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Noncommutative conference

18–20 August 2020, Ghent University

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Overcoming noncommutativity without commuting

International Conference on Noncommutative Analysis

18-20 August 2020

- Jean-Philippe Anker (Université d’Orléans, France)
- Ingrid Beltiţă (IMAR, Romania)
- Tommaso Bruno (Ghent University, Belgium)
- Duván Cardona (Ghent University, Belgium)
- Paula Cerejeiras (University of Aveiro, Portugal)
- Marianna Chatzakou (Imperial College London, UK)
- Gregory Chirikjian (Johns Hopkins University, USA, and NUS, Singapore)
- Radouan Daher (Université Hassan II de Casablanca, Morocco)
- Anthony Dooley (University of Technology Sydney, Australia)
- Hans Feichtinger (University of Vienna, Austria)

International Conference on Noncommutative Analysis
18-20 August 2020

- Hartmut Führ (Aachen University, Germany)
- Karlheinz Gröchenig (University of Vienna, Austria)
- Bernard Helffer (Université de Nantes, France)
- Uwe Kähler (Aveiro University, Portugal)
- Vishvesh Kumar (Ghent University, Belgium)
- El mehdi Loualid (University Chouaib Doukkali, El Jadida, Morocco)
- Franz Luef (Norwegian University of Science and Technology, Norway)
- Marius Mantoiu (University of Chile, Chile)
- Alessio Martini (University of Birmingham, UK)
- Kenneth Ross (University of Oregon, USA)
- David Rottensteiner (Ghent University, Belgium)

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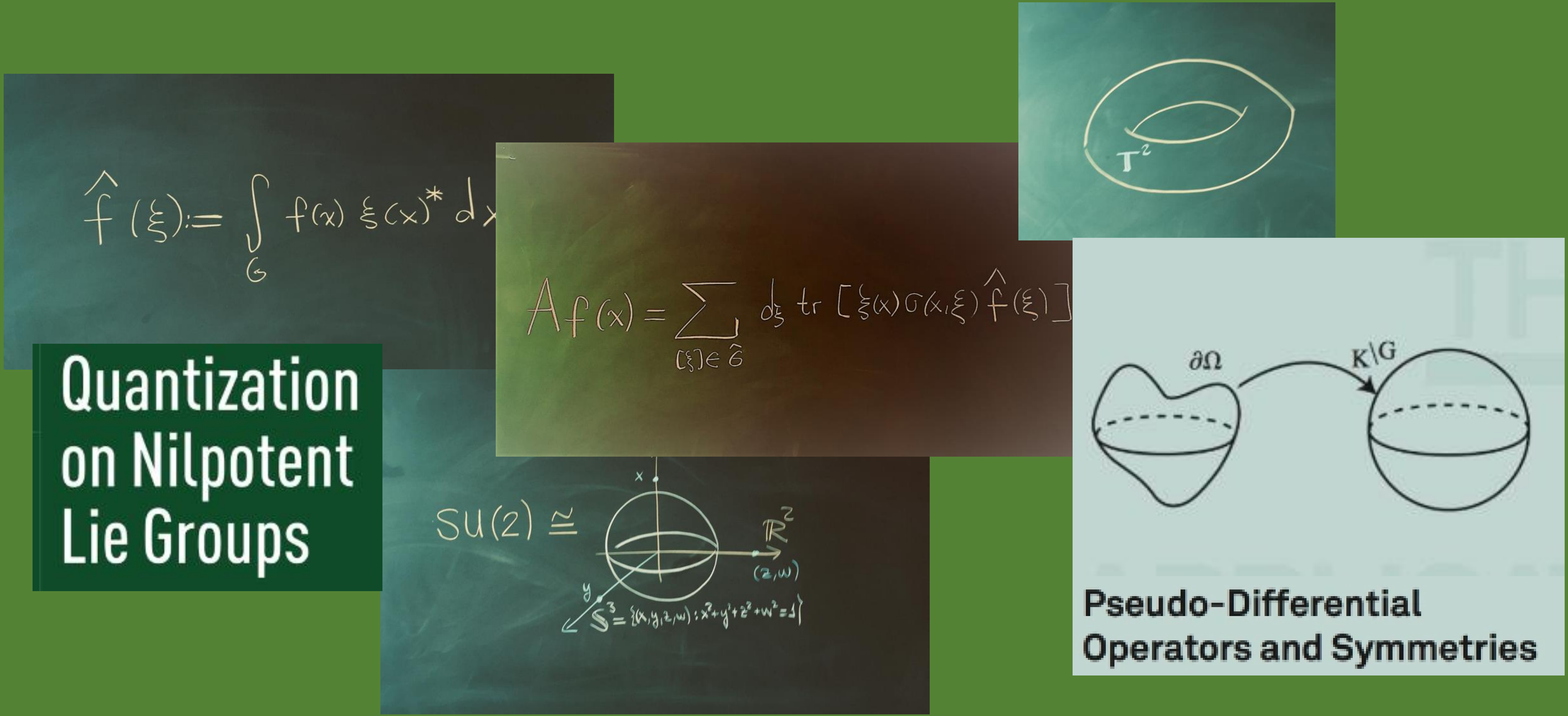
International Conference on Noncommutative Analysis

18-20 August 2020

- Margit Rösler (Paderborn University, Germany)
- Michael Ruzhansky (Ghent University and Queen Mary University of London)
- Bolys Sabitbek (Queen Mary University of London, UK)
- Faouaz Saadi (Université Hassan II de Casablanca, Morocco)
- Adam Sikora (Macquarie University, Australia)
- Fedor Sukochev (University of New South Wales)
- Gabor Szabo (KU Leuven, Belgium
- Sundaram Thangavelu (Indian Institute of Science, Bangalore, India)
- Othman Tyr (Université Hassan II de Casablanca, Morocco)
- Jeremy Tyson (University of Illinois at Urbana-Champaign, USA)
- Stefaan Vaes (KU Leuven, Belgium)
- Hong-Wei Zhang (Université d'Orléans, France)



Tuesday, 18 August

Noncommutative conference

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Fedor Sukochev

University of New South Wales, Australia

Singular traces and the density of states

Tuesday, 18 August

9:00 - 9:40

Abstract: The density of states is a non-negative measure associated to a Schrodinger operator H which is supported on its essential spectrum. Theoretical questions concerning the existence and properties of the density of states are of interest in solid state physics. We have recently found that quite generally the density of states measure can be computed by a formula involving a Dixmier trace, a tool from quantised calculus of A. Connes. This is a surprising new application of singular traces and methods from quantised calculus to mathematical physics which also uses recently developed techniques in operator integration theory. Joint work with N. Azamov, E. McDonald and D. Zanin.

Fedor Sukochev

University of New South Wales, Australia

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Margit Rösler

Paderborn University, Germany

Riesz distributions and the Wallach set in rational Dunkl theory

Tuesday, 18 August

9:40 - 10:20

Abstract: We introduce Riesz distributions associated with rational Dunkl operators of type A, which are closely related to the well-known Riesz distributions on symmetric cones. The study of these distributions relies on a suitable Laplace transform in the Dunkl setting. In particular, we shall present an analogue of a famous result of Gindikin for symmetric cones, which states that a Riesz distribution is actually a positive measure if and only if its index belongs to the so-called Wallach set. We further explain the relevance of this generalized Wallach set in connection with the existence of positive intertwining operators for root systems of type B.
Parts of the talk are based on joint work with Michael Voit, Dortmund.

Margit Rösler

Paderborn University, Germany

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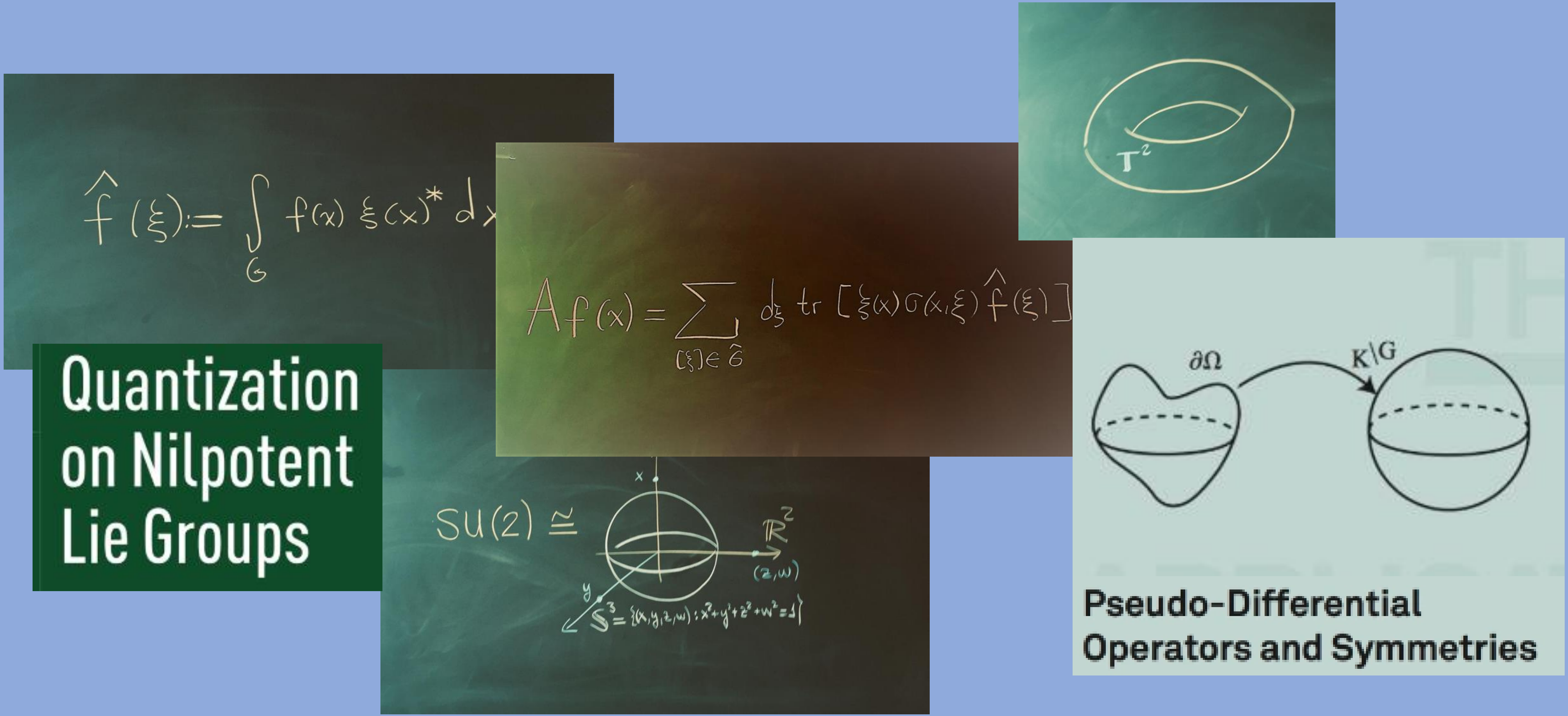

Break: 10:20-10:40

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Jean-Philippe Anker

Université d'Orléans, France

Bottom of the L^2 spectrum of the Laplacian on locally symmetric spaces

Tuesday, 18 August

10:40 - 11:20

Jean-Philippe Anker

Université d’Orléans, France

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Abstract: We estimate the bottom of the L^2 spectrum of the Laplacian on locally symmetric spaces in terms of the critical exponents of appropriate Poincaré series. Our main result is the higher rank analog of a characterization due to Elstrodt, Patterson, Sullivan and Corlette in rank one. It improves upon previous results obtained by Leuzinger in higher rank. This is joint work with Hong-Wei Zhang [arXiv:2006.06473].

Radouan Daher

Université Hassan II de Casablanca, Morocco

Some new results on q-Dunkl harmonic analysis

Tuesday, 18 August

11:20 - 12:00

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Abstract:

Using the q-harmonic analysis associated with the q-Dunkl operator, we prove an analog of Titchmarsh's theorem for functions satisfying the q-Dunkl Lipschitz condition, we look at problems in the q-version of approximation theory of functions in the space $L^2_{q,\alpha}(R^q)$, more precisely, we prove analogues of the direct and inverse Jackson's theorems. In the end, we show the q-analogue of the equivalence theorem between the modulus of smoothness and K-functional.

Radouan Daher

Université Hassan II
de Casablanca, Morocco

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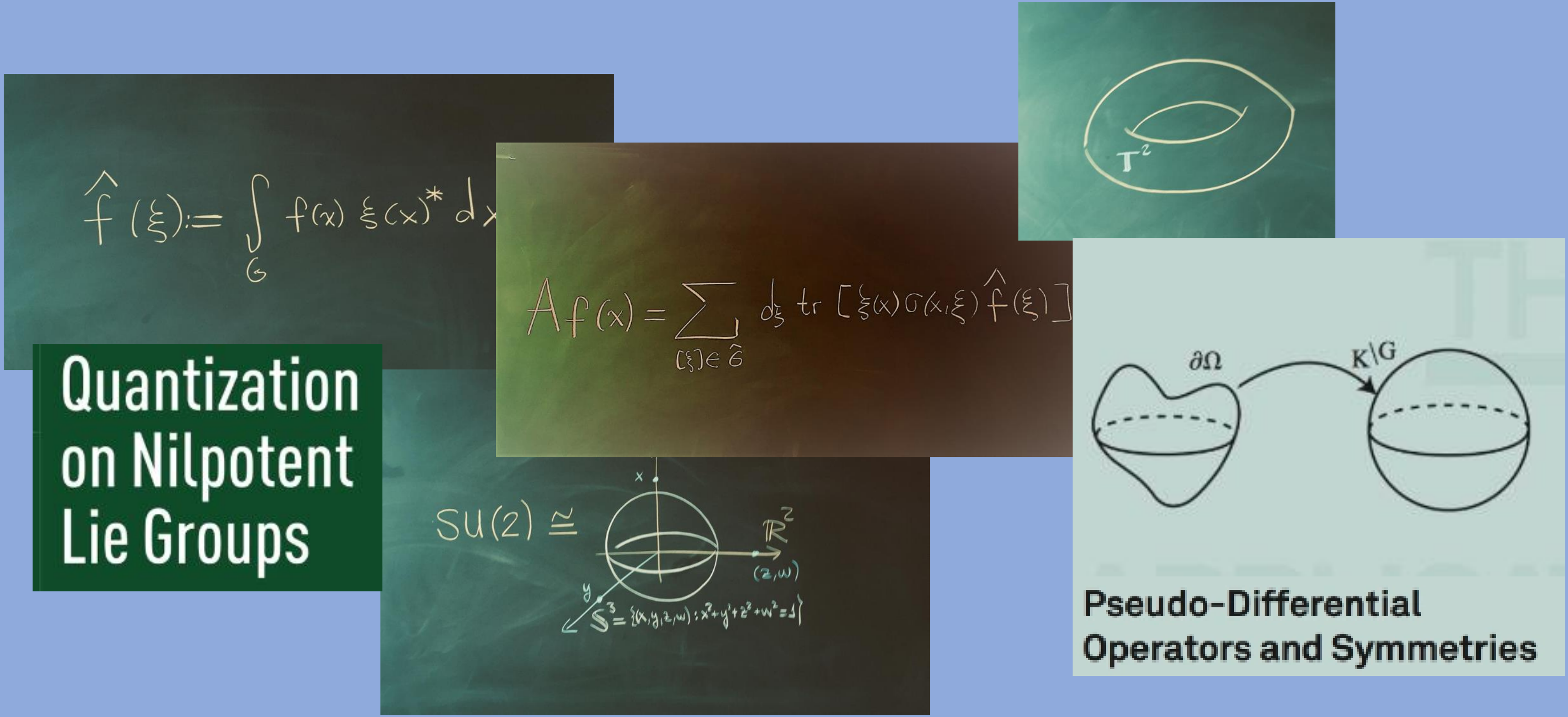
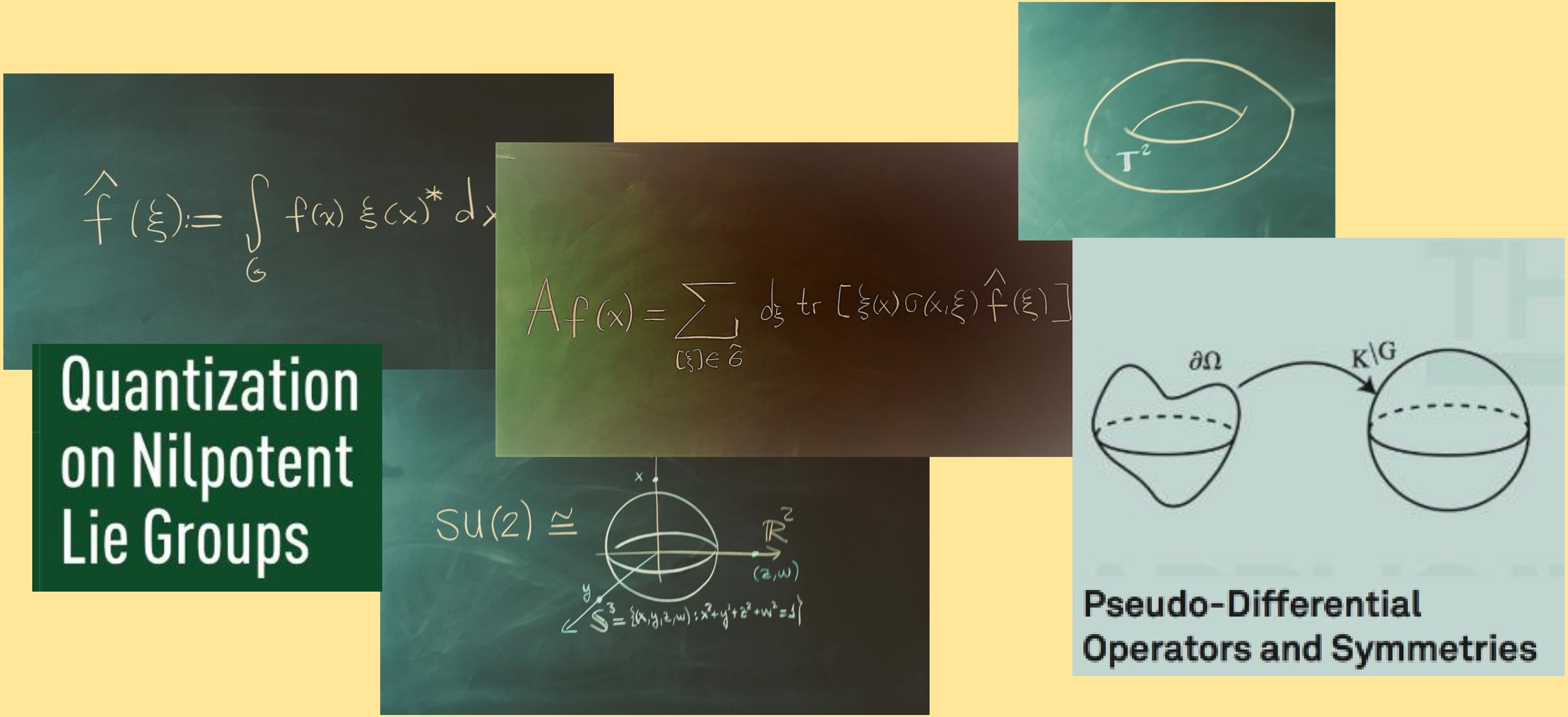


Break: 12:00-1:15



Blitztalk Session: 1:15-2:00

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Stefaan Vaes

KU Leuven, Belgium

Classification problems in operator algebras

Tuesday, 18 August

2:00 - 2:40

Steefan Vaes

KU Leuven, Belgium

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Abstract: The talk starts by an elementary introduction to the theory of operator algebras: C*-algebras and von Neumann algebras. I will review the fundamental classification results for amenable von Neumann algebras due to Connes and Haagerup. The second part of the talk will focus on similar classification problems for von Neumann algebras given by Bernoulli actions of discrete groups, highlighting the role of harmonic analytic properties of free groups, hyperbolic groups and property (T) groups.

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Marius Mantoiu

University of Chile, Chile

Some Spectral Results for Pseudodifferential Operators on Noncommutative Groups

Tuesday, 18 August

2:40 – 3:20

Abstract: We connect the global quantisation with operator-valued symbols on locally compact type I groups with a C^*-algebraic formalism involving dynamical systems. We use this connection to deduce several spectral results, for which the behavior of the symbol at infinity is relevant.

Marius Mantoiu

University of Chile, Chile

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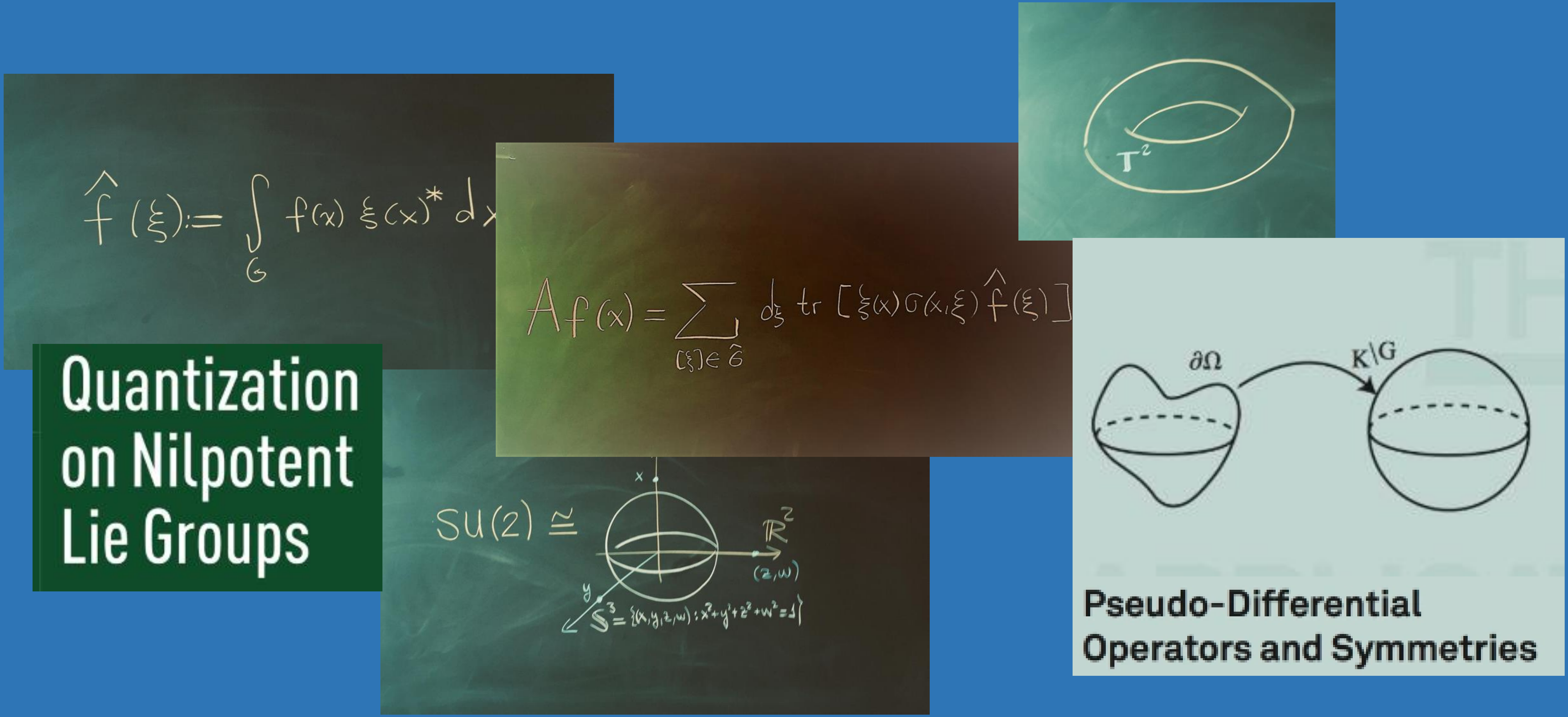

Wednesday, 19 August

Noncommutative conference

18–20 August 2020, Ghent University

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Sundaram Thangavelu

Indian Institute of Science,
Bangalore, India

Holomorphic extensions of eigenfunctions on NA groups

Wednesday, 19 August
9:00-9:40

Abstract: Let $X = G/K$ be a rank one Riemannian symmetric space of non-compact type. In view of the Iwasawa decomposition $G = NAK$ of the underlying semisimple Lie group, we can also view X as the solvable extension $S = NA$ of the Iwasawa group N which is known to be a *H*-type group. In this work we study the holomorphic extendability of eigenfunctions of the Laplace-Beltrami operator Δ_S to certain domains in the complexification of the nilpotent group *N*. We can also do the same for any *H*-type group *N* not necessarily an Iwasawa group. The results are accomplished by making use of the connection with solutions of the extension problem for the Laplacian or the sublaplacian on the corresponding *N*. This talk is based on joint work with Luz Roncal.

