphic line bundle $L \to \mathcal{B}$, $\mathcal{T}$ the generator of $t \mapsto e^{it}p$, $p \in M$, $\mathcal{V}$ be the CR structure of $M \subset L$, P be the Kohn Laplacian acting on $C^\infty(\bigwedge^q \overline{\mathcal{V}}^*)$ $(0,q)$-forms on M

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Circle bundle?

For $p \in M$: $\mathcal{O}_p =$ orbit of p,
say $p \sim p'$ iff $p' \in \overline{\mathcal{O}}_p$,
let $\wp : M \to B := M/\sim$.
The fibers of $\wp$ are tori.

Reason:
The closure of $\mathfrak{a}_t$ in $\mathrm{Iso}(M)$ is isomorphic to a torus $\mathbb{T}$.

There are open dense sets
$$M^{\mathrm{reg}} \subset M, \ B^{\mathrm{reg}} \subset B$$
such that $\pi : M^{\mathrm{reg}} \to B^{\mathrm{reg}}$ is a principal torus bundle.

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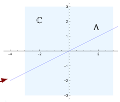

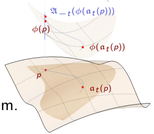

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(And there is an isomorphism $\mathcal{E}_{-m} \approx H^{0,q}(M; L^{\otimes m})$)

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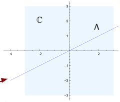

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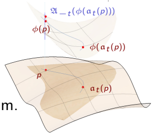

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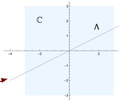

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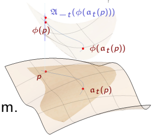

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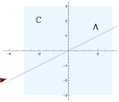

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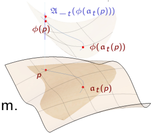

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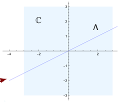

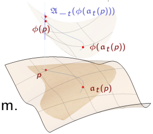

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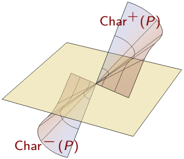

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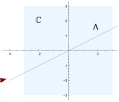

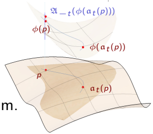

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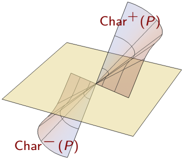

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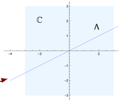

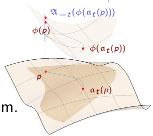

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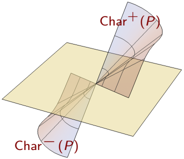

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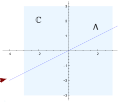

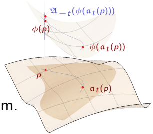

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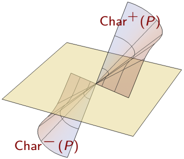

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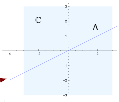

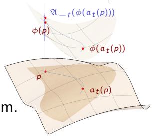

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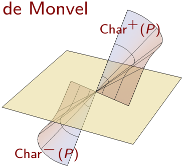

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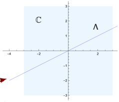

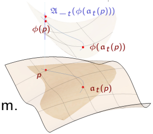

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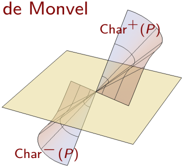

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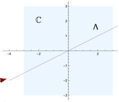

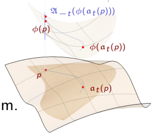

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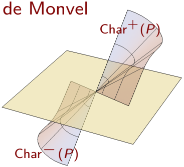
Example. Let B be a compact complex manifold, $\wp : M \to B$ the circle bundle of a holomorphic line bundle $L \to \mathcal{B}$, $\mathcal{T}$ the generator of $t \mapsto e^{it}p$, $p \in M$, $\mathcal{V}$ be the CR structure of $M \subset L$, P be the Kohn Laplacian acting on $C^\infty(\bigwedge^q \overline{\mathcal{V}}^*)$
$(0,q)$-forms on M

$$P + (-i\mathcal{L}_{\mathcal{T}})^2 \text{ is elliptic, } \ker P = \bigoplus_{m\in\mathbb{Z}} \mathcal{E}_m,\ -i\mathcal{L}_{\mathcal{T}}\phi = m\phi.$$

