Ghent Noncommutatiive Conference
August 20, 2020

# Degenerate Diffusions on Unimodular Lie Groups

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# Outline of this talk

**This is <u>not</u> a mathematics talk. Rather, it is a mathematical modeling talk which applies results from Lie theory and noncommutative harmonic analysis to Finance, Robotics, and DNA Statistical Mechanics.**

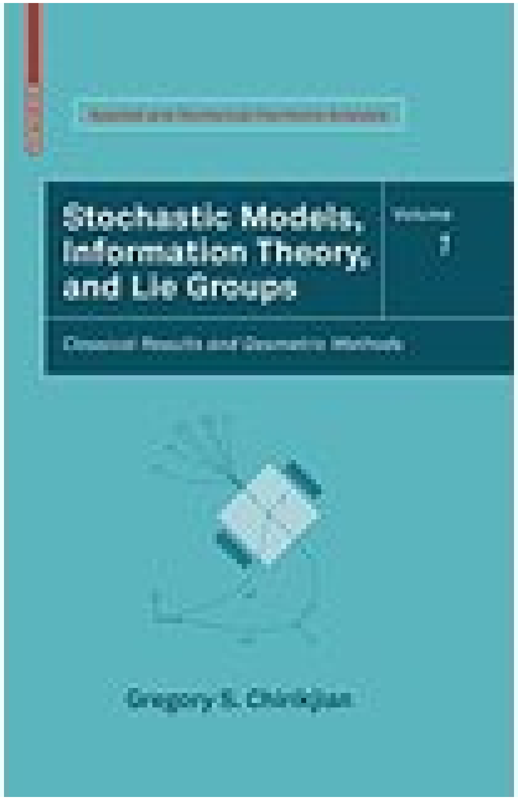
- **Background Mathematics and Terminology (Degenerate Diffusions on unimodular Lie groups, and how to describe them with Fourier and Gaussian methods)**
- **A Baby Example from Mathematical Finance**
- **Uncertainty Propagation in Nonholonomic Vehicles**
- **DNA and Filament Statistical Mechanics**
- **Mathematical Finance (Revisited)**

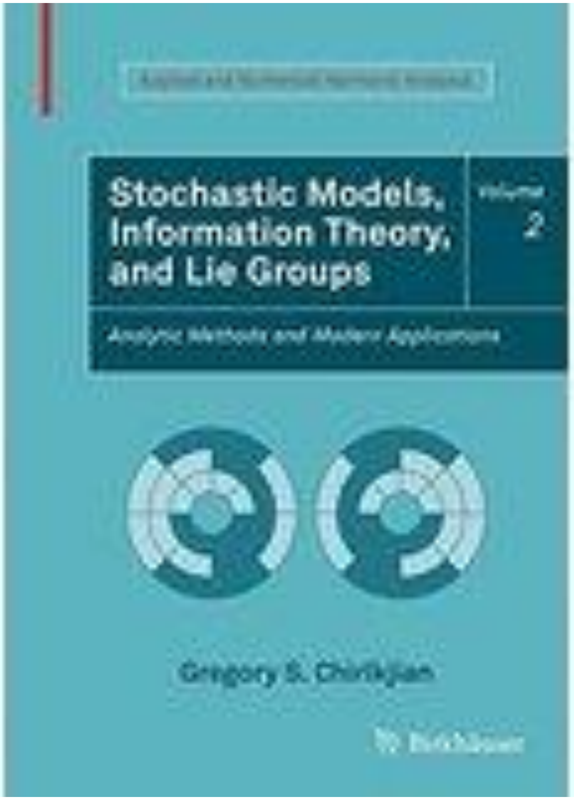
- **Acknowledgements: W. Park,Y. Zhou, A. Okamura, N. Cowan, Y. Wang, A.B. Kyatkin, Amitesh Jayaraman, …**