Sundaram Thangavelu

Indian Institute of Science, Bangalore, India

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Adam Sikora

Macquarie University, Australia

Riesz transforms on a class of non-doubling manifolds
(joint work with Andrew Hassell and Daniel Nix)

Wednesday, 19 August
9:40-10:20

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Adam Sikora

Macquarie University, Australia

Noncommutative conference

18–20 August 2020, Ghent University

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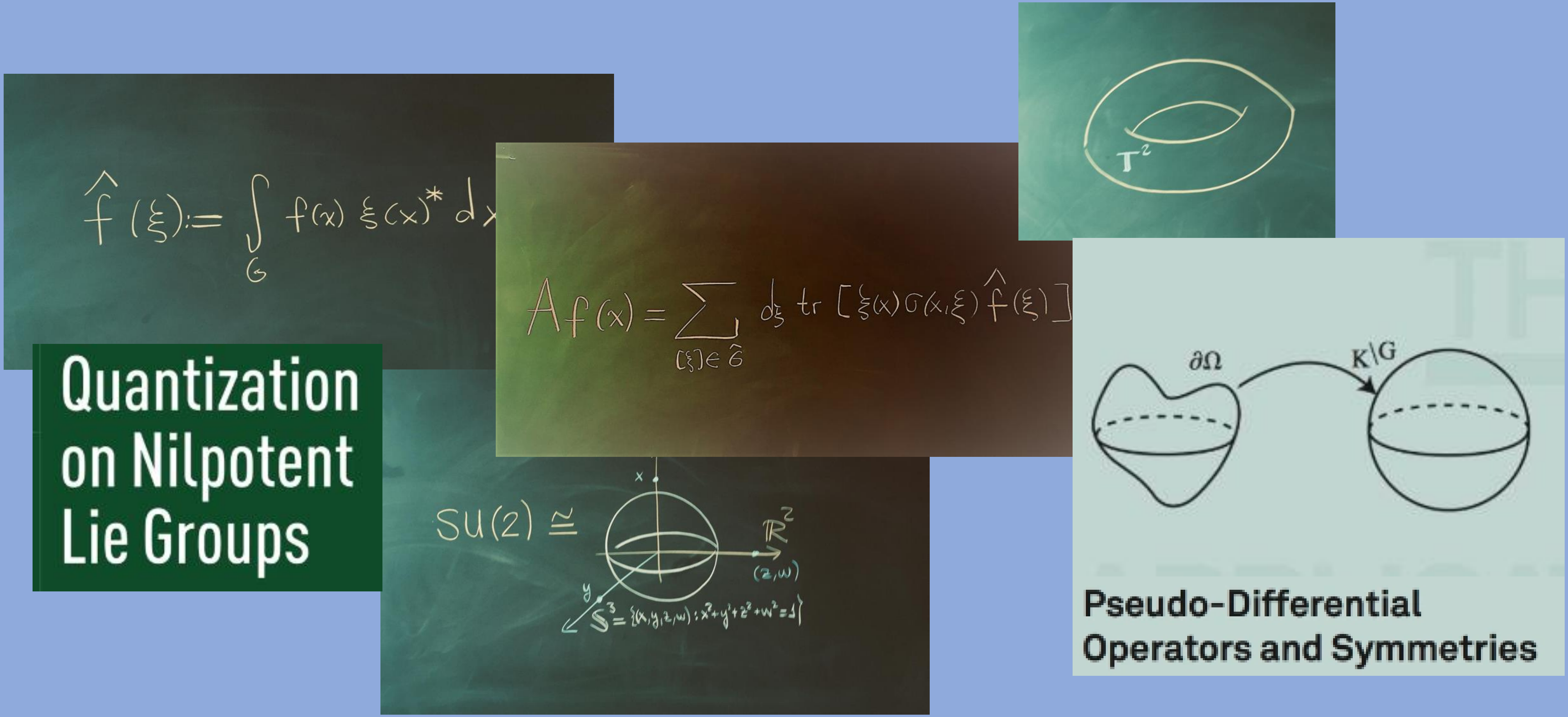
Abstract:

We consider a class of manifolds $\mathcal{M}$ obtained by taking the connected sum of a finite number of N-dimensional Riemannian manifolds of the form $(\mathbb{R}^{n_i}, \delta) \times (\mathcal{M}_i, g)$, where $\mathcal{M}_i$ is a compact manifold, with the product metric. The case of greatest interest is when the Euclidean dimensions n_i are not all equal. This means that the ends have different 'asymptotic dimension', and implies that the Riemannian manifold $\mathcal{M}$ is not a doubling space. We completely describe the range of exponents p for which the Riesz transform on $\mathcal{M}$ is a bounded operator on $L^p(\mathcal{M})$. Namely, under the assumption that each n_i is at least 3, we show that the Riesz transform is of weak type $(1,1)$, is continuous on L^p for all $p \in (1, \min_i n_i)$, and is unbounded on L^p otherwise. When $\min_i n_i = 2$ then the boundedness range is the interval $(1, 2]$.



Break: 10:20-10:40

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Bernard Helffer

Université de Nantes, France

Maximal estimates for the Kramers-Fokker-Planck operator with electromagnetic field (after Helffer-Nier, Karaki, Helffer-Karaki)

Wednesday, 19 August

10:40 - 11:20

Abstract: In continuation of a work by Z. Karaki on the torus, we consider the Kramers-Fokker-Planck operator (KFP) with an external electromagnetic field on R^d. We show a maximal type estimate on this operator using a nilpotent approach for vector field polynomial operators in connection with induced representations of a nilpotent graded Lie algebra. This estimate leads to an optimal characterization of the domain of the closure of the KFP operator.

Bernard Helffer

Université de Nantes, France

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Alessio Martini

University of Birmingham, UK

Spectral multipliers for spherical Grushin operators

Wednesday, 19 August
11:20-12:00

Abstract: We consider degenerate elliptic operators of Grushin type on the d-dimensional sphere, which are singular on a k-dimensional sphere for some $k < d$. For these operators we prove a spectral multiplier theorem of Mihlin-Hörmander type, which is optimal whenever $2k$ is not greater than d. The proof hinges on suitable weighted spectral cluster bounds, which in turn depend on precise estimates for ultraspherical polynomials.
This is joint work with Valentina Casarino and Paolo Ciatti.

Alessio Martini

University of Birmingham, UK

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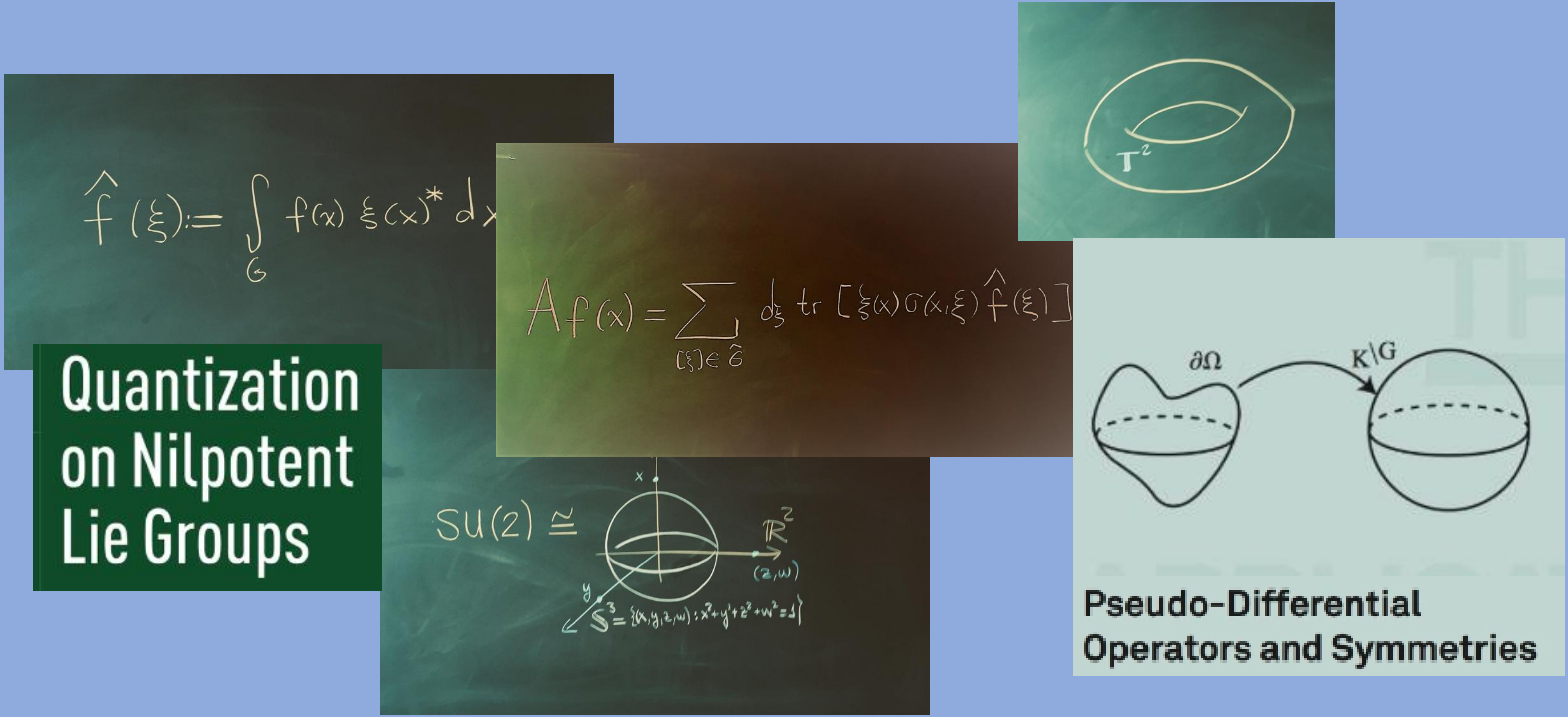

Break: 12:00-2:00

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Hans G. Feichtinger

University of Vienna, Austria

Invariant function spaces as double modules

Wednesday, 19 August

2:00-2:40

Abstract:

In this talk we would like to take a fresh look on double Banach modules $\boldsymbol{B}$ i.e. For simplicity, Banach spaces of tempered distributions which show invariance under translation and modulation. Hence Banach modules with respect to convolution and pointwise multiplication. This setting has been considered longago in [BdFe83] and more recently in [DiPiVi15-1] and [FeGu20]. These Banach spaces are invariant under translation as well as under modulation, or under the combined (!non-commutative) action of the reduced Heisenberg group (respectively the action of phase space $R^d \times \widehat{R^d}$ via a projective representation), known as Schrödinger representation. For these two module operations one can define a closure (hull) and kernel (essential part) operation. The talk will concentrate on the characterization of the resulting family of Banach modules which can be arranged in the form of a diagram). In important role for these characterizations is played by approximate units in Beurling algebras $L^1_w(R^d)$ on the one hand, and in the corresponding Fourier-Beurling algebras on the other hand. While [DiPiVi15-1] concentrates on the minimal spaces (the closure of $\mathcal{S}(R^d)$) dual spaces are always maximal. Consequently the diagram reduces to just one space for the case of reflexive Banach spaces, while it can have up to 6 different spaces for the choice $\boldsymbol{B} = C_0(R^d)$ resp. $L^\infty(R^d)$. The author suggests to investigate systematically the shape of this diagram for a large variety of Banach spaces arising in Fourier Analysis.

Hans G. Feichtinger

University of Vienna, Austria

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Uwe Kähler

Aveiro University, Portugal

Quaternionic Volterra operators and triangular representations

Wednesday, 19 August
2:40-3:20

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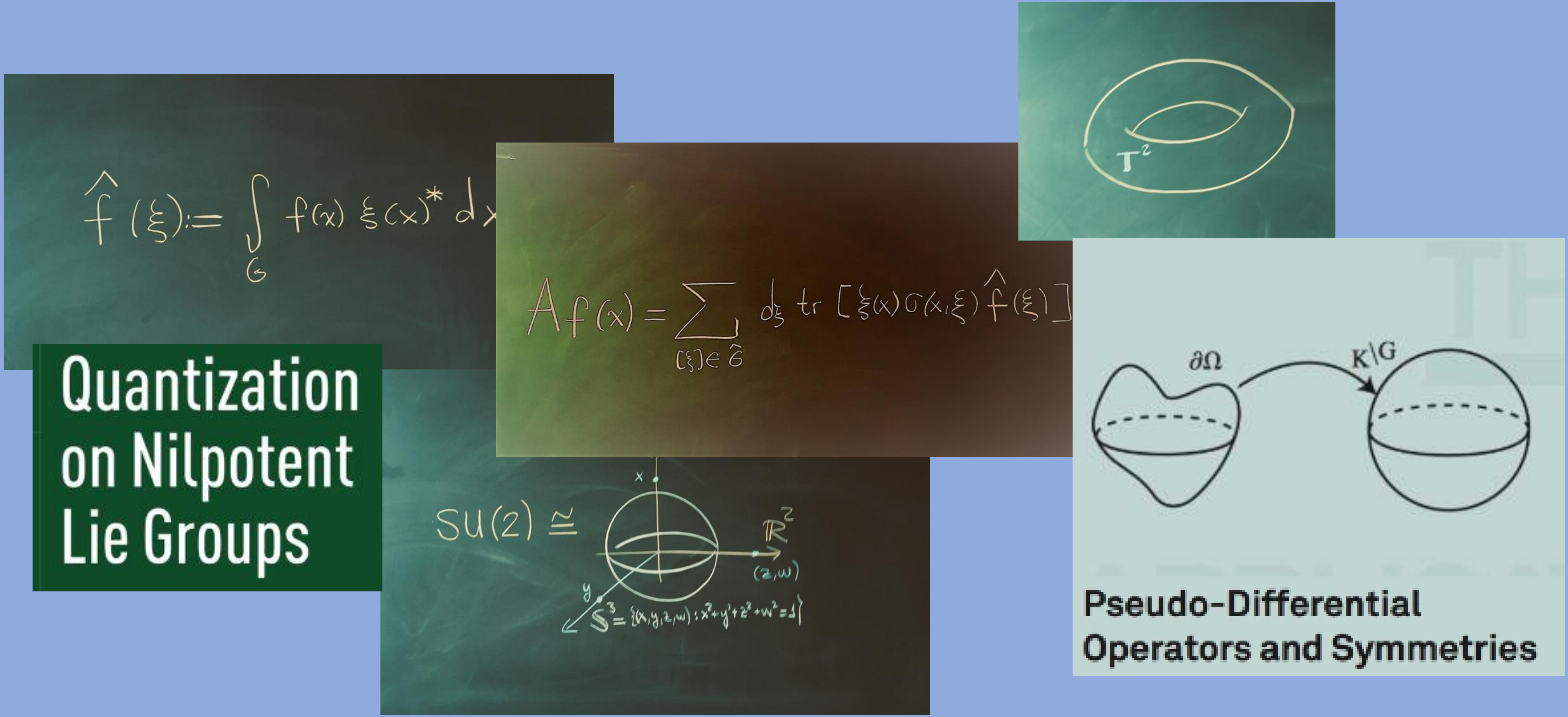
Abstract:

One of the principal problems in studying spectral theory for quaternionic or Clifford-algebra-valued operators lies in the fact that due to the noncommutativity many methods from classic spectral theory are not working. For instance, even in the simplest case of finite rank operators there are different notions of left and right spectrum. Hereby, the notion of a left spectrum has little practical use while the notion of a right spectrum is based on a nonlinear eigenvalue problem. In the present talk we will introduce the notion of S-spectrum as a natural way to consider a spectrum in a noncommutative setting. We will use it to discuss quaternionic Volterra operators and triangular representation of quaternionic operators similar to the classic approaches by Gohberg, Krein, Livsic, Brodskii and de Branges. To this end we introduce spectral integral representations with respect to quaternionic chains and discuss the concept of P-triangular operators in the quaternionic setting.

Uwe Kähler

Aveiro University, Portugal



Break: 3:20-3:40

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Hartmut Führ

Aachen University, Germany

Coarse geometric methods for generalized wavelet approximation theory

Wednesday, 19 August
3:40-4:20

Abstract: Generalized wavelet systems arise from the action of suitable semidirect products $R^d \times H$ on $L^2(R^d)$. Here H denotes a matrix group acting on functions by dilations. The approximation theoretic properties of these systems are coded in the associated *coorbit spaces*, defined by imposing suitably weighted L^p-norms on the transform side. Understanding the dependence of the scale of *coorbit spaces* on the dilation group is one of the basic problems of the theory, and generally not well understood. The talk translates this question into a problem from the domain of coarse geometry. Under suitable technical assumptions on two dilation groups H_1, H_2, the dual action of these groups induces a map ϕ: $H_1 \to H_2$, and it can be shown that H_1 and H_2 have the same scale of *coorbit spaces* if and only if ϕ is a quasi-isometry with respect to word metrics defined on H_1 and H_2. We present several applications of this argument. (Based on joint work with Rene Koch and Jordy van Velthoven).

Hartmut Führ

Aachen University, Germany

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Jeremy Tyson

University of Illinois at Urbana-Champaign, USA

Geometric measure theory and Fourier analysis on the sub-Riemannian Heisenberg group

Wednesday, 19 August
4:20-5:00

Abstract: I will discuss results in the geometry of sets, measures and mappings in the Heisenberg group equipped with a sub-Riemannian metric. I will start by reviewing some older work, joint with Balogh, Durand Cartagena, Faessler and Mattila, on dimension distortion under horizontal and vertical projection maps associated to canonical semidirect decompositions of the Heisenberg group. Next, I will survey more recent results on the densities of measures and the classification of uniform measures--in the spirit of classical theorems of Marstrand and Preiss--in the Heisenberg group equipped with the Koranyi metric. This work is joint with Chousionis and Magnani. Finally, I will briefly outline a recent approach to dimension estimates via energy integrals using the Heisenberg group Fourier transform, as formulated in Fourier coefficients language by Bahouri-Chemin-Danchin. This latter approach has been developed by my student Fernando Roman Garcia in his recent PhD thesis (UIUC, 2020).

Jeremy Tyson

University of Illinois at Urbana-Champaign, USA

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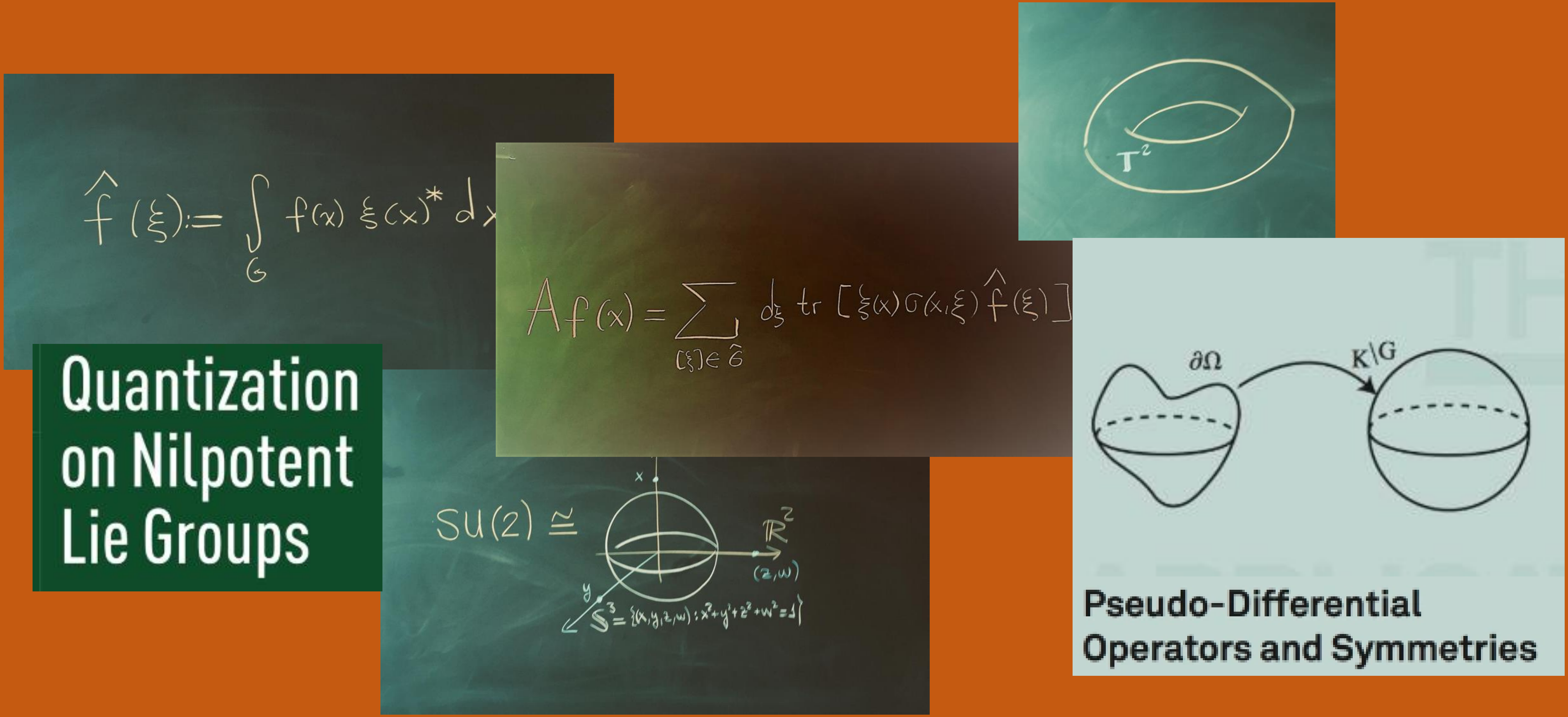

Thursday, 20 August

Anthony Dooley

University of Technology
Sydney, Australia

On the non-commutative Kirillov formula

Thursday, 20 August
9:00-9:40

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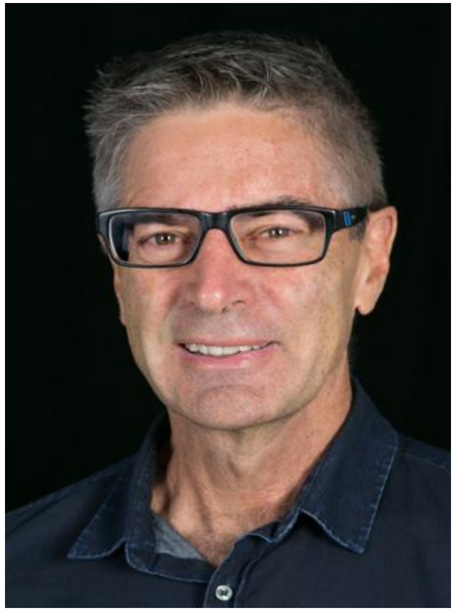

Abstract: The Kirillov orbit method associates to a coadjoint orbit of a Lie group G an irreducible representation of the group. One can calculate the trace of this representation from the Fourier transform of that orbit. However, if one starts not from the character, but from a general matrix entry, the Fourier transform is an interesting distribution on the Lie algebra. We describe the distributions which arise, their support, singular support and their convolutions.

Anthony Dooley

University of Technology Sydney, Australia

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Gregory S. Chirikjian

NUS, Singapore And
Johns Hopkins University, USA

Degenerate Diffusions on Unimodular Lie Groups

Thursday, 20 August
9:40-10:20

Abstract: In physical modeling problems, unimodular Lie groups such as the rotation group and group of handedness-preserving rigid-body motions of n-dimensional Euclidean space, $SE(n)$, play important roles. Degenerate diffusions on $SE(n)$ (i.e., diffusion equations in which fewer second-order Lie derivatives appear in the equation than the dimension of the group, $d = n(n+1)/2$), play important roles in DNA statistical mechanics when $n = 3$, and in phase noise in optical communication systems and robotic state estimation when n=2. The presenter has developed concepts of Gaussian distributions on these groups to accurately describe short-time solutions, and has applied existing methods of noncommutative harmonic analysis for long-time solutions. These methods apply to other broad classes of unimodular Lie groups as well. This will be reviewed together with a new result that the presenter published recently with coauthors: the cotangent bundle of a non-unimodular matrix Lie group can always be endowed with a group operation that makes the cotangent bundle a unimodular matrix Lie group. An example from mathematical finance in which the cotangent bundle of the affine group of the line illustrates this (https://www.mdpi.com/1099-4300/22/4/455).