(And there is an isomorphism $\mathcal{E}_{-m} \approx H^{0,q}(M; L^{\otimes m})$)



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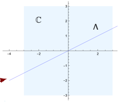

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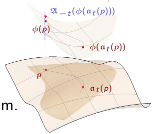

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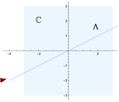

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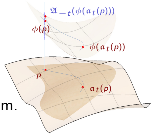

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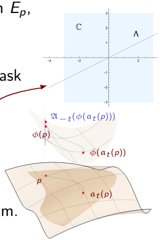

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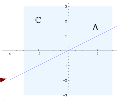

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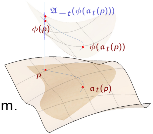

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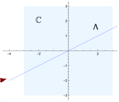

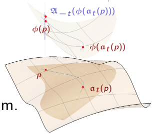

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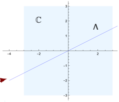

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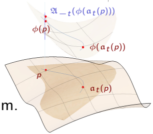

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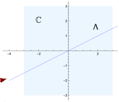

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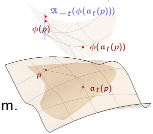

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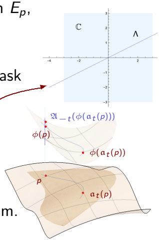

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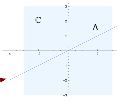

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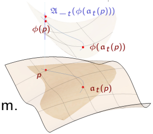

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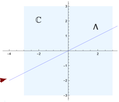

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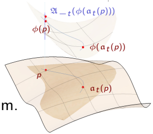

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(1) —, Hypoellipticity and vanishing theorems, Bull. Inst. Math. Acad. Sin. (N.S.) 8 (2013), 231–258.

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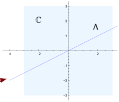

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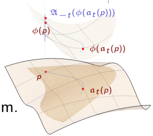

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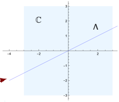

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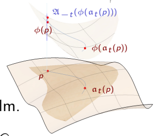

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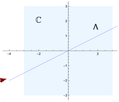

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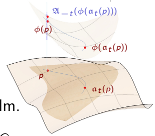

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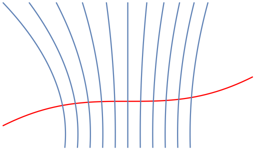

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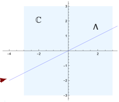

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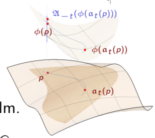

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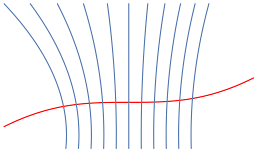

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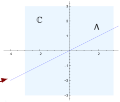

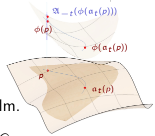

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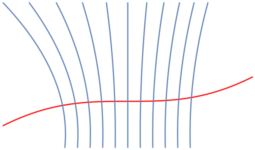

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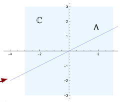

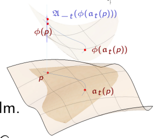

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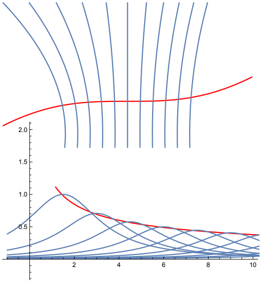

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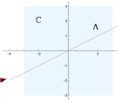

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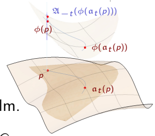

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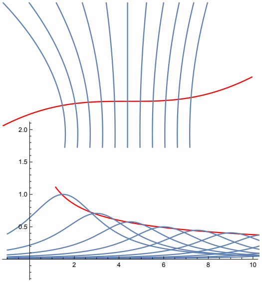

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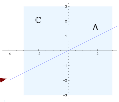

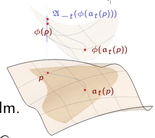

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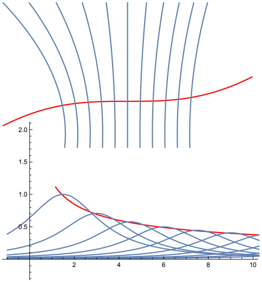

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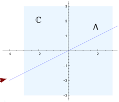

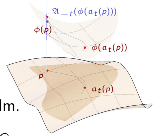

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Thank you