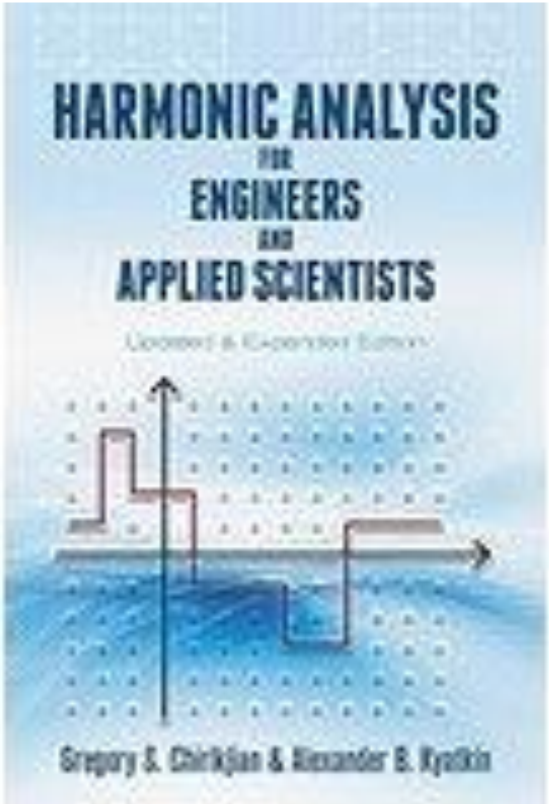
# Topic 1:

# Background Mathematics

# For Notation, See



# Diffusion Processes on Unimodular Lie Groups

$$\int_G f(g)dg = \int_G f(g \circ g_0)dg = \int_G f(g_0 \circ g)dg = \int_G f(g^{-1})dg$$

$$E_i^R f(g) \doteq \left. \frac{d}{dt} f(g \circ \exp(tE_i)) \right|_{t=0} \quad \text{and} \quad E_i^L f(g) \doteq \left. \frac{d}{dt} f(\exp(-tE_i) \circ g) \right|_{t=0}$$

$$\frac{\partial u}{\partial t} = -\sum_{i=1}^{N} m_i(t) E_i^R u + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} D_{ij}(t) E_i^R E_j^R u$$

$$\begin{aligned} u(g, t + \delta t) &= u(g,t) * u_f(g, \delta t) \\ &= \int_G u(h,t) u_f(h^{-1} \circ g, \delta t)\, dh \end{aligned}$$

**By `degenerate' what is meant is det D = 0**

# V and Hat Maps

$$(E_i, E_j) = e_i^T e_j \text{ where } E_i^\vee = e_i$$

$$[Ad(g)] = \left[(gE_1g^{-1})^\vee, \cdots, (gE_Ng^{-1})^\vee\right]$$

# Mean, Covariance, Gaussians

$$\int_G [\log(\mu(t)^{-1} \circ g)^{\vee}] u(g,t)\, dg = \mathbf{0}.$$

$$\Sigma(t) \doteq \int_G [\log(\mu(t)^{-1} \circ g)^{\vee}][\log(\mu(t)^{-1} \circ g)^{\vee}]^T u(g,t)\, dg$$

$$u_f(g, \delta t) = \frac{1}{(2\pi)^{N/2} |\det \Sigma(\delta t)|^{1/2}} \exp\left( -\frac{1}{2} [\log(\mu(\delta t)^{-1} \circ g)^{\vee}]^T \Sigma^{-1}(\delta t) [\log(\mu(\delta t)^{-1} \circ g)^{\vee}] \right)$$

# Propagation of Mean and Covariance Under Convolution on Unimodular Lie Groups

$$\mu(t+\delta t) = \mu(t) \circ \mu(\delta t),$$

$$\Sigma(t+\delta t) = \Sigma(\delta t) + [Ad(\mu(\delta t)^{-1})]\Sigma(t)[Ad(\mu(\delta t)^{-1})]^T$$

$$\mu(t) = \exp\left(\int_0^t m(\tau)^\wedge \, d\tau\right)$$

$$\Sigma(t) = \int_0^t [Ad(\mu(\tau)^{-1})]D(\tau)[Ad(\mu(\tau)^{-1})]^T \, d\tau$$