Gregory S. Chirikjian

NUS, Singapore And John Hopkins University, USA

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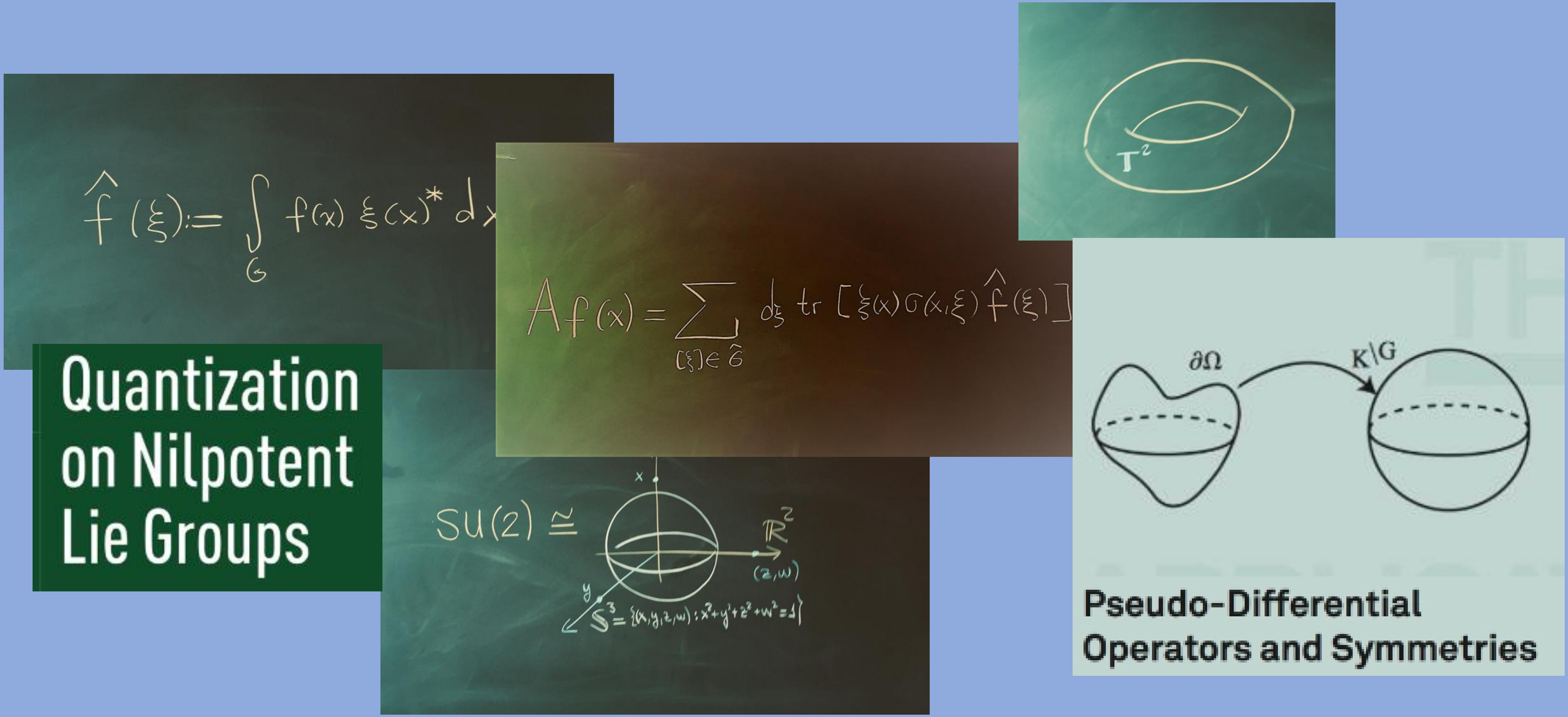

Break: 10:20-10:40

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Karlheinz Gröchenig

University of Vienna, Austria

New Function Spaces associated to Representations of Nilpotent Lie Groups and Generalized Time-Frequency Analysis

Thursday, 20 August
10:40-11:20

Abstract: We study function spaces that are related to square-integrable, irreducible, unitary representations of several low-dimensional nilpotent Lie groups. These are new examples of coorbit theory and yield new families of function spaces on R^d. The concrete realization of the representation suggests that these function spaces are useful for generalized time-frequency analysis or phase-space analysis.

Karlheinz Gröchenig

University of Vienna, Austria

Noncommutative conference

18–20 August 2020, Ghent University

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Franz Luef

Norwegian University of Science and Technology, Norway

Gaussian Gabor frames and Kaehler geometry

Thursday, 20 August
11:20-12:00

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Abstract: The commutation relations for the time-frequency shifts of the lattice points of a Gabor system yield to a skew-symmetric matrix that encodes the noncommutativity of the Gabor system. This skew-symmetric matrix induces a symplectic form and Kaehler structures compatible on the complex torus associated to the lattice. The Gaussians associated to the compatible Kaehler structures are parameterized by the Siegel upper half space.
We study Gabor systems generated by these Gaussians using tools from Kaehler geometry, such as the Seshadri constant of the complex torus of the adjoint lattice, to establish multi-variate analogs of the Lyubarskii-Seip-Wallsten theorem. This approach yields as a consequence for the univariate case analytic expressions for the frame bounds in terms of eta and theta functions.
This is joint work with Xu Wang (NTNU).

Franz Luef

Norwegian University of Science and Technology, Norway

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Ingrid Beltiţă

Institute of Mathematics of the Romanian Academy, Romania

C-algebras of solvable Lie groups and their finite-dimensional approximation properties*

Thursday, 20 August
12:00 - 12:40

Abstract: We present some finite-approximation properties of the C^*-algebras of certain simply connected Lie groups. We prove that, if their primitive ideal spectrum is T1, these C^*-algebras can be embedded in C^*-algebras that are inductive limits of finite-dimensional C^*-algebras. To this end, we show that if G is solvable and its action on the centre of $[G, G]$ has at least one imaginary root, then there are nonempty open and compact subsets of $Prim(G)$.

This is joint work with Daniel Beltiţă (IMAR).

Ingrid Beltiţă

Institute of Mathematics of the Romanian Academy, Romania

Noncommutative conference

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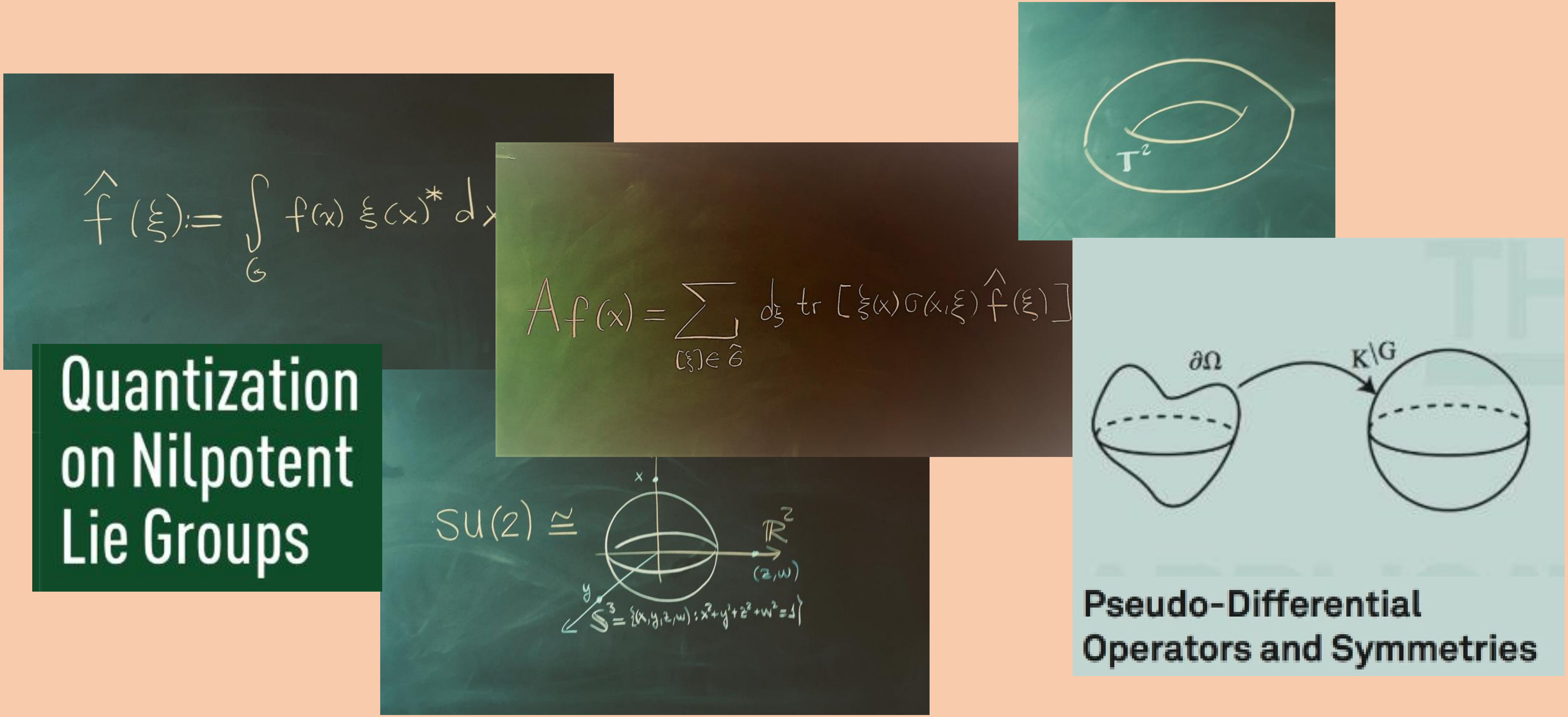
Quantization
on Nilpotent
Lie Groups
Pseudo-Differential
Operators and Symmetries
$\widehat{f}(\xi) := \int_G f(x)\,\xi(x)^*\,dx$
$Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \,\mathrm{tr}\,[\xi(x)\sigma(x,\xi)\widehat{f}(\xi)]$
$\mathbb{T}^2$
$\partial\Omega$
$K\backslash G$
$SU(2) \cong$
$\mathbb{R}^2$
(z,w)
$\mathbb{S}^3 = \{(x,y,z,w) : x^2+y^2+z^2+w^2=1\}$

Closing

Poster session.

$$Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \, \mathrm{tr}\left[\xi(x)\,\sigma(x,\xi)\,\widehat{f}(\xi)\right]$$

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Overcoming noncommutativity without commuting

Organisers:
Ghent Analysis & PDE Center

Hardy Inequalities on Homogeneous Groups

Michael Ruzhansky[1,2,3] and Durvudkhan Suragan[4]

[1]Imperial College London, [2]Ghent Analysis & PDE Center, Ghent University, [3]Queen Mary University of London, [4]Nazarbayev University

Description

This open access book provides an extensive treatment of Hardy inequalities and closely related topics from the point of view of Folland and Stein's homogeneous (Lie) groups. The place where Hardy inequalities and homogeneous groups meet is a beautiful area of mathematics with links to many other subjects. While describing the general theory of Hardy, Rellich, Caffarelli-Kohn-Nirenberg, Sobolev, and other inequalities in the setting of general homogeneous groups, the authors pay particular attention to the special class of stratified groups. In this environment, the theory of Hardy inequalities becomes intricately intertwined with the properties of sub-Laplacians and subelliptic partial differential equations. These topics constitute the core of this book and they are complemented by additional, closely related topics such as uncertainty principles, function spaces on homogeneous groups, the potential theory for stratified groups, and the potential theory for general Hörmander's sums of squares and their fundamental solutions.

Content

- Analysis on Homogeneous Groups;
- Hardy Inequalities on Homogeneous Groups;
- Rellich, Caffarelli-Kohn-Nirenberg, and Sobolev Type Inequalities;
- Fractional Hardy Inequalities;
- Integral Hardy Inequalities on Homogeneous Groups;
- Horizontal Inequalities on Stratified Groups;
- Hardy-Rellich Inequalities and Fundamental Solutions;
- Geometric Hardy Inequalities on Stratified Groups;
- Uncertainty Relations on Homogeneous Groups;
- Function Spaces on Homogeneous Groups;
- Elements of Potential Theory on Stratified Groups;
- Hardy and Rellich Inequalities for Sums of Squares of Vector Fields.

Hardy Inequality

Classical Hardy inequality in the Euclidean space $\mathbb{R}^n$: for all $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|_E^2} dx,$$

where the constant $(n-2)^2/4$ is sharp, $|\cdot|_E$ is the standard Euclidean norm, and ∇ is the standard gradient on $\mathbb{R}^n$.

Hardy inequality on homogeneous Lie groups $\mathbb{G}$: for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, we have

$$\int_{\mathbb{G}} |\mathcal{R} f(x)|^2 dx \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^2} dx,$$

where the constant $(Q-2)^2/4$ is sharp, Q is the homogeneous dimension of $\mathbb{G}$, $|\cdot|$ is any homogeneous quasi-norm on homogeneous Lie groups $\mathbb{G}$, and $\mathcal{R}$ is the radial derivative, $\mathcal{R}f(x) := \frac{\partial f(x)}{\partial |x|}$.

100 Years of Hardy Inequalities

G.H. Hardy and Harald Bohr (from Wikipedia page)

G.H. Hardy reported Harald Bohr as saying 'all analysts spend half their time hunting through the literature for inequalities which they want to use but cannot prove'.

Ferran Sunyer i Balaguer Prize

This monograph is **the winner of the 2018 Ferran Sunyer i Balaguer Prize**, a prestigious award for books of expository nature presenting the latest developments in an active area of research in mathematics. As can be attested as the winner of such an award, it is a vital contribution to literature of analysis not only because it presents a detailed account of the recent developments in the field, but also because the book is accessible to anyone with a basic level of understanding of analysis. Undergraduate and graduate students as well as researchers from any field of mathematical and physical sciences related to analysis involving functional inequalities or analysis of homogeneous groups will find the text beneficial to deepen their understanding.

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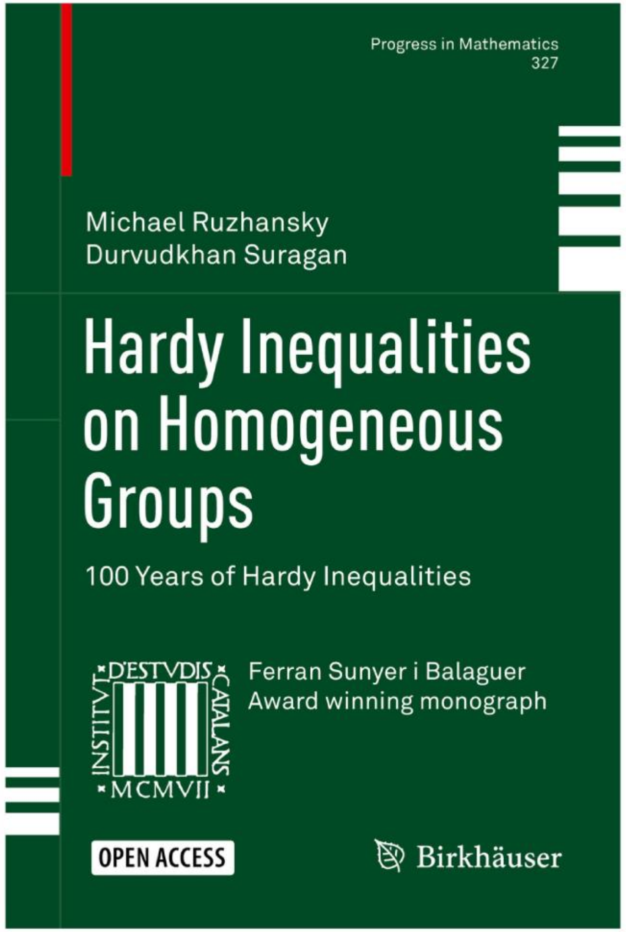

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L^p-bounds for pseudo-differential operators on graded Lie groups

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Abstract

We present the sharp L^p-estimates for pseudo-differential operators on arbitrary graded Lie groups proved by the authors in [1]. The results are presented within the setting of the global symbolic calculus on graded Lie groups by using the Fourier analysis associated to every graded Lie group which extends the usual one due to Hörmander on $\mathbb{R}^n$. The main result extends the classical Fefferman's sharp theorem on the L^p-boundedness of pseudo-differential operators for Hörmander classes on $\mathbb{R}^n$ to general graded Lie groups, also adding the borderline $\rho = \delta$.

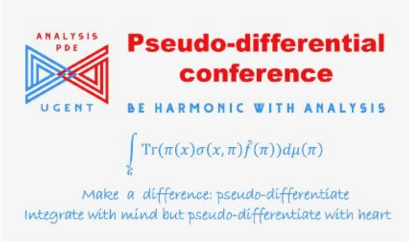
Introduction

- The investigation of the L^p boundedness of pseudo-differential operators is a crucial task for a large variety of problems in mathematical analysis and its applications, mainly due to its consequences for the regularity, approximation and existence of solutions on L^p-Sobolev spaces.
- There is an extensive literature on the subject, in particular, devoted to operators associated with symbols belonging to the Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, (see for instance, J.J. Kohn and L. Nirenberg [5], L. Hörmander [4] and C. Fefferman [2]).
- Our main goal is to extend a classical and sharp result by C. Fefferman [2] and to provide a critical order for the L^p-boundedness of pseudo-differential operators on graded Lie groups based on the quantization procedure developed by the third author and V. Fischer in [3].
- Our main estimate recover the sharp Fefferman theorem on $\mathbb{R}^n$, adding the critical case $\rho = \delta$.

Fourier analysis on nilpotent Lie groups

- Let G be a simply connected nilpotent Lie group and let $\widehat{G}$ be its unitary dual.
- Let $\exp_G : \mathfrak{g} \to G$, be the exponential mapping on G. The Schwartz class on G, is defined by those $f \in C^\infty(G)$, such that $f \circ \exp_G \in \mathscr{S}(\mathfrak{g})$, with $\mathfrak{g} \simeq \mathbb{R}^{\dim(G)}$.
  The Fourier transform of $f \in \mathscr{S}(G)$, at $\pi \in \widehat{G}$, is defined by:

$$\widehat{f}(\pi) = \int_G f(x)\pi(x)^* dx$$



Pseudo-differential operators on graded Lie groups

- Roughly speaking, a pseudo-differential operator is a continuous linear operator on $\mathscr{S}(G)$, defined by the (quantization) formula:

$$\mathrm{Op}(\sigma)f(x) = \int_{\widehat{G}} \mathrm{Tr}\left(\pi(x)\sigma(x,\pi)\widehat{f}(\pi)\right) d\mu(\pi)$$

  In such a case, we say that σ is the symbol associated with $\mathrm{Op}(\sigma)$. We have denoted by $d\mu(\pi)$ the Plancherel measure on $\widehat{G}$.
- A Rockland operator is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree $\nu = \nu_{\mathcal{R}}$ and such that, for every unitary irreducible non-trivial representation $\pi \in \widehat{G}$, $\pi(\mathcal{R})$ is injective on $\mathcal{H}^\infty_\pi$; $\sigma_{\mathcal{R}}(\pi) = \pi(\mathcal{R})$ is the symbol associated to $\mathcal{R}$.
- It can be shown that a Lie group G is graded if and only if there exists a differential Rockland operator on G.
- The basic example of graded Lie group is the Heisenberg group $\mathbb{H}^n$.

Hörmander classes on graded Lie groups

Hörmander classes on the phase space $G \times \widehat{G}$ can be defined by using Rockland operators. Indeed, the Hörmander class of order m, and of type (ρ, δ), $S^m_{\rho,\delta}(G \times \widehat{G})$, is defined by those *symbols* σ satisfying symbol inequalities of the kind:

L^p-L^p, H^1-L^1 and L^∞-BMO-boundedness of pseudo-differential operators

Theorem (Fefferman type estimates on Graded Lie groups)

Let G be a graded Lie group of homogeneous dimension Q. Let $A \equiv Op(\sigma) : C^\infty(G) \to \mathscr{D}'(G)$ be a pseudo-differential operator with symbol $\sigma \in S^{-m}_{\rho,\delta}(G \times \widehat{G})$, $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$. Then,

- *(a) if $m = \frac{Q(1-\rho)}{2}$, then A extends to a bounded operator from $L^\infty(G)$ to $BMO(G)$, from the Hardy space $H^1(G)$ to $L^1(G)$, and from $L^p(G)$ to $L^p(G)$ for all $1 < p < \infty$.*
- *(b) If $m \geq m_p := Q(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$, $1 < p < \infty$, then A extends to a bounded operator from $L^p(G)$ into $L^p(G)$.*

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L^p-L^q boundedness of pseudo-differential operators on smooth manifolds and its applications to nonlinear equations

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Abstract

We present the boundedness of global pseudo-differential operators on smooth manifolds obtained in [3]. By using the notion of global symbol [6, 7] we extend a classical condition of Hörmander type to guarantee the L^p-L^q-boundedness of global operators. First we investigate L^p-boundedness of pseudo-multipliers in view of the Hörmander-Mihlin condition. Later, we concentrate to settle L^p-L^q boundedness of the Fourier multipliers and pseudo-differential operators for the range $1 < p \leq 2 \leq q < \infty$. Finally, we present applications of our boundedness theorems to the well-posedness properties of different types of nonlinear partial differential equations.

Introduction

- The boundedness results for pseudo-differential operators in their own right are interesting and important but also these serve as crucial tools to tackle several important problems of mathematics, in particular, of nonlinear PDEs. We refer [8, 7] and references therein for available extensive literature on these topics.
- Our main purpose is to extend seminal and classical result of L. Hörmander on L^p-L^q boundedness for pseudo-differential operators [5] to manifolds using the nonharmonic Fourier analysis and global quantization developed the last two authors [6] (see also [4]). Recently, L^p-L^q boundedness results for Fourier multipliers have been established in [1, 2] in the context of unimodular locally compact groups.
- To prove L^p-L^q boundedness of global operators we establish and apply the Paley-inequality and Hausdorff-Young-Paley inequality in the setting of nonharmonic analysis.

Fourier analysis associated to a model operator L on M

- Let L be a pseudo-differential operator (need not be elliptic or self-adjoint) of order m on the interior M of a smooth manifold with boundary $\overline{M}$ in the sense of Hörmander.
- Assume that some boundary conditions (BC) are fixed and lead to a discrete spectrum with a family of eigenfunctions yielding a Riesz basis in $L^2(M)$.
- The discrete spectrum is $\{\lambda_\xi \in \mathbb{C} : \xi \in \mathcal{I}\}$ of L with corresponding eigenfunctions in $L^2(M)$ denoted by u_ξ which satisfy the boundary conditions (BC).
- The conjugate spectral problem is
$$L^* v_\xi = \overline{\lambda_\xi} v_\xi \text{ in } M, \quad \text{for all } \xi \in \mathcal{I},$$
which we equip with the conjugate boundary conditions $(\mathrm{BC})^*$. We further assume that the functions u_ξ, v_ξ are normalised and the systems $\{u_\xi\}_{\xi\in\mathcal{I}}$ and $\{v_\xi\}_{\xi\in\mathcal{I}}$ are bi-orthogonal.
- The space $C_L^\infty(\overline{M}) := \cap_{k=1}^\infty \mathrm{Dom}(L^k)$, where $\mathrm{Dom}(L^k) := \{f \in L^2(M) \,|\, L^j f \in \mathrm{Dom}(L),\ j = 0, 1, \cdots, k-1\}$, so that the boundary condition (BC) are satisfied by the operators L^j.
- The L-Fourier transform of $f \in C_L^\infty(\overline{M})$ is defined by
$$(\mathcal{F}_L f)(\xi) := \widehat{f}(\xi) := \int_{\overline{M}} f(x)\overline{v_\xi(x)}\, dx.$$
- We refer to [6, 4] for more details.

L-pseudo-differential operators on manifolds with boundary

- An L-pseudo-differential operator is a continuous linear operator $A : C_L^\infty(M) \to C_L^\infty(M)$, defined by
$$Af(x) = \sum_{\xi\in\mathcal{I}} u_\xi(x) m(x,\xi)\widehat{f}(\xi), \ f \in \mathrm{Dom}(A).$$
In this case, the function $m : M \times \mathcal{I} \to \mathbb{C}$, is called the L-symbol associated with A.
- Denote by L° the densely defined operator given by $L^\circ u_\xi = \overline{\lambda_\xi} u_\xi, \quad \xi \in \mathcal{I}$.
- An L-pseudo-differential operator A is called L-pseudo-multiplier if there exists a continuous function $\tau_m : \overline{M} \times \mathbb{R} \to \mathbb{C}$, such that for every $\xi \in \mathcal{I}$, and $x \in M$, we have $m(x,\xi) = \tau_m(x, |\lambda_\xi|)$. So, A is given by
$$Af(x) \equiv \tau_m(x, \sqrt{L^\circ L})f(x) := \sum_{\xi\in\mathcal{I}} u_\xi(x)\tau_m(x, |\lambda_\xi|)\widehat{f}(\xi).$$

Assumptions for H-M theorem

- There exist $-\infty < \gamma_p^{(1)}, \gamma_p^{(2)} < \infty$, satisfying
$$\|u_\xi\|_{L^p(M)} \lesssim |\lambda_\xi|^{\gamma_p^{(1)}}, \ \|v_\xi\|_{L^p(M)} \lesssim |\lambda_\xi|^{\gamma_p^{(2)}},$$
for $1 \leq p \leq \infty$.
- The operator $\sqrt{L^\circ L}$ satisfies the Weyl-eigenvalue counting formula
$$N(\lambda) := \sum_{\xi\in\mathcal{I} : |\lambda_\xi| \leq \lambda} = O(\lambda^Q), \ \lambda \to \infty,$$
where $Q > 0$. If $Q' > Q$, then $N(\lambda) = O(\lambda^{Q'})$, $\lambda \to \infty$, so that we assume that Q is the smallest real number with this property.