# Convolution Theorem and the Fourier Transform for Long Time Solutions

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1} \circ g) dh$$

$$F(f_1 * f_2) = F(f_2)F(f_1)$$

**Chirikjian, G.S., Kyatkin, A.B., Engineering Applications of Noncommutative Harmonic Analysis, CRC Press, 2001.**

# Finance

# *One-Asset Black-Scholes as a Diffusion on* $Gl^+(1)$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 a^2 \frac{\partial^2 V}{\partial a^2} + ra\frac{\partial V}{\partial a} - rV = 0.$$

$$Ef(a) = a\frac{\partial f}{\partial a}$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}E^2 V + \left(r - \frac{\sigma^2}{2}\right)EV - rV = 0.$$

# One-Asset Black-Scholes as a Diffusion on $Gl^+(1)$

$$V(a,t') = u(a,t')e^{-rt'}$$

$$\frac{\partial u}{\partial t'} = \left(r - \frac{\sigma^2}{2}\right) Eu + \frac{\sigma^2}{2} E^2 u$$

$$u_f(a,t') = \frac{1}{\sigma\sqrt{2\pi t'}} \exp\left(-\frac{1}{2\sigma^2 t'}[\log a + \left(r - \frac{\sigma^2}{2}\right) t']^2\right)$$

# Mobile Robots

# Planar Rigid-Body Motions

**Parameterization with Translation in Cartesian Coordinates**

$$g(x,y,\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$g(x_1,y_1,\theta_1) \circ g(x_2,y_2,\theta_2) =$$
$$g(x_1 + x_2\cos\theta_1 - y_2\sin\theta_1, y_1 + x_2\sin\theta_1 + y_2\cos\theta_1, \theta_1+\theta_2)$$

**Translation in Polar Coordinates Coordinates**

$$g(\phi,r,\theta) = \begin{pmatrix} \cos\phi & -\sin\phi & r\cos\theta \\ \sin\phi & \cos\phi & r\sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

**Exponential**

$$\begin{aligned} g(v_1,v_2,\alpha) &= \exp(X) \\ &= \begin{pmatrix} \cos\alpha & -\sin\alpha & t_1 \\ \sin\alpha & \cos\alpha & t_2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} t_1 &= [v_2(-1+\cos\alpha) + v_1\sin\alpha]/\alpha \\ t_2 &= [v_1(1-\cos\alpha) + v_2\sin\alpha]/\alpha. \end{aligned}$$

# Rigid-Body Motions in Euclidean Space

$$SE(n) = (\mathbb{R}^n, +) \rtimes SO(n)$$

$$g_1 \circ g_2 = (R_1, \mathbf{t}_1) \circ (R_2, \mathbf{t}_2) = (R_1 R_2, R_1 \mathbf{t}_2 + \mathbf{t}_1)$$

$$g^{-1} = (R^T, -R^T \mathbf{t}) \quad \text{and} \quad e = (\mathbb{I}, \mathbf{0})$$

# Stochastic Vehicle Models

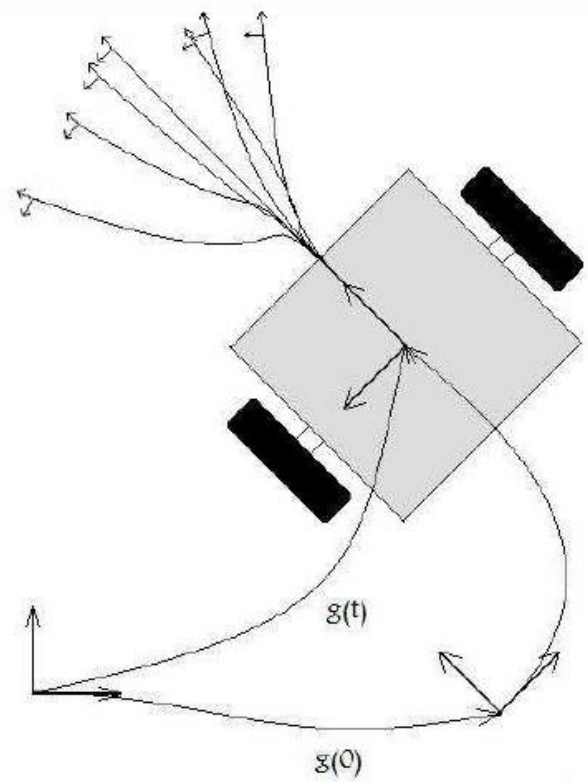
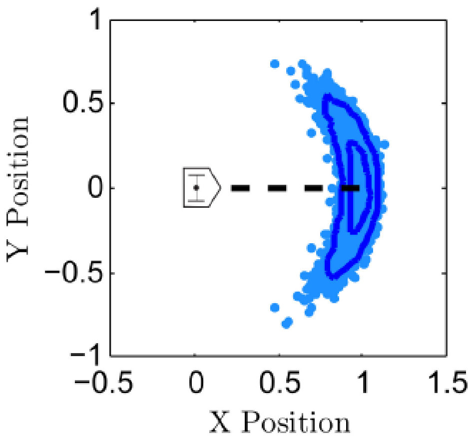



Fig. 0.1. A Kinematic Cart with an Uncertain Future Position and Orientation

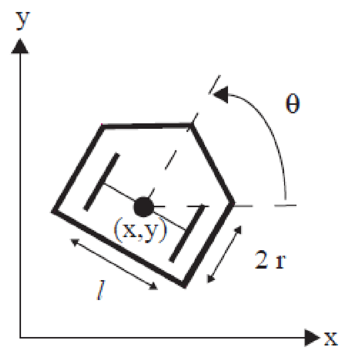
$$d\phi_1 = \omega(t)dt + \sqrt{D}dw_1$$
$$d\phi_2 = \omega(t)dt + \sqrt{D}dw_2$$

# SDE for the Kinematic Cart

**(Zhou and Chirikjian, ICRA 2003)**




(a)




(b)

$$\begin{pmatrix} dx \\ dy \\ d\theta \end{pmatrix} = \begin{pmatrix} \frac{r}{2}(\omega_1 + \omega_2)\cos\theta \\ \frac{r}{2}(\omega_1 + \omega_2)\sin\theta \\ \frac{r}{\ell}(\omega_1 - \omega_2) \end{pmatrix} dt + \sqrt{D} \begin{pmatrix} \frac{r}{2}\cos\theta & \frac{r}{2}\cos\theta \\ \frac{r}{2}\sin\theta & \frac{r}{2}\sin\theta \\ \frac{r}{\ell} & -\frac{r}{\ell} \end{pmatrix} \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix}$$

$$\begin{pmatrix} dx \\ \\ dy \\ \\ d\theta \end{pmatrix} = \begin{pmatrix} r\omega\cos\theta \\ \\ r\omega\sin\theta \\ \\ 0 \end{pmatrix} dt + \sqrt{D} \begin{pmatrix} \frac{r}{2}\cos\theta & \frac{r}{2}\cos\theta \\ \\ \frac{r}{2}\sin\theta & \frac{r}{2}\sin\theta \\ \\ \frac{r}{L} & -\frac{r}{L} \end{pmatrix} \begin{pmatrix} dw_1 \\ \\ dw_2 \end{pmatrix} \tag{0.4}$$

Corresponding to an SDE is a Fokker-Planck equation

$$\frac{\partial f}{\partial t} = -r\omega\cos\theta\frac{\partial f}{\partial x} - r\omega\sin\theta\frac{\partial f}{\partial y} +$$
$$\frac{D}{2}\left(\frac{r^2}{2}\cos^2\theta\frac{\partial^2 f}{\partial x^2} + \frac{r^2}{2}\sin 2\theta\frac{\partial^2 f}{\partial x\partial y} + \frac{r^2}{2}\sin^2\theta\frac{\partial^2 f}{\partial y^2} + \frac{2r^2}{L^2}\frac{\partial^2 f}{\partial \theta^2}\right).$$

There is a very clean coordinate-free way of writing these SDEs and FPEs. Namely,

$$\left(g^{-1}\frac{dg}{dt}\right)^{\vee} dt = r\omega\mathbf{e}_1 dt + \frac{r\sqrt{D}}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2/L & -2/L \end{pmatrix} d\mathbf{w}$$

# Definition of Operators

- **Let X be an infinitesimal planar rigid-body motion. Then**

$$\left(X^R f\right)(g) = \left.\frac{df\left(g e^{tX}\right)}{dt}\right|_{t=0}$$

- **$X^R$ can be thought of as the right directional derivative of *f* in the direction X. In particular, infinitesimal rigid-body motions in the plane are all combinations of:**

$$X_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\vee$ is the "vee operator" . The coordinate-free version of the Fokker-Planck equation is given below.