L^p-boundedness of L-pseudo-multipliers operators on M

Theorem (Hörmander-Mihlin (H-M) theorem for pseudo-multipliers)

Let M be a smooth manifold with boundary and let $A : C_L^\infty(M) \to C_L^\infty(M)$ be an L-pseudo-multiplier. Let us assume that τ_m satisfies the following Hörmander condition,
$$\|\tau_m\|_{l.u.\mathcal{H}^s} = \sup_{r>0, x\in\overline{M}} r^{(s-\frac{Q_m}{2})} \|\langle\cdot\rangle^s \mathscr{F}[\tau_m(x,\cdot)\psi(r^{-1}\cdot)]\|_{L^2(\mathbb{R})} < \infty,$$
for $s > \max\{1/2, \gamma_p + Q + (Q_m/2)\}$. Then $A \equiv T_m : L^p(M) \to L^p(M)$ extends to a bounded linear operator for all $1 < p < \infty$.

L^p-L^q-boundedness of L-Fourier multipliers operators on M

Theorem (Hörmander theorem for L-Fourier multipliers)

Let $1 < p \leq 2 \leq q < \infty$ and assume that
$$\sup_{\xi\in\mathcal{I}} \left|\frac{\|v_\xi\|_{L^\infty(M)}}{\|u_\xi\|_{L^\infty(M)}}\right| < \infty \quad \text{and} \quad \sup_{\xi\in\mathcal{I}} \left|\frac{\|u_\xi\|_{L^\infty(M)}}{\|v_\xi\|_{L^\infty(M)}}\right| < \infty.$$
Suppose that $A : C_L^\infty(M) \to C_L^\infty(M)$ is an L-Fourier multiplier with L-symbol $\sigma_{A,L}$ on M, that is, A satisfies
$$\mathcal{F}_L(Af)(\xi) = \sigma_{A,L}(\xi)\mathcal{F}_L f(\xi), \quad \text{for all } \xi \in \mathcal{I},$$
where $\sigma_{A,L} : \mathcal{I} \to \mathbb{C}$ is a function. Then we have
$$\|A\|_{\mathscr{B}(L^p(\overline{M}), L^q(\overline{M}))} \lesssim \sup_{s>0} s \left(\sum_{\substack{\xi\in\mathcal{I} \\ |\sigma_{A,L}(\xi)| \geq s}} \max\{\|u_\xi\|^2_{L^\infty(\overline{M})}, \|v_\xi\|^2_{L^\infty(\overline{M})}\} \right)^{\frac{1}{p}-\frac{1}{q}}$$

L^p-L^q-boundedness of L-pseudo-differential operators on M

Theorem (Hörmander theorem for L-pseudo-differential operators)

Let $1 < p \leq 2 \leq q < \infty$ and assume that
$$\sup_{\xi\in\mathcal{I}} \left|\frac{\|v_\xi\|_{L^\infty(M)}}{\|u_\xi\|_{L^\infty(M)}}\right| < \infty \quad \text{and} \quad \sup_{\xi\in\mathcal{I}} \left|\frac{\|u_\xi\|_{L^\infty(M)}}{\|v_\xi\|_{L^\infty(M)}}\right| < \infty.$$
Suppose that $A : C_L^\infty(M) \to C_L^\infty(M)$ is a continuous linear operators with L-symbol $\sigma_{A,L} : M \times \mathcal{I} \to \mathbb{C}$, where M is a compact manifold (with or without boundary), satisfying
$$\|\sigma_{A,L}\|_{(\beta)} := \sup_{s>0, y\in\overline{M}} s \left(\sum_{\substack{\xi\in\mathcal{I} \\ |\partial_y^\beta \sigma_{A,L}(y,\xi)| \geq s}} \max\{\|u_\xi\|^2_{L^\infty(\overline{M})}, \|v_\xi\|^2_{L^\infty(\overline{M})}\} \right)^{\frac{1}{p}-\frac{1}{q}} < \infty,$$
for all $|\beta| \leq \left[\frac{\dim(M)}{q}\right] + 1$, where ∂_y denotes the local partial derivative. If $\partial M \neq \emptyset$, let us assume additionally that $supp(\sigma_{A,L}) \subset \{(y,\xi) \in M \times \mathcal{I} : y \in M \setminus V\}$ where $V \subset M$ is an open neighbourhood of the boundary ∂M. Then A admits a bounded extension from $L^p(M)$ into $L^q(M)$.

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Applications to non-linear equations

We apply our L^p-L^q boundedness to establish the well-posedness properties the solutions of nonlinear equations in the space $L^\infty(0,T; L^2(M))$.

- In the nonlinear stationary problem case, we consider the following equation in $L^2(M)$
$$Au = |Bu|^p + f,$$
where $A, B : L^2(M) \to L^2(M)$ are linear operators and $1 \leq p < \infty$.
- In the case of the nonlinear heat equation, we study the Cauchy problem in the space $L^\infty(0,T; L^2(M))$
$$u_t(t) - |Bu(t)|^p = 0, u(0) = u_0,$$
where B is a linear operator in $L^2(M)$ and $1 \leq p < \infty$.
- In the non-linear wave equation case, we study the following initial value problem (IVP)
$$u_{tt}(t) - b(t)|Bu(t)|^p = 0,$$
$$u(0) = u_0, \ u_t(0) = u_1,$$
where b is a positive bounded function depending only on time, B is a linear operator in $L^2(M)$ and $1 \leq p < \infty$.

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The Harmonic Oscillator on The Heisenberg Group

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Abstract

We present a notion of harmonic oscillator on the Heisenberg group $\mathbf{H}_n$. This operator forms a natural analogue of the harmonic oscillator on $\mathbb{R}^n$.

Introduction

Our ansatz is based on a few reasonable assumptions: the harmonic oscillator on $\mathbf{H}_n$ should be a negative sum of squares of operators related to the sub-Laplacian on $\mathbf{H}_n$, essentially self-adjoint with purely discrete spectrum, and its eigenvectors should be smooth functions and form an orthonormal basis of $L^2(\mathbf{H}_n)$. This leads to a differential operator on $\mathbf{H}_n$ which is determined by the Dynin-Folland Lie algebra, a stratified 3-step nilpotent Lie algebra.

Ansatz

Our approach is motivated by the following three realizations of the classical harmonic oscillator $\mathcal{Q}_{\mathbb{R}^n}$ on $\mathbb{R}^n$:

(R1) the negative sum of squares $-\Delta + |x|^2$ of partial derivatives of order 1 and coordinate multiplication operators;

(R2) the Weyl and Kohn-Nirenberg quantizations on $\mathbb{R}^n$ of the symbol $\sigma(x,\xi) := |\xi|^2 + |x|^2$ with $x, \xi \in \mathbb{R}^n$;

(R3) the image $d\rho_1(-\mathcal{L}_{\mathbf{H}_n})$ of the negative sub-Laplacian $-\mathcal{L}_{\mathbf{H}_n}$ on $\mathbf{H}_n$ under the infinitesimal Schrödinger representation $d\rho_1$ (of Planck's constant equal to 1) of the Heisenberg Lie algebra $\mathfrak{h}_n$.

The Schrödinger representation ρ_1 of $\mathbf{H}_n$ acting on $L^2(\mathbb{R}^n)$ and the associated Lie algebra representation, naturally acting on $\mathcal{S}(\mathbb{R}^n)$, clearly relate each of the realisations (R1) - (R3) to the others. One can expect that similar realisations should be available for the canonical harmonic oscillator on $\mathbf{H}_n$.

Main Result

The harmonic oscillator $\mathcal{Q}_{\mathbf{H}_n}$ on the Heisenberg group $\mathbf{H}_n$ has a purely discrete spectrum $\mathrm{spec}(\mathcal{Q}_{\mathbf{H}_n}) \subset (0,\infty)$. The number of its eigenvalues, counted with multiplicities, which are less or equal to $\lambda > 0$ is asymptotically (as $\lambda \to \infty$) given by

$$N(\lambda) \sim \lambda^{\frac{6n+3}{2}},$$

and the magnitude of the eigenvalues is asymptotically equal to

$$\lambda_k \sim k^{\frac{2}{6n+3}} \text{ for } k = 1, 2, \ldots.$$

Moreover, the eigenvectors of $\mathcal{Q}_{\mathbf{H}_n}$ are in $\mathcal{S}(\mathbf{H}_n)$ and form an orthonormal basis of $L^2(\mathbf{H}_n)$.

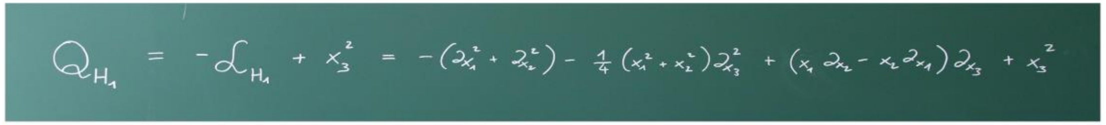

Figure 1: The Harmonic Oscillator on $\mathbf{H}_1$.

Definition

The Dynin-Folland Lie group $\mathbf{H}_{n,2} = \mathbb{R}^{2n+2} \rtimes \mathbf{H}_n$ acts on $f \in L^2(\mathbf{H}_n)$ by the unitary irreducible representation

$$(\pi(z,y,x)f)(t) = e^{iz} e^{i\langle t \cdot \frac{1}{2}x, y\rangle} f(t \cdot x),$$

where $t \cdot \frac{1}{2}x$ and $t \cdot x$ denote the $\mathbf{H}_n$-group products of the corresponding coordinate vectors.
For the basis $\{X_1, \ldots, Y_{2n+1}, Z\}$ of its Lie algebra $\mathfrak{h}_{n,2}$ we define the **harmonic oscillator** on $\mathbf{H}_n$ by

$$\mathcal{Q}_{\mathbf{H}_n} := d\pi(-\mathcal{L}_{\mathbf{H}_{n,2}}) = -d\pi(X_1)^2 - \ldots - d\pi(X_{2n})^2 - d\pi(Y_{2n+1})^2,$$

where $-\mathcal{L}_{\mathbf{H}_{n,2}}$ is the sub-Laplacian on $\mathbf{H}_{n,2}$. Its natural domain includes the smooth vectors $\mathcal{H}_\pi^\infty \cong \mathcal{S}(\mathbf{H}_n)$.

Interpretation

The essentially self-adjoint differential operator $\mathcal{Q}_{\mathbf{H}_n}$ on $\mathbf{H}_n$ admits analogues of (R1) – (R3):

(R1') the differential operator $-\mathcal{L}_{\mathbf{H}_n} + x_{2n+1}^2$;

(R2') i) the Dynin-Weyl quantization on $\mathbf{H}_n$ of the symbol $\sigma(x,\xi) := \xi_1^2 + \cdots + \xi_{2n}^2 + x_{2n+1}^2$ with $(x,\xi) \in \mathbb{R}^{2n+1} \times \widehat{\mathbb{R}}^{2n+1}$;
ii) the Kohn-Nirenberg quantization, in the sense of [3], of the operator-valued symbol on $\mathbf{H}_n \times \widehat{\mathbf{H}}_n$ $\sigma(x,\rho_\lambda) := -\rho_\lambda(X_1)^2 - \ldots - \rho_\lambda(X_{2n})^2 + x_{2n+1}^2$;

(R3') the image $d\pi(-\mathcal{L}_{\mathbf{H}_{n,2}})$ of the sub-Laplacian $\mathcal{L}_{\mathbf{H}_{n,2}}$ on $\mathbf{H}_{n,2}$, here by definition.

Methods

$\mathcal{Q}_{\mathbf{H}_n}$ has purely discrete spectrum in $(0,\infty)$ by [5]. The asymptotic growth rate of its eigenvalues is obtained via a powerful method developed in [8]. The number of eigenvalues (with multiplicities), asymptotically behaves like the volumes of certain subsets of the coadjoint orbit $\mathcal{O}_\pi \subset \mathfrak{h}_{n,2}^*$ corresponding to representation $\pi \in \widehat{\mathbf{H}}_n$. The subsets in question are determined (up to a multiplicative constant) by a (any) homogeneous quasi-norm on $\mathfrak{g}^*$. The flatness of $\mathcal{O}_\pi$ and a convenient choice of quasi-norm facilitate the computations substantially.

Remark

The power $\frac{6n+3}{2}$ intimately related to the homogeneous structure of $\mathfrak{h}_{n,2}$: the nominator $6n+3$ is the homogenous dimension of the first two strata $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \subseteq \mathfrak{h}_{n,2}$, while the denominator 2 is the homogeneous degree of $-\mathcal{L}_{\mathbf{H}_{n,2}}$.

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Smoothing and Strichartz estimates for Degenerate Schrödinger Operators

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Abstract

In what follows we shall present some recent results about the validity of smoothing and Strichartz-type estimates for time-degenerate Schrödinger operators. These results have important applications in the study of the local well-posedness of the initial value problem (IVP) associated with the operators under consideration.

Time degenerate Schrödinger-type Operators

We shall consider the following classes of degenerate Schrödinger-type operators

$$\mathcal{L}_a = i\partial_t + t^{\alpha}\Delta_x + b(t,x)\cdot\nabla_x, \qquad (1)$$

$$\mathcal{L}_c = i\partial_t + c'(t)\Delta_x, \qquad (2)$$

where $\alpha > 0$, $b(t,x) = (b_1(t,x), ..., b_n(t,x))$ is such that, for all $j = 1, ..., n$, $b_j \in C([0,T], C_b^{\infty}(\mathbb{R}^n))$, while $c \in C^1(\mathbb{R})$.

The class (1) was considered in [1] where both homogeneous and inhomogeneous weighted local smoothing estimates are derived. These estimates are also employed to obtain local well-posedness results for the associated nonlinear IVP (see [1]).

The class (2) was studied in [2] where global weighted homogeneous smoothing estimates are proved by means of comparison principles. For the class (2) weighted Strichartz estimates are proved in [2] where the application to the local well-posedness of the semilinear IVP is given.

The main difference between the operators of the form (1) and (2) and the other Schrödinger operators studied so far **is the presence of degeneracies**. Specifically, the degeneracies are given by the coefficient t^{α} and $c'(t)$ in (1) and (2) respectively.

Why to study smoothing and Strichartz estimates?

These estimates give us important information about the regularity properties of the solution of the IVP. In particular:

- **The homogeneous smoothing** effect describes a gain of smoothness of the homogeneous solution of the IVP with respect to the smoothness of the initial data.
- **The inhomogeneous smoothing** effect describes a gain of smoothness of the solution of the inhomogeneous IVP with respect to the regularity of the inhomogeneous data.
- **Strichartz estimates** describe a gain of integrability instead of a gain of smoothness of the solution of the IVP.

Additionally, **these estimates are fundamental in order to prove the local well-posedness of the corresponding semilinear and nonlinear IVP** through the standard fixed point argument.

What are comparison principles? (Following [3])

Question: Given two operators $P_a(t,x,D_t,D_x)$ and $P_b(t,x,D_t,D_x)$ depending on two functions a and b respectively, is it possible to compare (in a suitable sense) the solutions of the HIVP (homogeneous IVP) for P_a and P_b if a and b are comparable (in a suitable sense)?

This is essentially what the **comparison principles** we refer to do, that is, **they translate a relation between a and b in a relation between the solutions of the HIVP for P_a and P_b.**

Example (see [3]). Let $a, \tilde{a} \in C^1(\mathbb{R})$ be real valued and strictly monotone on the support of a measurable function χ, and let $\sigma, \tau \in C^0(\mathbb{R})$. Then, if $\forall\xi \in \operatorname{supp}\chi$ we have

$$\frac{|\sigma(\xi)|}{|a'(\xi)|^{1/2}} \leq C\frac{|\tau(\xi)|}{|\tilde{a}'(\xi)|^{1/2}}, \quad \text{then} \quad \|\chi(D_x)\sigma(D_x)e^{ita(D)}\varphi\|_{L^2(\mathbb{R}^2)} \leq C\|\chi(D_x)\tau(D_x)e^{it\tilde{a}(D)}\varphi\|_{L^2(\mathbb{R}^2)}$$

for all $\varphi = \varphi(x)$ smooth enough.

Weighted local smoothing effect for $\mathcal{L}_a$

We consider the IVP

$$\begin{cases} \partial_t u = it^{\alpha}\Delta_x u + ib(t,x)\cdot\nabla_x u + f(t,x) \\ u(0,x) = u_0(x). \end{cases} \qquad (3)$$

When $b \equiv 0$ one can proceed by using Fourier analysis. However, in the general case $b \not\equiv 0$, the use of pseudo-differential calculus is needed.

Theorem (F.-Staffilani)

Let $u_0 \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Assume that, for all $j = 1, ..., n$, b_j is such that $b_j \in C([0,T], C_b^{\infty}(\mathbb{R}^n))$ and there exists $\sigma > 1$ such that

$$|\text{Im}\,\partial_x^{\gamma} b_j(t,x)|, |\text{Re}\,\partial_x^{\gamma} b_j(t,x)| \lesssim t^{\alpha}\langle x\rangle^{-\sigma-|\gamma|}, \quad x \in \mathbb{R}^n, \qquad (4)$$

and denote by $\lambda(|x|) := \langle x\rangle^{-\sigma}$ and by $\Lambda := \langle\xi\rangle$.
Then

(i) If $f \in L^1([0,T]; H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0\leq t\leq T} \|u(t)\|_s \leq C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)}\left(\|u_0\|_s + \int_0^T \|f(t)\|_s dt\right);$$

(ii) If $f \in L^2([0,T]; H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist two positive constants C_1, C_2 such that

$$\sup_{0\leq t\leq T} \|u(t)\|_s^2 + \int_0^T\int_{\mathbb{R}^n} t^{\alpha}\left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)\,dx\,dt \leq C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)}\left(\|u_0\|_s^2 + \int_0^T \|f(t)\|_s^2 dt\right);$$

(iii) If $\Lambda^{s-1/2}f \in L^2([0,T]\times\mathbb{R}^n; t^{-\alpha}\lambda(|x|)^{-1}dt\,dx)$ then the IVP (3) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0\leq t\leq T} \|u(t)\|_s^2 + \int_0^T\int_{\mathbb{R}^n} t^{\alpha}\left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)\,dx\,dt \leq C_1 e^{C_2\frac{T^{\alpha+1}}{\alpha+1}}\left(\|u_0\|_s^2 + \int_0^T\int_{\mathbb{R}^n} t^{-\alpha}\lambda(|x|)^{-1}\left|\Lambda^{s-1/2}f\right|^2 dx\,dt\right).$$

Local well-posedness results for $\mathcal{L}_a$

We consider the nonlinear IVPs

$$\text{IVP1} = \begin{cases} \mathcal{L}_a u = \pm u|u|^{2k} \\ u(0,x) = u_0(x), \end{cases} \qquad \text{IVP2} = \begin{cases} \mathcal{L}_a u = \pm t^{\beta}\nabla u\cdot u^{2k}, \ \beta \geq \alpha > 0, \\ u(0,x) = u_0(x). \end{cases}$$

Theorem (F.-Staffilani). Let $\mathcal{L}_a$ be such that condition (4) is satisfied. Then the IVP1 is locally well posed in H^s for $s > n/2$.

Theorem (F.-Staffilani). Let $\mathcal{L}_a$ be such that condition (4) is satisfied with $\sigma = 2N$ (thus $\lambda(|x|) = \langle x\rangle^{-2N}$) for some $N \geq 1$, and $s > n + 4N + 3$ such that $s - 1/2 \in 2\mathbb{N}$. Let $H_\lambda^s := \{u_0 \in H^s(\mathbb{R}^n); \lambda(|x|)u_0 \in H^s(\mathbb{R}^n)\}$, then the IVP2 with $\beta \geq \alpha > 0$, is locally well posed in H_λ^s.

Global weighted smoothing and Strichartz estimates for $\mathcal{L}_c$

For operators of the form $\mathcal{L}_c$ several comparison principles are proved in [2]. These are used to obtain global smoothing estimates. We state below only one of them, namely the one corresponding to the suitable generalization of the standard (corresponding to the case $c'(t) = 1$) global homogeneous smoothing estimate. For more global smoothing estimates see [2].

Theorem (F.-Ruzhansky)

Let $n \geq 1$, $c \in C^1(\mathbb{R})$ be such that it vanishes at 0. Then, $\forall x \in \mathbb{R}^n$,

(i) If c is such that $\{t \in \mathbb{R}; c'(t) = 0\}$ is finite, then there exist a constant $C > 0$ such that, for all j,

$$\sup_{x_j} \||c'(t)|^{1/2}|D_{x_j}|^{1/2}e^{ic(t)\Delta_x}\varphi\|_{L^2(R_{x'}^{n-1}\times\mathbb{R}_t)} \leq C\|\varphi\|_{L_x^2(\mathbb{R}^n)}, \quad \forall\varphi \in L_x^2(\mathbb{R}^n);$$

(ii) If c is such that the set $\{t \in \mathbb{R}; c'(t) = 0\}$ is countable, then there exists a function $\tilde{c} \in C(\mathbb{R})$, and a positive constant C such that, for all j,

$$\sup_{x_j} \||\tilde{c}(t)c'(t)|^{1/2}|D_{x_j}|^{1/2}e^{ic(t)\Delta_x}\varphi\|_{L^2(R_{x'}^{n-1}\times\mathbb{R}_t)} \leq C\|\varphi\|_{L_x^2(\mathbb{R}^n)}, \quad \forall\varphi \in L_x^2(\mathbb{R}^n);$$

where $x' = (x_1, ..., x_{j-1}, x_{j+1}, ... x_n)$.

Both global and local Strichartz estimates are satisfied by $\mathcal{L}_c$. Below we give the statement of the local ones which are those employed to prove the local well-posedness of the semilinear IVP.

Theorem (F.-Ruzhansky)

Let $c \in C^1([0,T])$ be vanishing at 0 and such that $\sharp\{t \in [0,T]; c'(t) = 0\} = k \geq 1$. Then, on denoting by $L_t^q L_x^p := L^q([0,T]; L^p(\mathbb{R}^n))$, we have that for any (q,p) admissible pair $\left(\frac{2}{q} + \frac{n}{p} = \frac{n}{2}\right)$ such that $2 < q, p < \infty$, the following estimates hold

$$\||c'(t)|^{1/q}e^{ic(t)\Delta}\varphi\|_{L_t^q L_x^p} \leq C(n,q,p,k)\|\varphi\|_{L_x^2(\mathbb{R}^n)},$$

$$\|e^{ic(t)\Delta}\varphi\|_{L_t^{\infty}L_x^2} \leq \|\varphi\|_{L_x^2(\mathbb{R}^n)},$$

$$\||c'(t)|^{1/q}\int_0^t |c'(s)|e^{i(c(t)-c(s))\Delta}g(s)ds\|_{L_t^q L_x^p} \leq C(n,q,p,k)\||c'|^{1/q'}g\|_{L_t^{q'}L_x^{p'}},$$

$$\|\int_0^t |c'(s)|e^{i(c(t)-c(s))\Delta}g(s)ds\|_{L_t^{\infty}L_x^2} \leq C(n,q,p,k)\||c'|^{1/q'}g\|_{L_t^{q'}L_x^{p'}}.$$

Local well-posedness of the semilinear IVP for $\mathcal{L}_c$

We can now apply the previous results to obtain the local well-posedness of the semilinear IVP

$$\begin{cases} \partial_t u + ic'(t)\Delta u = \mu|c'(t)||u|^{p-1}u, \quad \mu \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \qquad (5)$$

Theorem (F.-Ruzhansky). Let $1 < p < \frac{4}{n} + 1$ and $c \in C^1([0,+\infty))$ be vanishing at 0 and it is either strictly monotone or such that $\sharp\{t \in [0,\tilde{T}]; c'(t) = 0\}$ is finite for any $\tilde{T} < \infty$. Then for all $u_0 \in L^2(\mathbb{R}^n)$ there exists $T = T(\|u_0\|_2, n, \mu, p) > 0$ such that there exists a unique solution u of the IVP (5) in the time interval $[0,T]$ with $u \in C([0,T]; L^2(\mathbb{R}^n)) \bigcap L_t^q([0,T]; L_x^{p+1}(\mathbb{R}^n))$ and $q = \frac{4(p+1)}{n(p-1)}$. Moreover the map $u_0 \mapsto u(\cdot,t)$, locally defined from $L^2(\mathbb{R}^n)$ to $C([0,T); L^2(\mathbb{R}^n))$, is continuous.