## Calculus on Euclidean Groups

Analogs of the usual partial derivatives in $\mathbb{R}^n$ can be defined in the Lie-group setting as

$$\tilde{X}_i f = \left[ \frac{d}{dt} f\left(g \circ e^{tX_i}\right) \right] \Big|_{t=0}, \; i = 1, 2, 3. \tag{0.5}$$

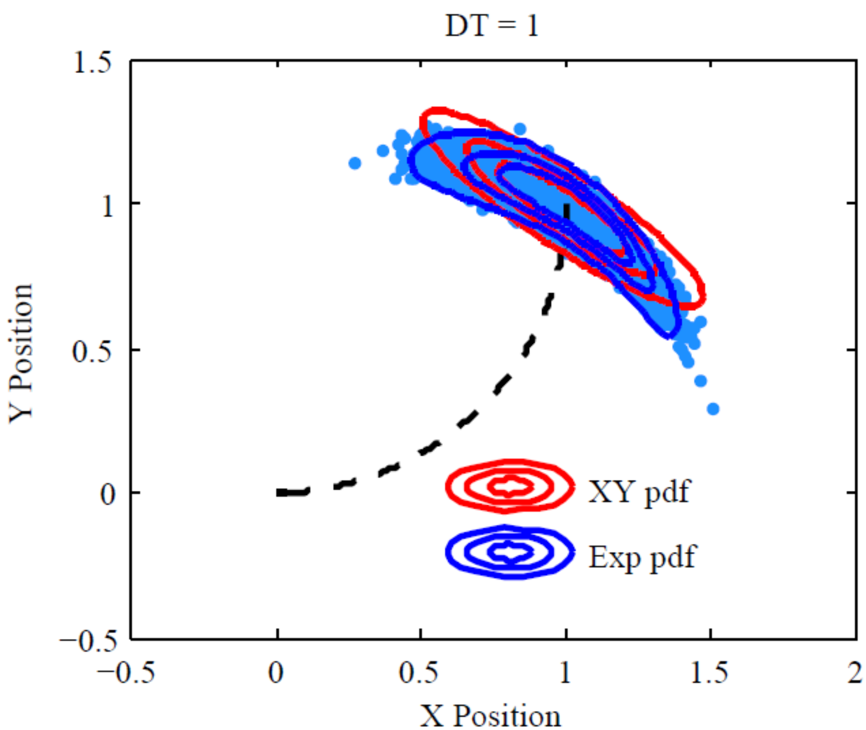
These are called Lie derivatives. The Fokker-Planck equation above can be written compactly in terms of these Lie derivatives as

$$\frac{\partial f}{\partial t} = -r\omega \tilde{X}_1 f + \frac{r^2 D}{4} (\tilde{X}_1)^2 f + \frac{r^2 D}{L^2} (\tilde{X}_3)^2 f. \tag{0.6}$$

# Exponential Coordinates for SE(2)

$$
\begin{aligned}
g(v_1, v_2, \alpha) &= \exp(X) \\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & t_1 \\ \sin\alpha & \cos\alpha & t_2 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
t_1 &= [v_2(-1 + \cos\alpha) + v_1 \sin\alpha]/\alpha \\
t_2 &= [v_1(1 - \cos\alpha) + v_2 \sin\alpha]/\alpha.
\end{aligned}
$$