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Hardy-Littlewood inequality and L^p-L^q Fourier multipliers on compact hypergroups

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Aim

- To prove Hardy-Littlewood inequality and Paley inequality for compact hypergroups [8].
- To establish Hörmander's L^p-L^q Fourier multiplier theorem on compact hypergroups for the range $1 < p \leq 2 \leq q < \infty$ [8].

Hypergroups: What & why?

- Roughly, a hypergroup K is a locally compact Hausdorff space with a convolution on the space $M_b(K)$ of regular bounded Borel measures on K with properties similar to those of group convolution.
- In non commutative setting, the analysis on hypergroups provides a natural extension of analysis on locally compact groups. While in commutative setting, they extend the theory of spherical functions and Gelfand pairs.
- Some of important examples are double coset spaces, the space of conjugacy classes on (Lie) groups and the space of group orbits.
- In particular, the results presented here are true for several interesting examples including Jacobi hypergroups with Jacobi polynomials as characters, compact hypergroup structure on the fundamental alcove with Heckman-Opdam polynomials as characters and multivariant disk hypergroups.
- A compact hypergroup can be countable infinite also ([6]). This property distinguishes them from compact groups.
- Unlike locally compact abelian groups, the support of the Plancherel measure on the dual space may not be full space in the case of commutative hypergroups.
- For more details on analysis on hypergroups and several interesting examples, see [4, 10, 6].

Fourier analysis on compact hypergroups

- Let K be compact hypergroup with normalised Haar measure λ. Denote by $\widehat{K}$ the dual space consisting of irreducible inequivalent continuous representations of K equipped with the discrete topology.
- Every $\pi \in \widehat{K}$ is finite dimensional but may not be unitary in contrast to compact groups case.
- In commutative setting also, the dual space $\widehat{K}$ may not have a hypergroup structure, in contrary to abelian groups.
- Denote the dimension and hyperdimension of $\pi \in \widehat{K}$ by d_π and k_π.
- For each $\pi \in \widehat{K}$, the Fourier transform $\widehat{f}$ of $f \in L^1(K)$ is defined as

$$\widehat{f}(\pi) = \int_K f(x)\bar{\pi}(x)\, d\lambda(x),$$

where $\bar{\pi}$ is the conjugate representation of π.
- We refer to [10, 4, 9] for more details on Fourier analysis and representation theory of compact hypergroups.

Methods

- To establish Hausdorff-Young-Paley inequality we first prove Paley inequality [2, 11] for compact hypergroups [8] and then we use weighted interpolation with Hausdorff-Young inequality [9].
- An application of Paley inequality gives Hardy-Littlewood inequality for compact hypergroups.
- We obtain Hörmander L^p-L^q boundedness of Fourier multiplier [7] in context of compact hypergroup with the help of the Hausdorff-Young-Paley inequality.

Literature

- Hardy-Littlewood inequality [5] was recently estalished for compact homogeneous spaces [2] and for compact quantum groups [1, 11].
- L^p-L^q boundedness of Fourier multipliers on locally compact unimodular groups was proved in [3] using von-Neumann algebra techniques.

Hausdorff-Young-Paley inequality on compact hypergroups

Theorem (Hausdorff-Young-Paley (Pitt) inequality)

Let K be a compact hypergroup and let $1 < p \leq b \leq p' < \infty$. If a positive sequence $\varphi(\pi), \pi \in K$, satisfies the condition

$$M_\varphi := \sup_{y>0} y \sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \geq y}} k_\pi^2 < \infty,$$

then we have

$$\left(\sum_{\pi \in \widehat{K}} k_\pi^2 \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \varphi(\pi)^{\frac{1}{b} - \frac{1}{p'}} \right)^b \right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b} - \frac{1}{p'}} \|f\|_{L^p(K)}.$$

Non-commutative version of Hardy-Littlewood inequality

Theorem (Hardy-Littlewood inequality for compact hypergroups)

Let $1 < p \leq 2$ and let K be a compact hypergroup. Assume that a sequence $\{\mu_\pi\}_{\pi \in K}$ grows sufficiently fast, that is,

$$\sum_{\pi \in \widehat{K}} \frac{k_\pi^2}{|\mu_\pi|^\beta} < \infty \quad \text{for some } \beta \geq 0.$$

Then we have

$$\sum_{\pi \in \widehat{K}} k_\pi^2 |\mu_\pi|^{\beta(p-2)} \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \right)^p \lesssim \|f\|_{L^p(K)}.$$

L^p-L^q-boundedness of Fourier multipliers on compact hypergroups

Theorem (Hörmander theorem for Fourier multipliers)

Let K be a compact hypergroup and let $1 < p \leq 2 \leq q < \infty$. Let A be a left Fourier multiplier with symbol σ_A, that is, A satisfies

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi), \quad \pi \in \widehat{K}.$$

Then we have

$$\|A\|_{L^p(K) \to L^q(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi \in \widehat{K} \\ \|\sigma_A(\pi)\|_{op} \geq y}} k_\pi^2 \right)^{\frac{1}{p} - \frac{1}{q}}.$$

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H-L inequality for Conj(SU(2))

If $1 < p \leq 2$ and $f \in L^p(\mathrm{Conj}(\mathrm{SU})(2))$, then we have

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p \leq C \|f\|_{L^p(\mathrm{Conj}(\mathrm{SU})(2))}$$

H-L for D-R hypergroups [6]

If $1 < p \leq 2$ then there exists a constant $C = C(p)$ such that

$$f(0) + \sum_{n \in \mathbb{N}} ((1-a)a^{-n})^{p(\frac{3}{2} - \frac{1}{p})} |\widehat{f}(n)|^p \leq C \|f\|_{L^p(H_a)}$$

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An extension of the Bessel-Wright transform in the class of Boehmians

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Abstract

In this paper, we first construct a suitable Boehmian space on which the Bessel-Wright transform can be defined and some desired properties are obtained in the class of Boehmians. Some convergence results are also established.

Introduction and preliminaries

The space of Boehmians is constructed using an algebraic approach that utilizes convolution and approximate identities or delta sequences. If the construction is applied to a function space and the multiplication is interpreted as convolution, the construction yields a space of generalized functions. Those spaces provide a natural setting for extensions of the Bessel-Wright transform newly introduced by Fitouhi et al. [3]. We cite here, as briefly as possible, some facts about harmonic analysis related to the Bessel-Wright operator $\Delta_{\alpha,\beta}$. For more details we refer to [3].

We consider, on $(0,\infty)$ the difference differential operator indexed by two parameters α and β

$$\Delta_{\alpha,\beta}f(x)=\frac{d^2f}{dx^2}(x)+\frac{2(\alpha+\beta)+1}{x}\frac{df}{dx}(x)+\frac{4\alpha\beta}{x^2}[f(x)-f(0)]. \quad (0.1)$$

These operators are very important in pure mathematics and especially in special functions and harmonic analysis. The Bessel-Wright operator admits as eigenfunction with $-\lambda^2$ as eigenvalue the Bessel-Wright function

$$j_{(\alpha,\beta)}(\lambda x)=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}\left(\frac{\lambda x}{2}\right)^{2n},\ (\lambda\in\mathbb{C}),$$

which is even and symmetric in α and β and coincides when $\alpha=0$ or $\beta=0$ with the normalized Bessel function given by

$$j_\alpha(\lambda x)=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}\left(\frac{\lambda x}{2}\right)^{2n},\ (\lambda\in\mathbb{C}).$$

Let $L^p_\alpha=L^p_\alpha(0,\infty)$ denote the class of measurable functions f on $(0,\infty)$ for which $\|f\|^p_\alpha<\infty$, where

$$\|f\|^p_\alpha=\left(\int_0^\infty|f(x)|^pd\mu_\alpha(x)\right)^{\frac{1}{p}},\ \ if\ p<\infty,$$

$$\|f\|_{\infty,\alpha}=\|f\|_\infty=ess\ sup_{x\in(0,\infty)}|f(x)|,$$

and $d\mu_\alpha(x)=x^{2\alpha+1}dx$.

The Bessel-Wright transform for $f\in L^p_\alpha$ is defined by

$$\mathcal{F}_{(\alpha,\beta)}(f)(\lambda)=c_\alpha\int_0^\infty f(x)j_{\alpha,\beta}(\lambda x)d\mu_\alpha(x) \quad (0.2)$$

where $c_\alpha=\frac{1}{2^\alpha\Gamma(\alpha+1)}$

The following two definitions are needed for our results.

Definition 0.1. The Mellin-type convolution product of first kind is defined by:

$$f\times g(y)=\int_0^\infty f(yx^{-1})x^{-1}g(x)dx. \quad (0.3)$$

Definition 0.2. Let $\alpha>-\frac{1}{2}$ and $f,g\in L^1(0,\infty)$.Then we define the product $\otimes$ of f and g by the integral

$$f\otimes g(y)=\int_0^\infty f(yt)g(t)d\mu_\alpha(t), \quad (0.4)$$

By using (0.3) and (0.4), we get the following proposition:

Proposition 0.1. *Let f, g, and h be integrable functions in $L^1(0,\infty)$ and let $y>0$. Then*

$$f\otimes(g\times h)(y)=(f\otimes g)\otimes h(y)$$

Proposition 0.2. *The Bessel-Wright transform $\mathcal{F}_{(\alpha,\beta)}$is a bounded linear operator from L^1_α to C_0.*

Generated Spaces of Boehmians

The class of Boehmians was introduced to generalize regular operators [2]. The minimal structure necessary for the abstract construction of Boehmian spaces consists of the following elements:

i. A topological vector space $\mathfrak{a}$

ii. A commutative semigroup $(\mathfrak{b},\bullet)$

iii. An operation $\star:\mathfrak{a}\times\mathfrak{b}\longrightarrow\mathfrak{a}$ such that, for each $x\in\mathfrak{a}$ and $s_1,s_2\in\mathfrak{b}$,

$$x\star(s_1\bullet s_2)=(x\star s_1)\star s_2.$$

iv. A collection $\Delta\subset\mathfrak{b}^{\mathbb{N}}$ such that:

a. if $x,y\in\mathfrak{a}$, $(s_n)\in\Delta$, $x\bullet s_n=y\bullet s_n$ for all n, then $x=y$;

b. if $(s_n),(t_n)\in\Delta$, then $(s_n\bullet t_n)\in\Delta$.

The elements of Δ are called delta sequences. Denote by Q the set

$$Q=\{(x_n,s_n):x_n\in\mathfrak{a},(s_n)\in\Delta,x_n\star s_m=x_m\star s_n\forall m,n\in\mathbb{N}\}.$$

If $(x_n,s_n),(y_n,t_n)\in Q$, $x_n\star t_m=y_m\star s_n\forall m,n\in\mathbb{N}$, then we say that $(x_n,s_n)\sim(y_n,t_n)$. The relation $\sim$ is an equivalence relation in Q. The space of equivalence classes in Q is denoted by $\mathfrak{B}$. The elements of $\mathfrak{B}$ are called Boehmians.

Between $\mathfrak{a}$ and $\mathfrak{B}$, there is a canonical embedding expressed as

$$x\rightarrow\frac{x\star s_n}{s_n}.$$

The operation $\star$ is extended to $\mathfrak{B}\times\mathfrak{b}$ as follows:

$$\text{If }\left[\frac{(f_n)}{(s_n)}\right]\in\mathfrak{B}\text{ and }\phi\in\mathfrak{b}\text{, then }\left[\frac{(f_n)}{(s_n)}\right]\star\phi=\left[\frac{(f_n)\star\phi}{s_n}\right].$$

We establish the following technical result.

Lemma 0.1. *Let $f\in L^1_\alpha(0,\infty)$ and $\psi\in D(0,\infty)$. Then*

$$\mathcal{F}_{(\alpha,\beta)}(f\times\psi)(\lambda)=(\mathcal{F}_{(\alpha,\beta)}f\otimes\psi)(\lambda).$$

The spaces generated here are the space $\mathfrak{B}_1=\mathfrak{B}_1(L^1_\alpha,(D,\times),\times,\Delta)$ and the space $\mathfrak{B}_2=\mathfrak{B}_2(L^1_\alpha,(D,\times),\otimes,\Delta)$. We denote by Δ, the set of delta sequences (δ_n) of $D(0,\infty)$ with the following properties:

$$\int_0^\infty\delta_n(x)dx=1, \quad (0.5)$$

$$\int_0^\infty|\delta_n(x)|dx<m, \quad (0.6)$$

where m is a positive real number

$$supp\ \delta_n(x)\rightarrow1,\ as\ n\rightarrow\infty. \quad (0.7)$$

Let us now establish that $\mathfrak{B}_1$ is a Boehmian space. We prefer to omit the proof for $\mathfrak{B}_2$ as its details are simlar.

Theorem 0.1. *Let $f\in L^1_\alpha(0,\infty)$, $\psi\in D(0,\infty)$ and $\alpha>-\frac{1}{2}$. Then $f\times\psi\in L^1_\alpha(0,\infty)$*

Theorem 0.2. *Let $f\in L^1_\alpha(0,\infty)$ and $\psi_1,\psi_2\in D(0,\infty)$, $\alpha>-\frac{1}{2}$.Then*

i. $f\times(\psi_1+\psi_2)=f\times\psi_1+f\times\psi_2$,

ii. $f\times(\psi_1\times\psi_2)=(f\times\psi_1)\times(\psi_2)$,

iii. $(\lambda f)\times\psi_1=\lambda(f\times\psi_1)=f\times(\lambda\psi_1),\lambda\in\mathbb{C}$

Theorem 0.3. *Let $f_n\rightarrow f\in L^1_\alpha(0,\infty)$ as $n\rightarrow\infty$ and let $\psi\in D(0,\infty)$, $\alpha>-\frac{1}{2}$. Then*

$$f_n\times\psi\rightarrow f\times\psi\ \ as\ \ n\rightarrow\infty$$

in $L^1_\alpha(0,\infty)$.

Theorem 0.4. *Let $f\in L^1_\alpha(0,\infty)$ and let $(\delta_n)\in\Delta$, $\alpha>-\frac{1}{2}$. Then*

$$f\times\delta_n\rightarrow f\ as\ n\rightarrow\infty$$

in $L^1_\alpha(0,\infty)$.

A sequence of Boehmians (ζ_n) in $\mathfrak{B}_1$ is said to be δ convergent to a Boehmian ζ in $\mathfrak{B}_1$ denoted by $\zeta_n\xrightarrow{\delta}\zeta$, if there exists a delta-sequence (δ_n) such that

$$(\zeta_n\times\delta_k),(\zeta\times\delta_k)\in L^1_\alpha\ \forall k,n\in\mathbb{N},$$

and

$$(\zeta_n\times\delta_k)\rightarrow(\zeta\times\delta_k)\ \ as\ n\rightarrow\infty,\ \ in\ \ L^1_\alpha,\ \forall k\in\mathbb{N}.$$

A sequence of Boehmians (ζ_n) in $\mathfrak{B}_1$ is said to be Δ convergent to a Boehmian ζ in $\mathfrak{B}_1$ denoted by $\zeta_n\xrightarrow{\Delta}\zeta$, if there exists a delta-sequence $(\delta_n)\in\Delta$ such that $(\zeta_n-\zeta)\times\delta_n\in L^1_\alpha\ \forall n\in\mathbb{N}$ and $(\zeta_n-\zeta)\times\delta_n\rightarrow0$ as $n\rightarrow\infty$ in L^1_α.

Similarly, the following theorems generate the Boehmian space $\mathfrak{B}_2$.

Theorem 0.5. *Let $f\in L^1_\alpha(0,\infty)$ and $\psi\in D(0,\infty)$. Then $f\otimes\psi\in L^1_\alpha(0,\infty)$.*

Theorem 0.6. *Let $f\in L^1_\alpha(0,\infty)$ and $\psi_1,\psi_2\in D(0,\infty)$. Then*

i. $f\otimes(\psi_1+\psi_2)=f\otimes\psi_1+f\otimes\psi_2$,

ii. $(\lambda f)\otimes\psi_1=\lambda(f\otimes\psi_1)=f\otimes(\lambda\psi_1)$, $\lambda\in\mathbb{C}$.

Theorem 0.7. *For $f\in L^1_\alpha(0,\infty)$ and $\psi_1,\psi_2\in D(0,\infty)$, the following relation is true:*

$$f\otimes(\psi_1\times\psi_2)=(f\otimes\psi_1)\otimes\psi_2.$$

Theorem 0.8. *i. Let $f_n\rightarrow f$ in $L^1_\alpha(0,\infty)$ as $n\rightarrow\infty$ and let $\psi\in D(0,\infty)$. Then $f_n\otimes\psi\rightarrow f\otimes\psi$ as $n\rightarrow\infty$.*

ii. Let $f_n\rightarrow f$ in $L^1_\alpha(0,\infty)$ and let $(\delta_n)\in\Delta$.Then $f_n\otimes\delta_n\rightarrow f$ as $n\rightarrow\infty$.

The Bessel-Wright Transform of a Boehmian

Let $\zeta\in\mathfrak{B}_1$ and $\zeta=[\frac{(f_n)}{(\delta_n)}]$. Then, for every $\alpha>-\frac{1}{2}$ we define the generalized Bessel-Wright transform of ζ as follows:

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\delta_n)}\right]\right)=\left[\frac{(\mathcal{F}_{(\alpha,\beta)}f_n)}{(\delta_n)}\right] \quad (0.8)$$

Theorem 0.9. *$\mathcal{F}^{ge}_{\alpha,\beta}$ is an isomorphism from $\mathfrak{B}_1$ into $\mathfrak{B}_2$.*

In addition, we now deduce the formula of extension of $\times$ to $\mathfrak{B}_1$ as follows:

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)\otimes\phi.$$

It can be proved as follows: By virtue of (0.8) we can write

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\left(\left[\frac{(\mathcal{F}^{ge}_{\alpha,\beta}(f_n\times\phi))}{(\omega_n)}\right]\right).$$

Hence, Lemma 0.1 gives

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\left(\left[\frac{(\mathcal{F}^{ge}_{\alpha,\beta}f_n\otimes\phi)}{(\omega_n)}\right]\right).$$

The definition of the product $\times$ implies that

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\left(\left[\frac{(\mathcal{F}^{ge}_{\alpha,\beta}f_n)}{(\omega_n)}\right]\right)\times\phi.$$

Thus, it follows from relation (0.8) that

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)\otimes\phi.$$

Hence, it is now possible to conclude that

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)\otimes\phi.$$

Theorem 0.10. *$\mathcal{F}^{ge}_{\alpha,\beta}:\mathfrak{B}_1\rightarrow\mathfrak{B}_2$ is continuous with respect to the $\delta-$convergence and $\Delta-$convergence.*

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Further informations

Imane Berkak, 3ème année du cycle doctoral.

Sujet de la thèse : Harmonic Analysis assosiated with the Bessel Wright operator.

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Van der Corput lemmas for Mittag-Leffler functions

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Introduction

In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals. This estimate was first obtained by the Dutch mathematician Johannes Gaultherus van der Corput [1] and named in his honour. While the paper [1] was published in Mathematische Annalen in 1921, he submitted it there on 17 December 1920 (from Utrecht). Therefore, it seems appropriate to us to dedicate this paper to the 100^{th} anniversary of this lemma.
Let us state the classical van der Corput lemmas as follows:

- **van der Corput lemma.** Suppose ϕ is a real-valued and smooth function in $[a,b]$. If ψ is a smooth function and $|\phi^{(k)}(x)| \geq 1$, $k \geq 1$, for all $x \in (a,b)$, then
$$\left|\int_a^b \exp\left(i\lambda\phi(x)\right)\psi(x)dx\right| \leq \frac{C}{\lambda^{1/k}}, \quad \lambda \to \infty, \qquad (1)$$
for $k=1$ and ϕ' is monotonic, or $k \geq 2$. Here C does not depend on λ.

Formulation of problem

The main goal of the present paper is to study van der Corput lemmas for the oscillatory integrals defined by (see [2], [3]) respectively:
$$I_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}\left(i\lambda\phi(x)\right)\psi(x)dx, \qquad (2)$$
and
$$\mathcal{I}_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}\left(i^{\alpha}\lambda\phi(x)\right)\psi(x)dx, \qquad (3)$$
where $\alpha > 0$, $\beta > 0$, ϕ is a phase and ψ is an amplitude, and λ is a positive real number that can vary. Here $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined as
$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R},$$
with the property that
$$E_{1,1}(z) = e^z. \qquad (4)$$

Main results for $I_{\alpha,\beta}$

van der Corput lemmas on $\mathbb{R}$: consider $I_{\alpha,\beta}$ defined by (2).
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha < 1$, $\beta > 0$, and $m = \operatorname{ess\,inf}_{x\in\mathbb{R}} |\phi(x)| > 0$, then
$$|I_{\alpha,\beta}(\lambda)| \leq \frac{M}{1+\lambda m}\|\psi\|_{L^1(\mathbb{R})}, \ \lambda \geq 1,$$
where M does not depend on ϕ, ψ and λ.
van der Corput lemmas on $I = [a,b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$.

- Let $0 < \alpha < 1$, $\beta > 0$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then
$$|I_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-\frac{1}{k}} \log^{\frac{1}{k}}(1+\lambda), \quad \lambda \geq 1,$$
where M_k does not depend on λ.
- Let $-\infty < a < b < +\infty$ and $I = [a,b] \subset \mathbb{R}$. Let $0 < \alpha \leq 1$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$. Then
$$|I_{\alpha,\alpha}(\lambda)| \leq M_k \lambda^{-1/k}, \quad \lambda \geq 1, \qquad (5)$$
for $k=1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ.

Main results for $\mathcal{I}_{\alpha,\beta}$

van der Corput lemmas on $\mathbb{R}$: consider $\mathcal{I}_{\alpha,\beta}$ defined by (3).
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha \leq 2$, $\beta > 1$, and $m = \operatorname{ess\,inf}_{x\in\mathbb{R}} |\phi(x)| > 0$, then
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq \frac{M}{1+\lambda m}\|\psi\|_{L^1(\mathbb{R})}, \ \lambda \geq 1, \ 0 < \alpha < 2, \ \beta \geq \alpha+1,$$
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq \frac{M}{(1+\lambda m)^{\frac{\beta-1}{\alpha}}}\|\psi\|_{L^1(\mathbb{R})}, \ \lambda \geq 1, \ 0 < \alpha \leq 2, \ 1 < \beta < \alpha+1,$$
where M does not depend on ϕ, ψ and λ.
van der Corput lemmas on $I = [a,b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$.

- Let $0 < \alpha < 2$, $\beta > 1$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-\frac{1}{k}} \log^{\frac{1}{k}}(1+\lambda), \quad \lambda \geq 1, \ 0 < \alpha < 2, \ \beta \geq \alpha+1,$$
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-\frac{1}{k}} (1+\lambda)^{\frac{\alpha+1-\beta}{\alpha k}}, \quad \lambda \geq 1, \ 0 < \alpha < 2, \ 1 < \beta < \alpha+1,$$
where M_k does not depend on λ.
- Let $-\infty < a < b < +\infty$ and $I = [a,b] \subset \mathbb{R}$. Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$. Then
$$|\mathcal{I}_{\alpha,\alpha}(\lambda)| \leq M_k \lambda^{-1/k}, \quad \lambda \geq 1, \qquad (6)$$
for $k=1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ.

Remark

The case of $\alpha = 1$ in (5) and (6) corresponds to the classical van der Corput lemma (1).

Conclusion

The main goal of the paper was to study van der Corput lemmas for the integrals defined by (2) and (3). Van der Corput type lemmas were obtained, for the different cases of parameters α and β.
As an immediate application of the obtained results, time-estimates of the solutions of time-fractional Klein-Gordon and Schrödinger equations and generalisations of the Riemann-Lebesgue lemma were also considered in [2] and [3].