**A. Long, K. Wolfe, M. Mashner, G. Chirikjian, ``The Banana Distribution is Gaussian" RSS 2012**

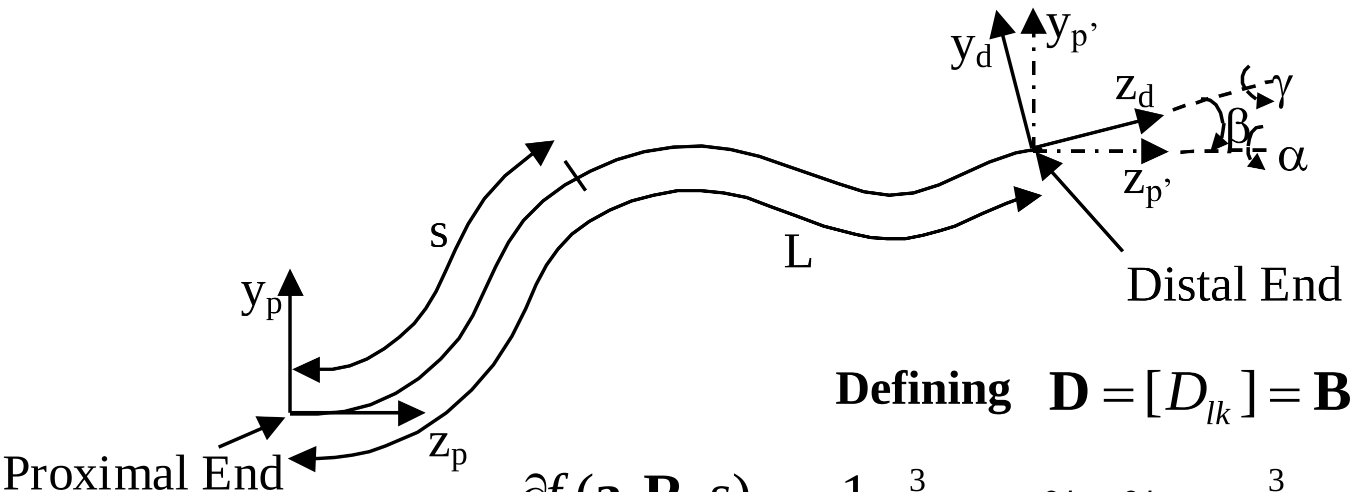
$$\int_G \log^\vee\left(\mu^{-1}\circ g\right) f(g)dg = \mathbf{0} \qquad \Sigma = \int_G \log^\vee(\mu^{-1}\circ g)[\log^\vee(\mu^{-1}\circ g)]^T f(g)dg$$

$$f(g;\mu,\Sigma) = \frac{1}{c(\Sigma)}\exp\left[-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y}\right] \qquad \mathbf{y} = \log(\mu^{-1}\circ g)^\vee$$

# DNA Statistical Mechanics

# A General Semiflexible Polymer Model

**A diffusion equation describing the PDF of relative pose between the frame of reference at arc length s and that at the proximal end of the chain**




**Defining** $\mathbf{D} = [D_{lk}] = \mathbf{B}^{-1} \quad \mathbf{d} = [d_l] = -\mathbf{B}^{-1}\mathbf{b}$

$$\frac{\partial f(\mathbf{a},\mathbf{R},s)}{\partial s} = \left(\frac{1}{2}\sum_{k,l=1}^{3} D_{lk}\tilde{X}_l^R \tilde{X}_k^R + \sum_{l=1}^{3} d_l \tilde{X}_l^R - \tilde{X}_6^R\right) f(\mathbf{a},\mathbf{R},s)$$

**Initial condition: f(a,R,0)= δ(a) δ(R)**

# Fourier Analysis of Motion

- **Fourier transform of a function of motion, *f(g)***

$$F(f) = \hat{f}(p) = \int_G f(g)U(g^{-1}, p)\,dg$$

- **Inverse Fourier transform of a function of motion**

$$F^{-1}(\hat{f}) = f(g) = \int trace(\hat{f}(p)U(g,p))\,p^{N-1}\,dp$$

**where *g* ∈*SE(N)* , *p* is a frequency parameter,**
***U(g,p)* is a matrix representation of *SE(N),* and**
***dg* is a volume element at *g*.**

# Operational Properties of SE(n) Fourier Transform

$$F\left(\tilde{X}_i^R f\right) = \int_G \frac{d}{dt}\left(f(g \circ \exp(t\tilde{X}_i))\right)\Big|_{t=0} U(g^{-1}, p)\, d(g)$$

$$\Downarrow_{h = g \circ \exp(t\tilde{X}_i)}$$

$$= \int_G f(h) \frac{d}{dt} U\left(\exp(t\tilde{X}_i) \circ h^{-1}, p\right)\Big|_{t=0} d(h)$$

$$\Downarrow_{U(g_1 \circ g_2, p) = U(g_1, p)\, U(g_2, p)}$$

$$= \left(\frac{d}{dt} U\left(\exp(t\tilde{X}_i), p\right)\Big|_{t=0}\right)\left(\int_G f(h) U(h^{-1}, p)\, d(h)\right)$$

$$= \eta(\tilde{X}_i, p)\, \hat{f}(p)$$

# Entries of $\eta(X_i, p)$ for i=1,2,3

$$\eta(\tilde{X}_i, p) = \left( \frac{d}{dt} U(\exp(t\tilde{X}_i), p) \right)\Big|_{t-0}$$

$$u^s_{l',m';l,m}(g,p) = \sum_{k=-l}^{l} [l', m' \mid p, s \mid l, m](\vec{a}) U^l_{km}(A)$$

$$\eta_{l',m';l,m}(\tilde{X}_1, p) = \frac{1}{2} c^l_{-m} \delta_{l,l'} \delta_{m'+1,m} - \frac{1}{2} c^l_m \delta_{l,l'} \delta_{m'-1,m}$$

$$\eta_{l',m';l,m}(\tilde{X}_2, p) = \frac{j}{2} c^l_{-m} \delta_{l,l'} \delta_{m'+1,m} + \frac{j}{2} c^l_m \delta_{l,l'} \delta_{m'-1,m}$$

$$\eta_{l',m';l,m}(\tilde{X}_3, p) = -jm \delta_{l,l'} \delta_{m',m}$$

# Entries of $\eta(X_i, p)$ for $l = 4,5,6$

$$\eta_{l',m';l,m}(\tilde{X}_4, p) = -\frac{jp}{2}\gamma^{s}_{l^{i},-m'}\delta_{m',m+1}\delta_{l'-1,l} + \frac{jp}{2}\lambda^{s}_{l,m}\delta_{m',m+1}\delta_{l',l} + \frac{jp}{2}\gamma^{s}_{l,m}\delta_{m',m+1}\delta_{l'+1,l}$$
$$+ \frac{jp}{2}\gamma^{s}_{l^{i},m'}\delta_{m',m-1}\delta_{l'-1,l} + \frac{jp}{2}\lambda^{s}_{l,-m}\delta_{m',m-1}\delta_{l',l} - \frac{jp}{2}\gamma^{s}_{l,-m}\delta_{m',m-1}\delta_{l'+1,l}$$

$$\eta_{l',m';l,m}(\tilde{X}_5, p) = -\frac{p}{2}\gamma^{s}_{l^{i},-m'}\delta_{m',m+1}\delta_{l'-1,l} + \frac{p}{2}\lambda^{s}_{l,m}\delta_{m',m+1}\delta_{l',l} + \frac{p}{2}\gamma^{s}_{l,m}\delta_{m',m+1}\delta_{l'+1,l}$$
$$- \frac{p}{2}\gamma^{s}_{l^{i},m'}\delta_{m',m-1}\delta_{l'-1,l} - \frac{p}{2}\lambda^{s}_{l,-m}\delta_{m',m-1}\delta_{l',l} + \frac{p}{2}\gamma^{s}_{l,-m}\delta_{m',m-1}\delta_{l'+1,l}$$

$$\eta_{l',m';l,m}(\tilde{X}_6, p) = jp\kappa^{s}_{l^{i},m'}\delta_{m',m}\delta_{l'-1,l} + jp\frac{sm}{l(l+1)}\delta_{m',m}\delta_{l',l} + jp\kappa^{s}_{l,m}\delta_{m',m}\delta_{l'+1,l}$$

# Solving for the evolving PDF Using the SE(3) FT

$$\frac{\partial f(\mathbf{a},\mathbf{R},s)}{\partial s} = \left(\frac{1}{2}\sum_{k,l=1}^{3} D_{lk}\tilde{X}_l^R \tilde{X}_k^R + \sum_{l=1}^{3} d_l \tilde{X}_l^R - \tilde{X}_6^R\right) f(\mathbf{a},\mathbf{R},s)$$

Applying SE(3) Fourier transform

$$\frac{d\hat{\mathbf{f}}^r}{ds} = \mathbf{B}^r \hat{\mathbf{f}}^r$$

where B is a constant matrix.