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Feature Extraction for License Plate Location Based on L_0-norm Smoothing

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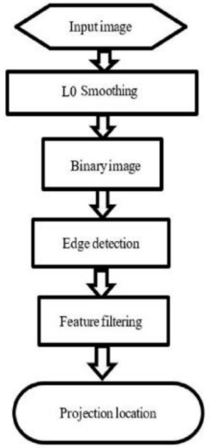
Overview

Abstact: The license plate location plays an important role in license plate recognition systems. As a premise of character recognition, the accuracy and robustness of license location directly determine the performance of the entire license plate recognition system. In many practical applications, license plate recognition systems mostly work outdoors, and the captured images inevitably suffer from different kinds of degradations caused by lighting, weather, and complex backgrounds, and so on. In general, it is a challenging problem under such a diverse, uncertain and complex environment and has also attracted considerable attention in both academic and industrial fields.

Main Idea: In regard to the complex background in license images, we first use an edge-aware filter, L^0-norm smoothing to remove the majority background textures but keep the license plate characters. Then, we take a series of feature filtering steps based on the geometrical textures and structures to furtherly to reduce the interference of pseudo-licenses. Finally, a simple projection location method is used to extract the position and size of the license plates. The whole location procedure is:



Contributions:

- Propose a practical license plate location system based on a series of feature extraction and filter steps.
- Use L_0 image smoothing algorithm to remove the background noise.
- Use a binarized image for fast multiscale resolution analysis.
- Take full use of textural information and projection location method to extract license plates.

Algorithm

L^0 Smoothing: In license plate images, license characters have high contrast textures, while the image background contains abundant low contrast details. We use the L_0-norm smoothing filters to suppress the details, which can be phased as,

$$\arg\min_f E(f) = \|f-g\|_2^2 + \lambda\|\nabla f\|_0, \qquad (1)$$

where g and f are the source and target images in $\mathbb{R}^N$, ∇ is gradient operator, $\|\cdot\|_2$ is L^2-norm, and $\|\cdot\|_0$ is so-called L^0-norm, counting the number of non-zero elements of an vector, which leads to a sparse regularization, and λ is a weight scalar.

L^0-norm minimization: The Eq.(1) is NP-hard to solve, we instead introduce two auxiliary variables h_i and v_i for the 2D discrete image, and rewrite it as,

$$\min_{f,h,v} E(f) = \sum_i (f_i - g_i)^2 + \lambda C(h,v) + \beta((\partial_x f_i - h_i)^2 + (\partial_y f_i - v_i)^2), \qquad (2)$$

where $C(h,v) = 1$ if $|h_i| + |v_i| \neq 0$, else 0. It is clear that the solution of Eq. (2) approximates that of Eq. (1) when $\beta \to +\infty$. We solve the Eq. (2) minimizing (h,v) and f alternatively.

1. Computing f: By fixing h_i and v_i and minimizing the function,

$$\min_f E(f) = \sum_i (f_i - g_i)^2 + \beta((\partial_x f_i - h_i)^2 + (\partial_y f_i - v_i)^2). \qquad (3)$$

The above function is quadratic and thus has a global minimum. Here we use fast Fourier transform (FFT) to accelerate the solver,

$$f = \mathscr{F}^{-1}\left\{\frac{\mathscr{F}(g) + \beta(\mathscr{F}(\partial_x) * \mathscr{F}(h) + \mathscr{F}(\partial_y) * \mathscr{F}(v))}{\mathscr{F}(1) + \beta(\mathscr{F}(\partial_x) * \mathscr{F}(\partial_x) + \mathscr{F}(\partial_y) * \mathscr{F}(\partial_y))}\right\}, \qquad (4)$$

where $*$ is a component-wise multiplication, $\mathscr{F}$ and $\mathscr{F}^{-1}$ denotes FFT and its inverse operators, and $\mathscr{F}(1)$ is the Fourier Transform of δ function.

2. computing (h,v): We solve (h,v) by minimizing

$$\min_{h,v} E(f) = \lambda C(h,v) + \beta((\partial_x f_i - h_i)^2 + (\partial_y f_i - v_i)^2), \qquad (5)$$

where $C(h,v)$ returns the number of non-zero elements of $|\partial_x f_i| + |\partial_y f_i|$. Eq. (5) can be spatially decomposed and solved fastly, because each element h_p and v_p can be estimated individually. It reaches its minimum E^* under the condition,

$$h_i, v_i = \begin{cases} (0,0), (\partial_y f_i + \partial_y f_i)^2 \leq \dfrac{\lambda}{\beta}, \\ 0, \text{ otherwise.} \end{cases} \qquad (6)$$

Post-processing: To extract and locate the license more accurately, we also introduce the following post-processing steps:

Binarized image: Let I be an image, and I_s is resized from I with a scale s, where $s \in S = \{s_{m,n}\} = \{2^{-m}, 2^{-n}\}_{m,n\in Z}$ are row and column scales. If we rearrange the images $\{I_s\}_{s\in S}$ on a 2-D plane in descending scales, a new image $B = \{I_s\}_{s\in S}$, called binarized image, is obtained, which provides a multiscale analysis of license image.

Feature Filtering: By using the L^0-norm smoothing, a mass of local textures can be removed and an optimal scale of the license is also available with binarized image. We further propose a feature filtering, that is, edge extraction, removing long lines, structure analysis, and texture density analysis to further remove the pseudo-licenses.

Projection Location: After the above steps, the license is easy to locate, we use a very simple projection location method to extract the position and size information.

Experiments & Results

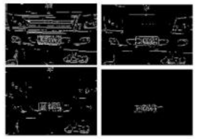
1. L^0-norm smoothing

- Left: input image, right: L^0-norm smoothing result. It is clear that the low-contrast details are significantly suppressed with L^0-norm smoothing.

2. Binarized image

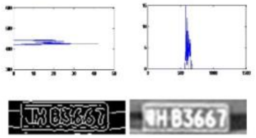
- The binarized image provides a multiscale analysis for licenses.

3. Feature filtering



- Edge extraction, removing lines, structural and textural analysis.

4. Projection results



- The horizontal and vertical projection histograms, and the output license plate.

Quantitative evaluation:

Method	Precision(%)	Recall (%)	F_1-score(%)
Top-hat	77.50	84.50	88.61
MSER&SIFT	83.73	90.47	86.97
Wavelet-based	90.30	94.03	95.00
CNN-based	97.80	95.30	96.01
Ours	96.10	95.40	95.30

- Quantitative evaluation on natural image datasets. We collected 1000 images with both simple and complex backgrounds.

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Spectral Geometry of Partial Differential Operators

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Description

The aim of Spectral Geometry of Partial Differential Operators is to provide a basic and self-contained introduction to the ideas underpinning spectral geometric inequalities arising in the theory of partial differential equations. Historically, one of the first inequalities of the spectral geometry was the minimisation problem of the first eigenvalue of the Dirichlet Laplacian. Nowadays, this type of inequalities of spectral geometry have expanded to many other cases with numerous applications in physics and other sciences. The main reason why the results are useful, beyond the intrinsic interest of geometric extremum problems, is that they produce a priori bounds for spectral invariants of (partial differential) operators on arbitrary domains.

Content

Spectral geometry

Let us consider **Riesz potential operators**

$$(\mathcal{R}_{\alpha,\Omega}f)(x) := \int_\Omega |x-y|^{\alpha-d} f(y)dy, \quad f \in L^2(\Omega), \ 0 < \alpha < d, \qquad (1)$$

where $\Omega \subset \mathbb{R}^d$ is a set with finite Lebesgue measure.

Rayleigh-Faber-Krahn inequality: The ball Ω^* is the maximiser of the first eigenvalue of the operator $\mathcal{R}_{\alpha,\Omega}$ among all domains of a given volume, i.e.

$$0 < \lambda_1(\Omega) \le \lambda_1(\Omega^*)$$

for an arbitrary domain $\Omega \subset \mathbb{R}^d$ with $|\Omega| = |\Omega^*|$.

Hong-Krahn-Szegö inequity: The maximum of the second eigenvalue $\lambda_2(\Omega)$ of $\mathcal{R}_{\alpha,\Omega}$ among all sets $\Omega \subset \mathbb{R}^d$ with a given measure is approached by the union of two identical balls with mutual distance going to infinity.

Richard Feynman's drum

The picture from planksip.org

It is proved that the deepest bass note is produced by the circular drum among all drums of the same area (as the circular drum). Moreover, one can show that among all bodies of a given volume in the three-dimensional space with constant density, the ball has the gravitational field of the highest energy.

Rex is minimising the strain energy by circling

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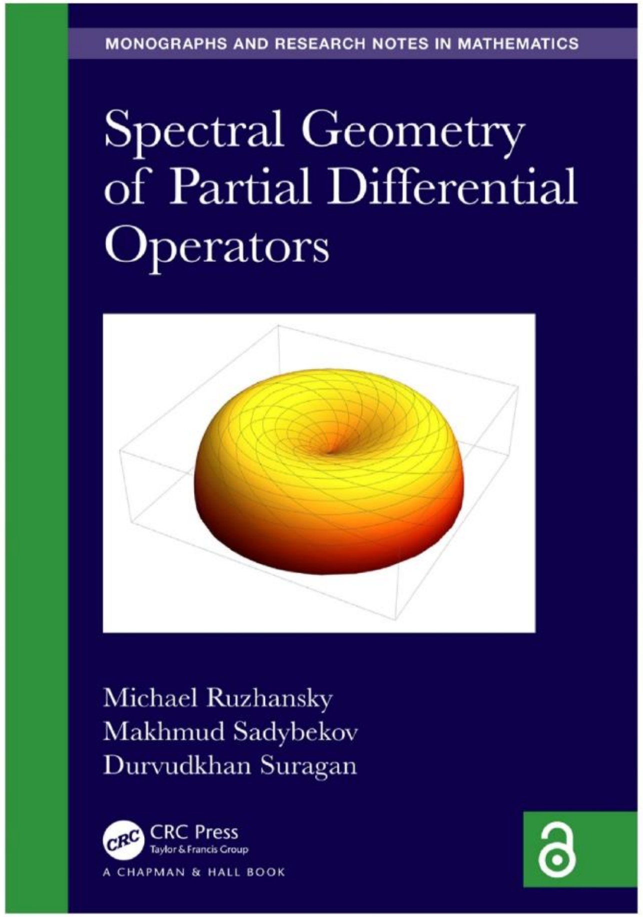

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Subelliptic pseudo-differential operators on compact Lie groups

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Abstract

In this work we extend the theory of global pseudo-differential operators on compact Lie groups to a subelliptic context. More precisely, given a compact Lie group G, and the sub-Laplacian $\mathcal{L}$ associated to a system of vector fields $X = \{X_1, \cdots, X_k\}$ satisfying the Hörmander condition, we introduce a (subelliptic) pseudo-differential calculus associated to $\mathcal{L}$, based on the matrix-valued quantisation process developed in [2].

Introduction

- In modern mathematics, the theory of pseudo-differential operators is a powerful branch in the analysis of linear partial differential operators due to its interactions with several areas of mathematics.
- For instance, from the point of view of the theory of PDE, pseudo-differential operators are used e.g.
  - I. to study the global/local solvability of several partial differential problems.
  - II. To understand the mapping properties of certain singular integral operators.
  - III. To understand the propagation of singularities in distribution theory, and in the construction of fundamental solutions and parametrices.
  - IV. To compute some geometric invariants arising in the index theory.
- Here we describe how in [1], we associate to every sub-Riemannian structure of a compact Lie group a pseudo-differential calculus.

Compact Lie groups

- $$\text{Compact Lie Group} = \begin{cases} \text{Closed manifold} \\ + \\ \text{topological group} \end{cases}$$
- Examples: $\mathbb{T}^n$, $\mathrm{SU}(n)$, $\mathrm{Spin}(n)$.

Fourier analysis on compact Lie groups

- If G is a compact Lie group, its unitary dual G consists of all equivalence classes of continuous irreducible unitary representations of G.
- Unitary Representation: for each ξ from the equivalence class $[\xi] \in G$ we have
$$\xi \in \mathrm{HOM}(G, U(H_\xi))$$
for some (finite-dimensional) vector space $H_\xi \cong \mathbb{C}^{d_\xi}$. We denote by $d_\xi = dim(H_\xi)$ the dimension of the representation ξ.
$$\widehat{f}(\xi) := \int_G f(x)\xi(x)^* dx, \quad \xi \in [\xi] \in \widehat{G}.$$

Pseudo-differential operators on compact Lie groups

- Every continuous linear operator T from $C^\infty(G)$ to itself admits a representation in the following (quantization) formula:
$$Tf(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)a(x,\xi)\widehat{f}(\xi)],$$
- The matrix-valued function a defined on the phase space $G \times G$ is called the symbol of a. For every ξ, $a(x,\xi) := \xi(x)^*(A\xi)(x)$ is a matrix of size $d_\xi \times d_\xi$.

Laplacian vs sub-Laplacian

- The positive sub-Laplacian $\mathcal{L}$ associated to a system of vector fields $X = \{X_1, \cdots, X_k\}$ satisfying the Hörmander condition, is defined by $\mathcal{L} = -X_1^2 - \cdots - X_k^2$.
- If $X' := \{X_1, \cdots, X_n\}$ is a basis of the Lie algebra $\mathfrak{g}$, of G, the positive Laplacian $\mathcal{L}_G$ is defined by $\mathcal{L} = -X_1^2 - \cdots - X_n^2$.

Subelliptic pseudo-differential calculus

The Laplacian generates an elliptic pseudo-differential calculus on every compact Lie group [2]. The elliptic Hörmander classes for this calculus are denoted by $\mathscr{S}^m_{\rho,\delta}(G\times G)$. The theory can be extended to sub-Riemannian structures induced by any choice of a sub-Laplacian [1]. The subelliptic classes are denoted by $S^{m,\mathcal{L}}_{\rho,\delta}(G \times G)$.

Main results and applications of the theory

- The subelliptic pseudo-differential calculus is closed under compositions, inverses, parametrices, complex powers, etc.
- (Boundedness properties)
  - I. (Fefferman estimates). Let Q be the Hausdorff dimension of G associated to the control distance associated to the sub-Laplacian $\mathcal{L}$. If $m \geqslant m_p := Q(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$, then
  $$\mathrm{Op}(S^{-m,\mathcal{L}}_{\rho,\delta}(G \times G)) \subset \mathscr{B}(L^p(G))\, 1 < p < \infty.$$
  - II. (Calderón-Vaillancourt Theorem) For $0 \leqslant \delta \leqslant \rho \leqslant 1/2\kappa$, or $0 \leq \delta < \rho \leq 1$,
  $$\mathrm{Op}(S^{0,\mathcal{L}}_{\rho,\delta}(G \times G)) \subset \mathscr{B}(L^2(G))\, 1 < p < \infty.$$
- (Gårding Inequality)
$$\mathrm{Re}(a(x,D)u,u) \geqslant C_1\|u\|_{L^{2,\mathcal{L}}_{\frac{m}{2}}(G)} - C_2\|u\|^2_{L^2(G)}.$$
- Well posedness for the Cauchy problem
$$(\mathrm{PVI}) : \begin{cases} \frac{\partial v}{\partial t} = K(t,x,D)v + f, \\ v(0) = u_0, v \in \mathscr{D}'((0,T) \times G) \end{cases} \quad (1)$$
- Asymptotic expansions in spectral geometry
$$\mathbf{Tr}(A\psi(tE)) = t^{-\frac{Q+m}{q}}\left(\sum_{k=0}^{\infty} a_k t^k\right) + \frac{c_Q}{q}\int_0^\infty \psi(s) \times \frac{ds}{s},$$
- L^p-boundedness of Fourier integral operators.
- Dixmier traces, classification in Schatten-von Neumann classes.
- Sharp-L^p-estimates for oscillatory Fourier multipliers.

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GHENT ANALYSIS PDE

On a class of anharmonic oscillators

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Abstact

We study a class of anharmonic oscillators within the framework of the Weyl-Hörmander calculus. By associating a Hörmander metric to a given anharmonic oscillator we extend the so-called *Shubin classes* associated to the harmonic oscillator and the corresponding pseudo-differential calculus. Spectral properties of negative powers of anharmonic oscillators, as well as of the operator itself, are derived.

Introduction

In the study of the **Schrödinger equation** $i\partial_t\psi = -\Delta\psi + V(X)\psi$ the analysis of the energy levels is often reduced to the corresponding eigenvalue problem for the operator $-\Delta + V(x)$.
Spectral properties of the anharmonic oscillator, on $\mathbb{R}$ or more generally on $\mathbb{R}^n$, with different potentials V have been studied (c.f. [1],[2],[4]) by several authors in the last 40 years. However, the exact solution of the eigenvalue problem is still unknown.
Here we consider a more general case on $\mathbb{R}^n$ where a **prototype** is of the form

$$\mathcal{A} = (-\Delta)^l + |x|^{2k}, \quad \text{where} \quad k,l \geq 1 \quad \text{integers}, \tag{1}$$

and **more generally** we consider operators of the form

$$T = q(D) + p(x), \tag{2}$$

where p, q are special polynomials on $\mathbb{R}^n$. In particular we write $p \in \mathcal{P}_{2k}, q \in \mathcal{P}_{2l}$, for some integers $k,l \geq 1$, where we have defined

$$\mathcal{P}_{2k} = \left\{ p : \mathbb{R}^n \to \mathbb{R}, \text{with } \deg(p) = 2k \text{, and } \lim \inf_{|x|\to\infty} \frac{p(x)}{|x|^{2k}} > 0 \right\}.$$

Therefore, for $p \in \mathcal{P}_{2k}$ (and similarly for q), there exists $p_0 > 0$ such that

$$p(x) + p_0 > 0, \quad \text{for every} \quad x \in \mathbb{R}^n.$$

Weyl-Hörmander classes associated to the anharmonic oscillators

- **In the case of the general anharmonic oscillator** T as in (2) with (rescalled) symbol τ we have
$$\tau(x,\xi) = p(x) + q(\xi) \in S(M^{p,q}, g^{p,q}),$$
where the Hörmander metric $g^{p,q}$, and the g-weight $M^{p,q}$ are given by
$$g^{p,q}_{x,\xi}(dx, d\xi) = \frac{dx^2}{\underbrace{(p_0 + q_0 + p(x) + q(\xi))^{\frac{1}{k}}}_{>0}} + \frac{d\xi^2}{(p_0 + q_0 + p(x) + q(\xi))^{\frac{1}{l}}}, \tag{3}$$
and
$$M^{p,q}(x,\xi) = p_0 + q_0 + p(x) + q(\xi) \tag{4}$$
- **In the case of the prototype of the anharmonic oscillator** $\mathcal{A}$ as in (1) the metric (3) is equivalent to the metric
$$g^{k,l}_{x,\xi}(dx, d\xi) = \frac{dx^2}{(1 + |x|^{2k} + |\xi|^{2l})^{\frac{1}{k}}} + \frac{d\xi^2}{(1 + |x|^{2k} + |\xi|^{2l})^{\frac{1}{l}}}. \tag{5}$$
- **In the symmetric case where** $k = l$ in (1) the metric associated to the operator $\mathcal{A}$ is equivalent to
$$g_{x,\xi}(dx, d\xi) = \frac{dx^2}{1 + |x|^2 + |\xi|^2} + \frac{d\xi^2}{1 + |x|^2 + |\xi|^2},$$

which corresponds to the symplectic metric defining the *Shubin classes* associated to the *harmonic oscillator*.

Associated symbol classes and operators

Let $m \in \mathbb{R}$. We say that the function $a \in C^\infty(\mathbb{R}^{2n})$ is in the **class of symbols** $\Sigma^m_{k,l}(\mathbb{R}^n)$, if

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}\Lambda(x,\xi)^{m - \frac{|\alpha|}{k} - \frac{|\beta|}{l}}, \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}^n,$$

where we have denoted $\Lambda(x,\xi) := (1 + |x|^{2k} + |\xi|^{2l})^{\frac{1}{2}}$, for $k,l \geq 1$ integers. The **associated operators** are denoted by $\Psi\Sigma^m_{k,l}(\mathbb{R}^n)$, and in particular

$$\Psi\Sigma^m_{k,l}(\mathbb{R}^n) = Op^w(\Sigma^m_{k,l}(\mathbb{R}^n)).$$

For example the **prototype anharmonic oscillator** $\mathcal{A}$ as in (1) is an operator in $\Psi\Sigma^2_{k,l}(\mathbb{R}^n)$.

Pseudo-differential calculus on $\Sigma^m_{k,l}(\mathbb{R}^n)$

The following can be viewed as a consequence of the Weyl-Hörmander calculus:

- The class of operators $\cup_{m\in\mathbb{R}}\Psi\Sigma^m_{k,l}(\mathbb{R}^n)$ forms an algebra of operators that is stable under taking the adjoint.
- Let $m_1, m_2 \in \mathbb{R}$ and let $k,l \geq 1$ integers. If $a \in \Sigma^{m_1}_{k,l}$ and $b \in \Sigma^{m_2}_{k,l}$, then there exists $c \in \Sigma^{m_1+m_2}_{k,l}$ such that $Op^w(a) \circ Op^w(b) = Op^w(c)$. Moreover, we have the asymptotic formula
$$c \sim \sum_\alpha \frac{(2\pi i)^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha a)(\partial_x^\alpha b).$$
- The operators in $\Psi\Sigma^0_{k,l}$ extend boundedly to $L^2(\mathbb{R}^n)$. Furthermore, there exists $C > 0$ and $N \in \mathbb{N}$ such that if $A = Op^w(a) \in \Psi\Sigma^0_{k,l}$, then
$$\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C\|a\|_{\Sigma^0_{k,l};N},$$
where $\|\cdot\|_{\Sigma^0_{k,l};N}$, denotes the inherited seminorm in the class in the class of symbols $\Sigma^0_{k,l}(\mathbb{R}^n)(\equiv S(1, g^{k,l}))$.

Associated Sobolev spaces

Using the functional calculus on (the compact, positive operator) $\mathcal{A}$ as in (1), we define the operator $\mathcal{A}^{\frac{m}{2}}$, for $m \in \mathbb{R}$, by

$$\mathcal{A}^{\frac{m}{2}}u = \sum_{j=1}^{\infty} \lambda_j^{\frac{m}{2}}(\phi_j, u)_{L^2}\phi_j, \quad \text{for} \quad u \in Dom\left(\mathcal{A}^{\frac{m}{2}}\right),$$

where $(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ made of eigenfunctions of $\mathcal{A}$, and $(\lambda_j)_{j\in\mathbb{N}}$ the corresponding eigenvalues.

The **Sobolev spaces related to** $\mathcal{A}$, denoted by $\mathcal{A}^m_{k,l}(\mathbb{R}^n)$, for $m \in \mathbb{R}$, $k,l \geq 1$, integers, is the subspace $\mathcal{S}'(\mathbb{R}^n)$ that is the completion of $Dom\left(\mathcal{A}^{\frac{m}{2}}\right)$ for the norm

$$\|u\|_{\mathcal{A}_m} := \|\mathcal{A}^{\frac{m}{2}}u\|_{L^2(\mathbb{R}^n)}.$$

Continuity Properties

The identification of the Sobolev spaces $\mathcal{A}^m_{k,l}(\mathbb{R}^n)$ with suitable Sobolev spaces $H(M,g)$ in the Weyl-Hörmander setting, and the general theory yield:

- Let $m \in \mathbb{R}$ and $k,l \geq 1$ integers. If $a \in \Sigma^m_{k,l}(\mathbb{R}^n)$, then
$$Op^w(a) : \mathcal{A}^s_{k,l}(\mathbb{R}^n) \xrightarrow{\text{cont.}} \mathcal{A}^{s-m}_{k,l}(\mathbb{R}^n), \quad \forall s \in \mathbb{R}.$$

More generally, equivalence of quantizations in our particular case yield:

- For $a \in \Sigma^m_{k,l}(\mathbb{R}^n)$ $(m \in \mathbb{R}, k,l \geq 1)$ we have
$$Op^\tau(a) : \mathcal{A}^s_{k,l}(\mathbb{R}^n) \xrightarrow{\text{cont.}} \mathcal{A}^{s-m}_{k,l}(\mathbb{R}^n), \quad \forall s \in \mathbb{R}, \forall \tau \in \mathbb{R}.$$

Anarmonic oscillators and Schatten-classes of operators

For $1 \leq r < \infty$, we denote by $S_r(L^2(\mathbb{R}^n))$ the r^{th}-Shatten class of operators.

- For $g = g^{p,q}$ as in (3), and for $a \in S(\lambda_g^{-\mu}, g)$, $\mu > \frac{n}{r}$, we have
$$Op^\tau(a) \in S_r(L^2(\mathbb{R}^n)), \quad \text{for all} \quad \tau \in \mathbb{R}.$$
- For the operator $\mathcal{A}$ as in (1), or more generally, for the operator T as in (2), and for $\mu > \frac{n(k+l)}{2klr}$ (for k,l as in (1), or accordingly to the choices of p,q as in (2)) we have
$$T^{-\mu}, \mathcal{A}^{-\mu} \in S_r(L^2(\mathbb{R}^n)).$$

Eigenvalue asymptotics for negative powers of operators

The general theory on the Schatten-classes $S_r(L^2(\mathbb{R}^n))$ and the Weyl-inequality (see [5])

$$\sum_{j=1}^{\infty} | \underbrace{\lambda_j(T)^r}_{\text{eigen. of } T} | \leq \sum_{j=1}^{\infty} \underbrace{s_j(T)^r}_{\text{sing. val. of } T}, \quad r > 0,$$

imply:

- For $g = g^{p,q}$ as in (3), and for $a \in S(\lambda_g^{-\mu}, g)$, $\mu > \frac{n}{r}$, and for any $\tau \in \mathbb{R}$ we have
$$\lambda_j(Op^\tau(a)) = o(j^{-\frac{1}{r}}), \quad \text{as} \quad j \to \infty.$$
- For the operator $\mathcal{A}$ as in (1), or more generally, for the operator T as in (2), and for $\mu > \frac{n(k+l)}{2klr}$ (for k,l as in (1), or accordingly to the choices of p,q as in (2)) we have
$$\lambda_j(\mathcal{A}^{-\mu}) = o(j^{-r}), \quad \text{as} \quad j \to \infty.$$

Rate of growth of the eigenvalues of the anharmonic oscillator $\mathcal{A}$

Let k,l be as in (1) and $r > \frac{n(k+l)}{2kl}$. Then for every $N \in \mathbb{N}$ there exists $N_0 \in \mathbb{N}$ such that

$$Nj^{\frac{1}{r}} \leq \lambda_j(\mathcal{A}), \quad \text{for} \quad j \geq N_0.$$

Thus, the eigenvalues $\lambda_j(\mathcal{A})$ are at least of growth

$$j^{\frac{1}{r}}, \quad \text{as} \quad j \to \infty.$$

Asymptotic expansions of $\lambda_j(\mathcal{A})$ have also been studied in [1] and [2].

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Dunkl-Hausdorff operators on $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ and $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$

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Abstract

In the present paper, we have studied the Dunkl–Hausdorff operators $\mathcal{H}_{\alpha,\varphi}$ on the Dunkl-type homogeneous weighted Herz spaces $\dot{K}^{\beta,p}_{\alpha,q}$ and Dunkl Herz-type Hardy space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$. We have determined simple sufficient conditions for these operators to be bounded on these spaces. As applications we provide necessary and sufficient conditions for Dunkl-Cesàro operator and sufficient conditions for Dunkl-Hardy operator to be bounded on the homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$.

Introduction

Let $\varphi \in L^1(\mathbb{R})$. First of all, we will start by recalling the definition of the Dunkl-Hausdorff operator (see [4, 6, 7, 8, 12])

$$\mathcal{H}_{\alpha,\varphi}f(x) := \int_0^\infty \frac{\varphi(t)}{t^{2\alpha+2}} f(\frac{x}{t})dt \qquad (0.1)$$

When $\alpha = -1/2$, The operator $\mathcal{H}_{\alpha,\varphi}$ is the famous Hausdorff operator

$$\mathcal{H}_{\varphi}f(x) = \int_0^\infty \frac{\varphi(t)}{t} f(\frac{x}{t})dt,$$

from which several well known operators can be deduced for suitable choices of φ, e.g., for $\varphi(t) = \frac{1}{t}\chi_{(1,\infty)}(t)$, the operator $\mathcal{H}_\varphi$ reduces to the standard Hardy averaging operator

$$\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

while for $\varphi(t) = \chi_{[0,1]}(t)$, it reduces to the adjoint of Hardy averaging operator

$$\mathcal{H}^* f(x) = \int_x^\infty \frac{f(t)}{t}\,dt.$$

For more details of its historical development, background and some applications, the reader can see a recent survey article [11] by E.Liflyand which contains the main results on Hausdorff operators in various settings and bibliography up 2013.

1 Boundedness of $\mathcal{H}_{\alpha,\varphi}$ on the homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$.

Let $\beta \in \mathbb{R}$, $0 < p < +\infty$, and $1 \le q < +\infty$. The homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$ is the space constituted by all the functions $f \in L^q_\alpha(\mathbb{R})_{loc}$, such that

$$\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})} := \Big(\sum_{k=-\infty}^{+\infty} 2^{2(\alpha+1)\beta kp}\|f\chi_k\|^p_{L^q_\alpha(\mathbb{R})}\Big)^{\frac{1}{p}} < +\infty,$$

where χ_k is the characteristic function of the set

$$A_k = \{x \in \mathbb{R};\, 2^{k-1} \le |x| \le 2^k\} \text{ for } k \in \mathbb{Z},$$

and $L^q_\alpha(\mathbb{R})_{loc}$ is the space $L^q_{loc}(\mathbb{R}, |x|^{2\alpha+1}dx)$.
Note that $\dot{K}^{\beta,0}_{\alpha,q}(\mathbb{R}) = L^q_\alpha(\mathbb{R})$.

The main result of this subsection is the following .

Theorem 1 Let $\alpha > \frac{-1}{2}$, $\beta \in \mathbb{R}$, $1 < p < +\infty$, $1 \le q < +\infty$, and φ a measurable function on $\mathbb{R}$ such that

$$C_{q,\alpha,\beta,\varphi} = \int_0^\infty |\varphi(t)| t^{2(\alpha+1)(\beta-1+\frac{1}{q})}dt < \infty.$$

Then, the Dunkl-Hausdorff operator $\mathcal{H}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself, i.e.,

$$\|\mathcal{H}_{\alpha,\varphi}f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})} \lesssim C_{q,\alpha,\beta,\varphi}\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}.$$

If φ is supported in the interval $[0,1]$, then $\mathcal{H}_{\alpha,\varphi}$ reduces to the Dunkl-Cesàro operator $\mathcal{C}_{\alpha,\varphi}$ defined by

$$\mathcal{C}_\varphi f(x) := \int_0^1 \frac{\varphi(t)}{t^{2\alpha+2}} f(\frac{x}{t})dt, \quad x \in \mathbb{R},$$

(see [4, 1]).

Corollary 1.
Let $\alpha \ge \frac{-1}{2}$, $\beta \in \mathbb{R}^\times$, $1 < p < +\infty$, $1 \le q < +\infty$, and φ a non-negative measurable function defined on $[0,1]$.
Then, the generalized Cesàro operator $\mathcal{C}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself if and only if

$$C_{q,\alpha,\varphi} = \int_0^1 \varphi(t)t^{2(\alpha+1)(\beta+\frac{1}{q}-1)}dt < \infty.$$

If $\varphi(t) = \frac{\chi_{(1,\infty)}(t)}{t}$, then (0.1) is of the following form:

$$\mathcal{H}_\alpha f(x) = \frac{1}{x^{2\alpha+1}}\int_0^x f(\xi)d\mu_\alpha(\xi).$$

In this case, $\mathcal{H}_{\alpha,\varphi}$ reduces to the Dunkl-Hardy operator for which we deduce this result.

Corollary 2. Let $\alpha \ge \frac{-1}{2}$, $1 < p < +\infty$, $1 \le q < +\infty$, $0 < \beta < 1 - \frac{1}{q}$, and $\varphi(t) = \frac{\chi_{(1,\infty)}(t)}{t}$.
Then, the Dunkl-Hardy operator $\mathcal{H}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself and we have

$$\|\mathcal{H}_{\alpha,\varphi}f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})} \le \frac{1}{2(\alpha+1)(\beta-1+\frac{1}{q})}\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}.$$

2 Boundedness of $\mathcal{H}_{\alpha,\varphi}$ on the Dunkl Herz-type Hardy space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$.

Let $\alpha \ge \frac{-1}{2}$, $N \in \mathbb{N}$, $\beta \in \mathbb{R}$, $0 < p < +\infty$, and $1 \le q < +\infty$. The Herz-type Hardy space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$ is the space of distributions $f \in S'(\mathbb{R})$ such that $G_{\alpha,N}(f) \in \dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$. Moreover, we have

$$\|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})} = \|G_{\alpha,N}(f)\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}$$

In the sequel, we are interested in the spaces $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$, when $\beta \ge 1 - 1/q$. Now, we turn to the atomic characterization of the space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$.

Let $\alpha \ge \frac{-1}{2}$, $1 \le q \le \infty$, and $\beta \ge 1 - 1/q$. A measurable function a on $\mathbb{R}$ is called a (central) $(\beta;q)$-atom if it satisfies:

1. $supp\, a \subset [-r,r]$, for a certain $r > 0$,
2. $\|a\|_{q,\alpha} \le r^{-2(\alpha+1)\beta}$,
3. $\int_{\mathbb{R}} a(x)x^k d\mu_\alpha(x) = 0$, $k = 0, 1, \cdots, 2s+1$,

where s is the integer part of $(\alpha+1)(\beta-1+1/q)$.

Theorem 2 Let $\alpha > \frac{-1}{2}$, $0 < p < +\infty$, $1 \le q < +\infty$, $\beta > 1 - 1/q$, and $N \in \mathbb{N}$; $N > 2(2s+3+\alpha)$. Then $f \in H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$ if and only if there exist, for all $j \in \mathbb{N}\setminus\{0\}$, an $(\beta;q)$-atom a_j and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=1}^\infty |\lambda_j|^p < \infty$ and $f = \sum_{j=1}^\infty \lambda_j a_j$. Moreover,

$$\|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})} = \inf\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{1/p}$$

where the infinimum is taken over all atomic decompositions of f . The main result of this subsection is the following theorem.

Theorem 3 Let $\alpha > \frac{-1}{2}$, $0 < p \leqslant 1 < q < \infty$, $\beta > 1 - 1/q$, and $N \in \mathbb{N}$; $N > 2(2s+3+\alpha)$.

(i) For $0 < p < 1$, let

$$C_{p,\sigma} = \int_{\mathbb{R}} t^{2(\alpha+1)(\beta-1+1/q)}\varphi(t)(1+|\log_2|t||)^\sigma dt.$$

If for some $\sigma > \frac{1-p}{p}$, $C_p := C_{p,\sigma} < \infty$, then

$$\|\mathcal{H}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}}.$$

(ii) For $p = 1$, let

$$C_1 = \int_{\mathbb{R}} t^{2(\alpha+1)(\beta-1+1/q)}\varphi(t)dt.$$

If $C_1 < \infty$, then

$$\|\mathcal{H}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}}.$$

We now return to the example of the Dunkl-Cesàro operator $\mathcal{C}_{\alpha,\varphi}$.

Corollary 3 Let $\alpha \ge \frac{-1}{2}$, $0 < p \leqslant 1 < q < \infty$, $\beta \ge 1 - 1/q$, and $N \in \mathbb{N}$; $N > 2(2s+3+\alpha)$.

(i) For $0 < p < 1$, let

$$C_{p,\sigma} = \int_0^1 t^{2(\alpha+1)(\beta-1+1/q)}\varphi(t)(1+|\log_2|t||)^\sigma dt.$$

If for some $\sigma > \frac{1-p}{p}$, $C_p := C_{p,\sigma} < \infty$, then

$$\|\mathcal{C}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}}.$$

(ii) For $p = 1$, let

$$C_1 = \int_0^1 t^{2(\alpha+1)(\beta-1+1/q)}\varphi(t)dt.$$

If $C_1 < \infty$, then

$$\|\mathcal{C}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}}.$$

Conclusion

We had introduced and studied recently in [4, 6, 7, 8] the Dunkl–Hausdorff operators on the weighted Lebesgue spaces $L^1_\alpha(\mathbb{R})$, Dunkl-type Hardy spaces $H^1_\alpha(\mathbb{R})$, Dunkl-type spaces of functions of bounded mean oscillation $BMO_\alpha(\mathbb{R})$, and Dunkl-type Sobolev spaces $W^{p,r}_\alpha(\mathbb{R})$. Gasmi and his collaborators in [10] introduced a new weighted Herz spaces associated with the Dunkl operators on $\mathbb{R}$. Also they characterize by atomic decompositions the corresponding Herz-type Hardy spaces. Motivated by this results, this paper aims to investigate the Dunkl–Hausdorff operators on these spaces in the spirit of those in [5]. As applications we provide necessary and sufficient conditions for Dunkl-Cesàro operator and sufficient conditions for Dunkl-Hardy operator to be bounded on the homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$.

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Hyperbolic systems with non-diagonalisable principal part

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Introduction

We consider

$$\begin{cases} D_t u = A(t,x,D_x)u + B(t,x,D_x)u + f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $n \geq 1$, $m \geq 2$ and $D_t = -\mathrm{i}\partial_t$, $D_x = -\mathrm{i}\partial_x$. We assume that $A(t,x,D_x) = (a_{ij}(t,x,D_x))_{i,j=1}^m$ is an $m \times m$ matrix of continuously time dependent pseudo-differential operators of order 1, i.e., $a_{ij} \in C([0,T], \Psi^1_{1,0}(\mathbb{R}^n))$ and that $B(t,x,D_x) = (b_{ij}(t,x,D_x))_{i,j=1}^m$ is an $m \times m$ matrix of pseudo-differential operators of order 0, i.e., $b_{ij} \in C([0,T], \Psi^0_{1,0}(\mathbb{R}^n))$. We also assume that the matrix A is upper triangular and hyperbolic, i.e.,

$$\begin{aligned} A(t,x,D_x) &= \Lambda(t,x,D_x) + N(t,x,D_x) \\ &= \mathrm{diag}(\lambda_1(t,x,D_x), \lambda_2(t,x,D_x), \ldots, \lambda_m(t,x,D_x)) + N(t,x,D_x) \end{aligned}$$

with real eigenvalues $\lambda_1(t,x,\xi), \lambda_2(t,x,\xi), \ldots, \lambda_m(t,x,\xi)$ of $A(t,x,\xi)$ and

$$N(t,x,D_x) = \begin{pmatrix} 0 & a_{12}(t,x,D_x) & a_{13}(t,x,D_x) & \cdots & a_{1m}(t,x,D_x) \\ 0 & 0 & a_{23}(t,x,D_x) & \cdots & a_{2m}(t,x,D_x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{m-1m}(t,x,D_x) \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Furthermore, we introduce the following two hypotheses:

(H1) For the coefficients of the lower order term $B(t,x,D_x)$:

the lower order terms b_{ij} belong to $C([0,T], \Psi^{j-i})$ for $i > j$.

(H2) For some theorems, we assume that A does not depend on t, i.e. $A = A(x,D_x)$ and satisfies: there exists $M \in \mathbb{N}$ such that if $\lambda_j(x,\xi) = \lambda_k(x,\xi)$ for some $j,k \in \{1,\ldots,m\}$ and $\lambda_j(x,\xi)$ and $\lambda_k(x,\xi)$ are not identically equal near (x,ξ) then there exists some $N \leq M$ such that

$$\lambda_j(x,\xi) = \lambda_k(x,\xi) \Rightarrow H^N_{\lambda_j}(\lambda_k) := \{\lambda_j, \{\lambda_j, \ldots, \{\lambda_j, \lambda_k\}\} \ldots\}(x,\xi) \neq 0, \tag{H2}$$

where the Poisson bracket $\{\cdot,\cdot\}$ in $H^N_{\lambda_j}$ is iterated N times.

Well-posedness

In Garetto et al. [2018], we prove a well-posedness result for (1) under hypothesis (H1):

Theorem 1. *Consider the Cauchy problem (1), where $A(t,x,D_x)$ and $B(t,x,D_x)$ are as described in the introduction and $B(t,x,D_x)$ satisfies (H1). If now $u^0_k \in H^{s+k-1}(\mathbb{R}^n)$ and $f_k \in C([0,T], H^{s+k-1})$ for $k = 1,\ldots,m$, then (1) has a unique anisotropic Sobolev solution u, i.e., $u_k \in C([0,T], H^{s+k-1})$ for $k = 1,\ldots,m$.*

This theorem is proved by making use of the triangular form, solving the last equation and then iteratively building the solution of the system from the solutions to scalar equations. For each characteristic λ_j of A, we denote by $G^0_j\theta$ and $G_j g$ the respective solution solution to

$$\begin{cases} D_t w = \lambda_j(t,x,D_x)w + b_{jj}(t,x,D_x)w, \\ w(0,x) = \theta(x), \end{cases} \text{ and } \begin{cases} D_t w = \lambda_j(t,x,D_x)w + b_{jj}(t,x,D_x)w + g(t,x), \\ w(0,x) = 0. \end{cases}$$

The operators G^0_j and G_j can be microlocally represented by Fourier integral operators

$$G^0_j\theta(t,x) = \int e^{\mathrm{i}\varphi_j(t,x,\xi)} a_j(t,x,\xi)\widehat{\theta}(\xi)d\xi$$

and

$$G_j g(t,x) = \int_0^t \int e^{\mathrm{i}\varphi_j(t,s,x,\xi)} A_j(t,s,x,\xi)\widehat{g}(s,\xi)d\xi ds = \int_0^t \mathcal{E}_j(t,s)g(s,x)ds,$$

where

$$\mathcal{E}_j(t,s)g(s,x) = \int e^{\mathrm{i}\varphi_j(t,s,x,\xi)} A_j(s,x,\xi)\widehat{g}(s,\xi)d\xi$$

with $\varphi_j(t,s,x,\xi)$ solving the eikonal equation

$$\begin{cases} \partial_t\varphi_j = \lambda_j(t,x,\nabla_x\varphi_j), \\ \varphi_j(s,s,x,\xi) = x\cdot\xi, \end{cases}, \quad \varphi_j(t,x,\xi) := \varphi_j(t,0,x,\xi).$$

The amplitudes $A_{j,-k}(t,s,x,\xi)$ of order $-k$, $k \in \mathbb{N}$, giving $A_j \sim \sum_{k=0}^\infty A_{j,-k}$, and they satisfy the usual transport equations with initial data at $t = s$, and we have $a_j(t,x,\xi) = A_j(t,0,x,\xi)$.

The components of the solution u of (1) is given by a composition of the operators described above together with principal part coefficients and lower order coefficients. That is where hypothesis (H1) comes into play. For example in the case $m = 2$., we get

$$\begin{cases} u_1 = U^0_1 + G_1((a_{12} + b_{12})u_2), \\ u_2 = U^0_2 + G_2(b_{21}u_1), \end{cases} \quad U^0_j = G^0_j u^0_j + G_j(f_j), \quad j = 1,2.$$

That then gives

$$\begin{aligned} u_1 &= \tilde{U}^0_1 + G_1(a_{12}G_2(b_{21}u_1)) + G_1(b_{12}G_2(b_{21}u_1)) \\ \tilde{U}^0_1 &= G^0_1 u^0_1 + G_1(f_1) + G_1((a_{12} + b_{12})U^0_2). \end{aligned}$$

One then gets to the final result by setting up a fixed point problem to which Banach's fixed point theorem can be applied. A general time interval $[0,T]$ can be iteratively covered since the estimates involved for the G's do only depend on the coefficients and not the initial data.

Solution representations and regularity results

In the case of $A = A(x,D_x)$, asking in addition to (H1) on the lower order terms also (H2) for the principal part, the solutions of (1) can be represented explicitly modulo some smoothing operators. Here, we state the principal results and refer to Garetto et al. [2020] for the details and proofs.

Theorem 2. *Consider (1) with $A = A(x,D_x)$ and $B(t,x,D_x)$ satisfying properties described above and let in addition (H1) and (H2) be satisfied. Let u_0 and f have components u^0_j and f_j, respectively, with $u^0_j \in H^{s+j-1}(\mathbb{R}^n)$ and $f_j \in C([0,T], H^{s+j-1})$ for $j = 1,\ldots,m$. Then, for any $N \in \mathbb{N}$, the components u_j, $j = 1,\ldots,m$, of the solution u are given by*

$$u_j(t,x) = \sum_{l=1}^m \left(\mathcal{H}^{l-j}_{j,l}(t) + R_{j,l}(t)\right)u^0_l + \left(\mathcal{K}^{l-j}_{j,l}(t) + S_{j,l}(t)\right)f_l,$$

where $R_{j,l}$, $S_{j,l} \in \mathcal{L}(H^s, C([0,T], H^{s+N-l+j}))$ and the operators $\mathcal{H}^{l-j}_{j,l}$, $\mathcal{K}^{l-j}_{j,l} \in \mathcal{L}(C([0,T], H^s), C([0,T], H^{s-l+j}))$ are integrated Fourier Integral Operators of order $l-j$.

Using the explicit solution representations, we get

Theorem 3. *Let $p \in (1,\infty)$ and $\alpha = (n-1)|\frac{1}{p} - \frac{1}{2}|$. Consider (1) under (H1) and (H2). Then, for any compactly supported $u_0 \in L^p_\alpha \cap L^2_{comp}$, the solution $u = u(t,x)$ of the Cauchy problem (1) satisfies $u(t,\cdot) \in L^p_{loc}$, for all $t \in [0,T]$. Moreover, there is a positive constant C_T such that*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^p_{loc}} \leq C_T\|u_0\|_{L^p_\alpha}.$$

Local estimates can be obtained in other spaces as well, for $s \in \mathbb{R}$ and α as above. In detail, assuming u_0 below is compactly supported, we have that $u_0 \in L^p_{s+\alpha}$ implies $u(t,\cdot) \in L^p_s$; that $u_0 \in C^{s+\frac{n-1}{2}}$ implies $u(t,\cdot) \in C^s$; and, for $1 < p \leq q \leq 2$, that $u_0 \in L^p_{s-\frac{1}{q}+\frac{n}{p}-\frac{n-1}{2}}$ implies $u(t,\cdot) \in L^q_s$.

Propagation of singularities

Operators of the form $(\prod_\sigma G_{\sigma(i)})^k$, which appear in the solution representation above in the operators $\mathcal{H}^{l-j}_{j,l}$, are of the general form

$$Q_l = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} D(\bar{t})H(\bar{t})\, dt_l \ldots dt_1, \quad H(\bar{t}) = e^{\mathrm{i}\lambda_{j_1}t_1}e^{\mathrm{i}\lambda_{j_2}(t_2-t_1)} \cdot \ldots \cdot e^{\mathrm{i}\lambda_{j_l}(t_l-t_{l-1})}e^{-\mathrm{i}\lambda_{j_l}t_l}.$$

For these operators, we have $Q_l \in \mathcal{L}(H^s, H^{s+N(l)})$, where $N(l) \to +\infty$ as $l \to +\infty$. The singularities propagate along broken Hamiltonian flows.

Let $J = \{j_1, \ldots, j_{l+1}\}$, $1 \leq j_k \leq m$, $j_k \neq j_{k+1}$. From the definition of $H(\bar{t})$, we have that its canonical relation $\Lambda^{\bar{t}} \subseteq T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ is given by

$$\Lambda^{\bar{t}} = \left\{(x,p,y,\xi) : (x,p) = \Psi^{\bar{t}}(y,\xi)\right\}, \quad \Psi^{\bar{t}} = \Phi^{t_1}_{j_1} \circ \cdots \circ \Phi^{t_l - t_{l-1}}_{j_l} \circ \Phi^{-t_l}_{j_{l-1}}$$

and the Φ^t_j are the transformations corresponding to a shift by t along the trajectories of the Hamiltonian flow defined by the λ_j.

Let $\Phi_J(t,x,\xi)$ be the corresponding broken Hamiltonian flow. It means that points follow bicharacteristics of λ_{j_1} until meeting the characteristic of λ_{j_2}, and then continue along the bicharacteristic of λ_{j_2}, etc.

Operators of the type Q_l can be rewritten as standard Fourier integral operators where the domain of integration is not the whole space but a simplex. For those operators, following arguments of Hörmander, we get $WF(Iu) \subset \bigcup_j \Lambda_j(WF(u))$. For details see Kamotski and Ruzhansky [2007], Garetto et al. [2020].

Corollary 1. *Let $n \geq 1$, $m \geq 2$, and consider (1) with $A = A(x,D_x)$ under hypotheses (H1) and (H2). Then, see above, we have an explicit representation of the solution u. Consequently, up to any Sobolev order (depending on N), the wave front set of u_j is given by*

$$WF(u_j(t,\cdot)) \subset \left(\bigcup_{l=1}^m WF(\mathcal{H}^{l-j}_{j,l}(t)u^0_l)\right) \bigcup \left(\bigcup_{l=1}^m WF(\mathcal{K}^{l-j}_{j,l}(t)f_l)\right), \tag{2}$$

with each of the wave front sets for terms in the right hand side of (2) given by the propagation along the broken Hamiltonian flow as described above.

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Geometric Hardy inequalities on starshaped sets

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ANALYSIS PDE UGENT

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Abstract

This work [2] is devoted to deriving the geometric Hardy inequalities on the starshaped sets in Carnot groups with some examples on Heisenberg and Engel groups. Also, the geometric Hardy inequalities are obtained on half-spaces for general vector fields, with the exmaple on the Grushin plane.

Preliminaries

Sub-Riemannian manifold: Let M be a smooth manifold of dimension n, with a given family of vector fields $\{X_k\}_{k=1}^N$, $n \geq N$, defined on M and satisfying the Hörmander condition. They induce a sub-Riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on the associated space $\mathcal{H}_x = \operatorname{span}(X_1(x), \dots, X_N(x))$. The triple $(M, \mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a sub-Riemannian manifold. Note that, unlike for Carnot groups, in general, it is not possible to define dilations, translations, the homogeneous norm and the distance in the setting of general sub-Riemannian manifolds. Let us denote the operator of the sum of squares of vector fields by

$$\mathcal{L} := \sum_{k=1}^N X_k^2,$$

where $\nabla_X := (X_1, \dots, X_N)$.

Grushin plane: One of the important examples of a sub-Riemannian manifold is the Grushin plane. The Grushin plane is the space $\mathbb{R}^2$ with vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \text{ and } X_2 = x_1\frac{\partial}{\partial x_2}.$$

Carnot groups: Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ be a stratified Lie group (or a homogeneous Carnot group or just a Carnot group), with the dilation structure δ_λ and Jacobian generators $X_1, \dots, X_N$, so that N is the dimension of the first stratum of $\mathbb{G}$. Let us denote by Q the homogeneous dimension of $\mathbb{G}$. We refer to the recent book [4] for extensive discussions of stratified (Carnot) Lie groups and their properties.

Heisenberg groups: The Heisenberg group is the most common example of a step 2 stratified group (Carnot group). The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}_1$ is spanned by

$$X_1 := \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_3}, \quad X_2 := \frac{\partial}{\partial x_2} - 2x_1\frac{\partial}{\partial x_3},$$

with their (non-zero) commutator $[X_1, X_2] = -4\frac{\partial}{\partial x_3}$.

Engel groups: The Engel group is a well-known example of a step 3 stratified group (Carnot group). Let $\mathbb{E}$ be the Engel group, with the vector fields

$$X_1 := \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \left(\frac{x_3}{2} + \frac{x_1x_2}{12}\right)\frac{\partial}{\partial x_4},$$

$$X_2 := \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2}{12}\frac{\partial}{\partial x_4},$$

$$X_3 := [X_1, X_2] = \frac{\partial}{\partial x_3} + \frac{x_1}{2}\frac{\partial}{\partial x_4}, \quad X_4 := [X_1, X_3] = \frac{\partial}{\partial x_4}.$$

Definition (Starshapedness [1])

Let $\Omega \subset \mathbb{G}$ be a C^1 domain containing the identity e. Then Ω is starshaped with respect to e if for every $x \in \partial\Omega$ one has

$$\langle Z(x), n(x)\rangle \geq 0, \tag{1}$$

where n is the Riemannian outer normal to $\partial\Omega$.

When the strict inequality holds, then Ω is said to be strictly starshaped with respect to e.

Here the vector fields Z are the infinitesimal generators of this group automorphism. This vector field Z takes the form

$$Z = \sum_{i=1}^N x_i'\frac{\partial}{\partial x_i'} + 2\sum_{l=1}^{N_2} x_{2,l}\frac{\partial}{\partial x_{2,l}} + \dots + r\sum_{l=1}^{N_r} x_{r,l}\frac{\partial}{\partial x_{r,l}}. \tag{2}$$

Then for $x' \in \mathbb{R}^N$ and $x^{(i)} \in \mathbb{R}^{N_i}$ with $i = 2, \dots, r$, we have

$$\langle Z(x), n(x)\rangle = x'n' + 2x^{(2)}n^{(2)} + \ldots + rx^{(r)}n^{(r)}, \tag{3}$$

and $Z(x) = (x', 2x^{(2)}, \cdots, rx^{(r)})$, since $n(x) := (n', n^{(2)}, \dots, n^{(r)})$ with $n' \in \mathbb{R}^N$ and $n^{(i)} \in \mathbb{R}^{N_i}$, $i = 2, \dots, r$.

Geometric Hardy inequality on the starshaped sets

Let Ω be a starshaped set on a Carnot group. Then for every $\gamma \in \mathbb{R}$ and $p > 1$ we have the following Hardy inequality

$$\int_\Omega |\nabla_H f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma)\int_\Omega \frac{|\nabla_H\langle Z(x), n(x)\rangle|^p}{|\langle Z(x), n(x)\rangle|^p}|f(x)|^p dx$$
$$+\gamma\int_\Omega \frac{\mathcal{L}_p(\langle Z(x), n(x)\rangle)}{|\langle Z(x), n(x)\rangle|^{p-1}}|f(x)|^p dx, \tag{4}$$

for every function $f \in C_0^\infty(\Omega)$.

Geometric Hardy inequalities on $\mathbb{H}^*$ and $\mathbb{E}^*$

Let $\mathbb{H}^*$ and $\mathbb{E}^*$ be starshaped sets on the Heisenberg group $\mathbb{H}_1$ and the Engel group. Then for every function $f \in C_0^\infty(\mathbb{H}^*)$, we have the following Hardy inequalities

$$\int_{\mathbb{H}^*} |\nabla_H f(x)|^2 dx \geq \frac{1}{4}\int_{\mathbb{H}^*} \frac{|(n_1 + 4x_2n_3, n_2 - 4x_1n_3)|^2}{|x_1n_1 + x_2n_2 + 2x_3n_3|^2}|f(x)|^2 dx, \tag{5}$$

and for all $\gamma \in \mathbb{R}$, and $f \in C_0^\infty(\mathbb{E}^*)$,

$$\int_{\mathbb{E}^*} |\nabla_H f(x)|^2 dx \geq -(|\gamma|^2 + \gamma)\int_{\mathbb{E}^*} \frac{|\nabla_H\langle Z(x), n(x)\rangle|^2}{\langle Z(x), n(x)\rangle^2}|f(x)|^2 dx \tag{6}$$
$$+\frac{\gamma}{2}\int_{\mathbb{E}^*} \frac{x_2n_4}{\langle Z(x), n(x)\rangle}|f(x)|^2 dx.$$

Hardy inequality on the half-spaces of M

Let us define the half-space of a sub-Riemannian manifold by

$$\Omega^+ := \{x \in \mathbb{R}^n : \langle x, n(x)\rangle > d\},$$

where $n = n(x) \in \mathbb{R}^n$ is the Riemannian outer unit normal to $\partial\Omega^+$ and $d \in \mathbb{R}$. The Euclidean distance to the boundary $\partial\Omega^+$ is denoted by $dist(x, \partial\Omega^+)$ and defined by

$$dist(x, \partial\Omega^+) := \langle x, n(x)\rangle - d.$$

Let M be a sub-Riemannian manifold, let $\Omega^+ \subset M$ be a half-space and let $X_1, \dots, X_N$ be the general vector fields. Then for every $\gamma \in \mathbb{R}$ and every $p > 1$, we have the following Hardy inequality

$$\int_{\Omega^+} |\nabla_X f|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma)\int_{\Omega^+} \frac{|\nabla_X dist(x, \partial\Omega^+)|^p}{dist(x, \partial\Omega^+)^p}|f|^p dx$$
$$+\gamma\int_{\Omega^+} \frac{\mathcal{L}_p(dist(x, \partial\Omega^+))}{dist(x, \partial\Omega^+)^{p-1}}|f|^p dx,$$

for every function $f \in C_0^\infty(\Omega^+)$.

Note that above inequality was obtained for the Carnot groups by the authors in [3], but here we extend it to general sub-Riemannian manifolds.

Let Ω^+ be a half-space in the Grushin plane G. Then for every function $f \in C_0^\infty(\Omega^+)$ and every $p > 1$, we have the following Hardy inequality

$$\int_{\Omega^+} |\nabla_X f|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma)\int_{\Omega^+} \frac{(n_1^2 + x_1^2n_2^2)^{p/2}}{(x_1n_1 + x_2n_2 - d)^p}|f|^p dx$$
$$+(p-2)\gamma\int_{\Omega^+} \frac{|\nabla_X dist(x, \partial\Omega^+)|^{p-4}n_1n_2^2x_1}{(x_1n_1 + x_2n_2 - d)^{p-1}}|f|^p dx.$$

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Hardy and Hardy-Sobolev inequalities on Heisenberg groups

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Abstract

This work [3] is devoted to present the geometric Hardy and Hardy-Sobolev inequalities for the sub-Laplacian in the half-spaces of the Heisenberg group with a sharp constant. This result answers a conjecture posed by S. Larson in [2]. As a consequence, a geometric Hardy-Sobolev-Maz'ya inequality is recovered.

Preliminaries on the Heisenberg group:

Let $\mathbb{H}^n$ be the Heisenberg group, that is, the set $\mathbb{R}^{2n+1}$ equipped with the group law

$$\xi \circ \widetilde{\xi} := (x + \widetilde{x}, y + \widetilde{y}, t + \widetilde{t} + 2\sum_{i=1}^{n}(\widetilde{x}_i y_i - x_i \widetilde{y}_i)),$$

where $\xi := (x, y, t) \in \mathbb{H}^n$, $x := (x_1, \ldots, x_n)$, $y := (y_1, \ldots, y_n)$, and $\xi^{-1} = -\xi$ is the inverse element of ξ with respect to the group law. The dilation operation of the Heisenberg group with respect to the group law has the following form $\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t)$ for $\lambda > 0$.
The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}^n$ is spanned by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \text{ and } Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

and with their (non-zero) commutator $[X_i, Y_i] = -4\frac{\partial}{\partial t}$. The horizontal gradient of $\mathbb{H}^n$ is $\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$
Let us define the half-space of the Heisenberg group by

$$\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu \rangle > d\},$$

where $\nu := (\nu_x, \nu_y, \nu_t)$ with $\nu_x, \nu_y \in \mathbb{R}^n$ and $\nu_t \in \mathbb{R}$ is the Riemannian outer unit normal to $\partial\mathbb{H}^+$ (see [1]) and $d \in \mathbb{R}$. The Euclidean distance to the boundary $\partial\mathbb{H}^+$ is defined by

$$dist(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d.$$

Let us define

$$X_i(\xi) = (\underbrace{0, \ldots, \overset{i}{1}, \ldots, 0}_{n}, \underbrace{0, \ldots, 0}_{n}, 2y_i),$$

$$Y_i(\xi) = (\underbrace{0, \ldots, 0}_{n}, \underbrace{0, \ldots, \overset{i}{1}, \ldots, 0}_{n}, -2x_i).$$

Then we have

$$\langle X_i(\xi), \nu \rangle = \nu_{x,i} + 2y_i \nu_t, \qquad \langle Y_i(\xi), \nu \rangle = \nu_{y,i} - 2x_i \nu_t,$$

where $\xi := (x, y, t)$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\nu := (\nu_x, \nu_y, \nu_t)$.

Introduction

The Hardy inequality in the half-space on the Heisenberg group was shown by Luan and Young in [4] as follows

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi. \tag{1}$$

An alternative proof of this inequality was given by Larson in [2], where the author generalised it to any half-space of the Heisenberg group,

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4} \int_{\mathbb{H}^+} \frac{\sum_{i=1}^{n} \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2}{dist(\xi, \partial\mathbb{H}^+)^2} |u|^2 d\xi,$$

where X_i and Y_i (for $i = 1, \ldots, n$) are left-invariant vector fields on the Heisenberg group, ν is the Riemannian outer unit normal to the boundary. Also, there is the L^p-generalisation of the above inequality

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\sum_{i=1}^{n} |\langle X_i(\xi), \nu \rangle|^p + |\langle Y_i(\xi), \nu \rangle|^p}{dist(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi. \tag{2}$$

Conjecture posed by S. Larson

A more natural weight in the right-hand side of (2) would be

$$\frac{\left(\sum_{i=1}^{n} \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2\right)^{p/2}}{dist(\xi, \partial\mathbb{H}^+)^p}.$$

L^p-Hardy inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all functions $u \in C_0^\infty(\mathbb{H}^+)$ and $p > 1$ we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq C \int_{\mathbb{H}^+} \frac{\left(\sum_{i=1}^{n} \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2\right)^{p/2}}{dist(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi, \tag{3}$$

where the constant $C := \left(\frac{p-1}{p}\right)^p$ is sharp.

L^2-Hardy inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all functions $u \in C_0^\infty(\mathbb{H}^+)$ we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi, \tag{4}$$

where the constant is sharp.
This corollary can be proved by considering $p = 2$, $dist(\xi, \partial\mathbb{H}^+) = t$ and $\nu = (0, 0, 1)$.

Hardy-Sobolev inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$ and $2 \leq p < Q$ with $Q = 2n + 1$, there exists some $C_1 > 0$ such that we have

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - C \int_{\mathbb{H}^+} \frac{\left(\sum_{i=1}^{n} \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2\right)^{p/2}}{dist(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi\right)^{\frac{1}{p}} \geq C_1 \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{\frac{1}{p^*}},$$

where $p^* := Qp/(Q - p)$ and the constant $C := \left(\frac{p-1}{p}\right)^p$.

Hardy-Sobolev-Maz'ya inequality

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$, there exists some $C > 0$ such that we have

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi\right)^{\frac{1}{2}} \geq C \left(\int_{\mathbb{H}^+} |u|^{2^*} d\xi\right)^{\frac{1}{2^*}},$$

where $2^* := 2Q/(Q - 2)$ with the homogeneous dimension $Q = 2n + 1$.

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GHENT ANALYSIS PDE

Noncommutative conference

16–20 August 2020, Ghent University

Be harmonic with analysis
Overcoming noncommutativity without commuting

Calderón's type reproducing formula related to the q-Dunkl two wavelet theory

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Abstract

In this paper, using some elements of the q-harmonic analysis associated to the q-Dunkl operator introduced by N. Bettaibi et al. in [1], for fixed $0 < q < 1$, the notion of a q-Dunkl two-wavelet is introduced. The resolution of the identity formula for the q-Dunkl continuous wavelet transform is then formalized and proved. Calderón's type reproducing formula in the context of the q-Dunkl two wavelet theory is proved.

Introduction

Calderón formula [3] involving convolution related to the Fourier transform is useful in obtaining reconstruction formula for wavelet transform besides many other applications in decomposition of certain function spaces. It is expressed as follows:

$$\phi(\xi) = \frac{1}{c_{\varphi,\phi}} \int_0^{+\infty} \Phi * \varphi_t * \phi_t(\xi) \frac{dt}{t}, \quad \xi \in \mathbb{R}, \tag{1}$$

where

$$\varphi_t(x) := \frac{1}{t}\varphi\left(\frac{x}{t}\right), \quad \phi_t(x) := \frac{1}{t}\phi\left(\frac{x}{t}\right), \quad \forall x \in \mathbb{R},$$

$c_{\varphi,\phi}$ is a constant depending on functions φ, ϕ and $*$ denotes a convolution operation.

The aim of this paper is to give a q-version of Calderón's type reproducing formula (1) in the context of q-Dunkl harmonic analysis, more precisely, we study similar question when in (1), the classical convolution $*$ is replaced by a generalized q-Dunkl convolution $*_{q,\alpha}$ on the real line generated by the q-Dunkl differential operator $\Lambda_{q,\alpha}$, $\alpha \geq -1/2$.

Preliminaries and q-Notations

Throughout this paper we assume that $\alpha \geq -1/2$ and $0 < q < 1$.

1. We denote

$$\mathbb{R}_q = \{\pm q^n,\ n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n,\ n \in \mathbb{Z}\} \quad \text{and} \quad \widetilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}.$$

2. For complex number a, the q-shifted factorials are defined by:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \ n = 1, 2, \ldots, \quad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

and we also denote for all $x \in \mathbb{C}$ and $n \in \mathbb{N}$

$$[x]_q = \frac{1-q^x}{1-q}, \quad [n]_q! = [1]_q \times [2]_q \ldots \times [n]_q = \frac{(q;q)_n}{(1-q)^n}.$$

3. The q-Jackson integrals from 0 to a and from $-\infty$ to $+\infty$ are defined by

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{+\infty} q^n f(aq^n),$$

$$\int_{-\infty}^{+\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{+\infty} q^n [f(q^n) + f(-q^n)].$$

4. We denote by $L^p_{q,\alpha}(\mathbb{R}_q)$, $p \in [1, +\infty]$, the set of all real functions on $\mathbb{R}_q$ for which

$$\|f\|_{q,p,\alpha} = \begin{cases} \left(\int_{-\infty}^{+\infty} |f(x)|^p |x|^{2\alpha+1} d_q x\right)^{1/p} < +\infty & \text{if } 1 \leq p < +\infty, \\ \operatorname{ess\,sup}_{x \in \mathbb{R}_q} |f(x)| < +\infty & \text{if } p = +\infty, \end{cases}$$

5. The q-Dunkl translation operator is defined for $f \in L^1_{q,\alpha}(\mathbb{R}_q)$ and $x, y \in \mathbb{R}_q$ by

$$T_x^{D,\alpha}(f)(y) = c_{q,\alpha} \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda) \Psi_{q,\alpha}(\lambda x) \Psi_{q,\alpha}(\lambda y) |\lambda|^{2\alpha+1} d_q \lambda.$$

The q-Dunkl translation operators allow us to define a q-Dunkl convolution product $*_{q,\alpha}$ as follows: for all $f, g \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$f *_{q,\alpha} g(x) = c_{q,\alpha} \int_{-\infty}^{\infty} T_x^{D,\alpha}(f)(-y) g(y) |y|^{2\alpha+1} d_q y, \quad \forall x, y \in \mathbb{R}_q,$$

provided the q-integral exists.

q-Dunkl Harmonic analysis associated with $\Lambda_{q,\alpha}$

1. The q-Dunkl operator $\Lambda_{q,\alpha}$ is defined by

$$\Lambda_{q,\alpha}(f)(x) = \partial_q[\mathcal{H}_{q,\alpha}(f)](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{2x},$$

where

$$\mathcal{H}_{q,\alpha} : f = f_e + f_o \mapsto f_e + q^{2\alpha+1} f_o,$$

with f_e and f_o are respectively, the even and the odd parts of f.
The q-Dunkl kernel is defined on $\mathbb{R}_q$ by

$$\Psi_{q,\alpha}(z) = j_\alpha(z; q^2) + \frac{iz}{[2\alpha+2]_q} j_{\alpha+1}(z; q^2), \quad z \in \mathbb{R}_q,$$

where $j_\alpha(\cdot, q^2)$ is the normalized third Jackson's q-Bessel function given by

$$j_\alpha(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} \left(\frac{x}{1+q}\right)^{2n}.$$

2. the function $\Psi_{q,\alpha}(\lambda\cdot)$, $\lambda \in \mathbb{C}$ is the unique analytic solution of the q-differential-difference equation:

$$\begin{cases} \Lambda_{q,\alpha}(f) = i\lambda f, \\ f(0) = 1. \end{cases}$$

3. The q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$ is defined on $L^1_{q,\alpha}(\mathbb{R}_q)$ by

$$\mathcal{F}_D^{q,\alpha}(f)(\lambda) = c_{q,\alpha} \int_{-\infty}^{\infty} f(x) \Psi_{q,\alpha}(-\lambda x) |x|^{2\alpha+1} d_q x, \quad \forall \lambda \in \mathbb{R}_q,$$

where

$$c_{q,\alpha} = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}.$$

4. The q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ onto itself and extends uniquely to an isometric isomorphism on $L^2_{q,\alpha}(\mathbb{R}_q)$ with:

$$\|\mathcal{F}_D^{q,\alpha}(f)\|_{q,2,\alpha} = \|f\|_{q,2,\alpha}.$$

5. If $f \in L^1_{q,\alpha}(\mathbb{R}_q)$ such that $\mathcal{F}_D^{q,\alpha}(f) \in L^1_{q,\alpha}(\mathbb{R}_q)$, then the q-inversion formula holds and we have

$$f(x) = \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda) \Psi_{q,\alpha}(\lambda x) d\mu_{q,\alpha}(\lambda).$$

q-Dunkl two-wavelet theory

Definition 1 (q-Dunkl wavelet)
A q-Dunkl wavelet is a square q-integrable function h on $\mathbb{R}_q$ satisfying the following admissibility condition

$$0 < C_{\alpha,h} = \int_0^{+\infty} |\mathcal{F}_D^{q,\alpha}(h)(a)|^2 \frac{d_q a}{a} = \int_0^{+\infty} |\mathcal{F}_D^{q,\alpha}(h)(-a)|^2 \frac{d_q a}{a} < \infty.$$

Definition 2 (q-Dunkl two-wavelet)
Let u and v be in $L^2_{q,\alpha}(\mathbb{R}_q)$. We say that the pair (u, v) is a q-Dunkl two-wavelet on $\mathbb{R}_q$ if for almost all $\lambda \in \mathbb{R}_q$, we have

$$C_{\alpha,u,v} = \int_0^{+\infty} \mathcal{F}_D^{q,\alpha}(v)(a\lambda) \overline{\mathcal{F}_D^{q,\alpha}(u)(a\lambda)} \frac{d_q a}{a} < \infty.$$

Definition 3 (The continuous q-wavelet transform)
Let h be a q-Dunkl wavelet on $\mathbb{R}_q$ in $L^2_{q,\alpha}(\mathbb{R}_q)$. We define the continuous q-wavelet transform associated with the q-Dunkl operator for all $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ by

$$\Psi_{q,h}^{\alpha,D}(f)(a, x) = c_{q,\alpha} \int_{-\infty}^{+\infty} f(\lambda) \overline{h_{a,x}(\lambda)} |\lambda|^{2\alpha+1} d_q \lambda,$$

where

$$h_{a,x}(\lambda) = a^{\alpha+1} T_x^{D,\alpha}(h_a)(\lambda) \text{ and } h_a(x) = \frac{1}{a^{2\alpha+2}} h\left(\frac{x}{a}\right), \quad \forall x \in \mathbb{R}_q.$$

Theorem 1 (Parseval formula)
Let (u, v) be a q-Dunkl two-wavelet. Then for all f and g in $L^2_{q,\alpha}(\mathbb{R}_q)$, there holds

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \Psi_{q,u}^{\alpha,D}(f)(a, x) \overline{\Psi_{q,v}^{\alpha,D}(g)(a, x)} d\mu_{q,\alpha}(a, x)$$
$$= C_{\alpha,u,v} \int_{-\infty}^{+\infty} f(x) \overline{g(x)} |x|^{2\alpha+1} d_q x.$$

Theorem 3 (Inversion formula) Let (u, v) be a q-Dunkl two-wavelet. For all f in $L^1_{q,\alpha}(\mathbb{R}_q)$ such that $\mathcal{F}_D^{q,\alpha}(f)$ belongs to $L^1_{q,\alpha}(\mathbb{R}_q)$, we have

$$f(\lambda) = \frac{c_{q,\alpha}}{C_{\alpha,u,v}} \int_0^{+\infty} \int_{-\infty}^{+\infty} \Psi_{q,u}^{\alpha,D}(f)(a, x) v_{a,x}(\lambda) d\mu_{q,\alpha}(a, x), \quad \forall \lambda \in \mathbb{R}_q.$$

Calderón's type reproducing formula in the context of the q-Dunkl two-wavelet

Theorem 1 (q-Calderón's type formula) Let u and v be two q-Dunkl wavelets in $L^2_{q,\alpha}(\mathbb{R}_q)$ such that (u, v) is a q-Dunkl two-wavelet, $C_{\alpha,u,v} \neq 0$, and $\mathcal{F}_D^{q,\alpha}(u)$ and $\mathcal{F}_D^{q,\alpha}(v)$ both belong to $L^\infty_{q,\alpha}(\mathbb{R}_q)$. Then, for all f in $L^2_{q,\alpha}(\mathbb{R}_q)$ and $0 < \varepsilon < \delta < \infty$, the function

$$f^{\varepsilon,\delta}(\lambda) = \frac{c_{q,\alpha}}{C_{\alpha,u,v}} \int_\varepsilon^\delta \int_{-\infty}^{+\infty} \Psi_{q,u}^{\alpha,D}(f)(a, x) v_{a,x}(\lambda) |x|^{2\alpha+1} d_q x \frac{d_q a}{a^{2\alpha+1}}$$

belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$, and satisfies

$$\lim_{\varepsilon \to 0,\, \delta \to +\infty} \|f^{\varepsilon,\delta} - f\|_{q,2,\alpha} = 0.$$

Conclusion

In this paper, using some new elements of q-harmonic analysis related to the q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$ introduced by in [1], we define and study the q-Dunkl two wavelet and the continuous q-wavelet transform associated with this q-harmonic analysis. In addition to several properties, we establish a Plancherel formula and an inversion theorem for this transform. As applications, we prove a Calderón's type reproducing formula in the context of the q-Dunkl two wavelet theory.

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