Solving ODE

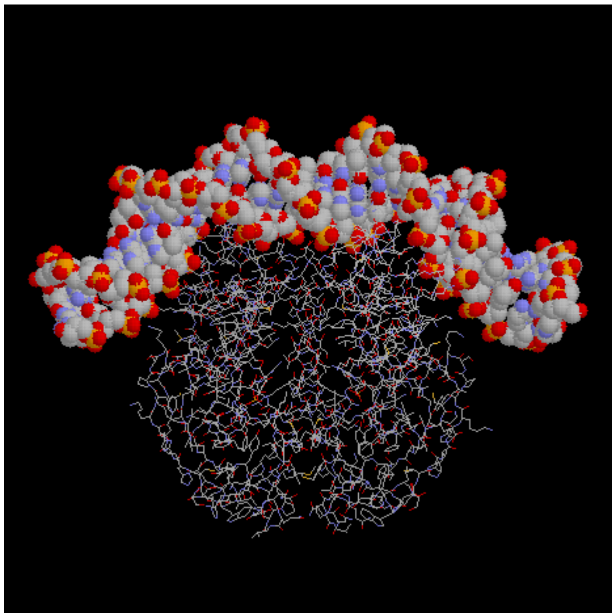
$$\hat{\mathbf{f}}^r(p,s) = e^{s\mathbf{B}^r}$$

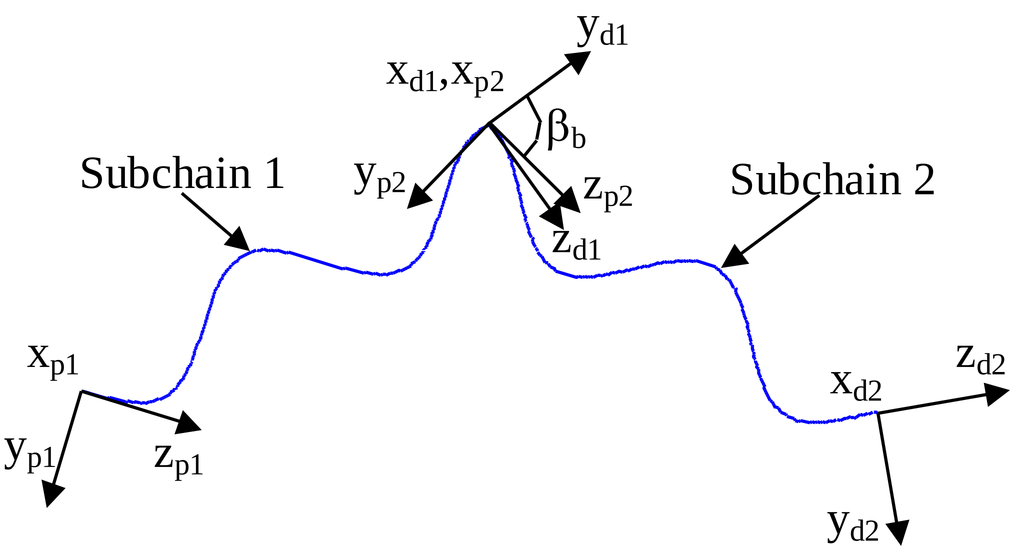
Applying inverse transform

$$f(\mathbf{a},\mathbf{R},s) = \frac{1}{2\pi^2}\sum_{r=-\infty}^{\infty}\sum_{l'=|r|}^{\infty}\sum_{l=|r|}^{\infty}\sum_{m'=-l'}^{l'}\sum_{m=-l}^{l}\int_0^{\infty} \hat{f}^r_{l,m;l',m'}(p)U^r_{l',m';l,m}(\mathbf{a},\mathbf{R};p)p^2 dp$$

# A General Algorithm for Bent or Twisted Macromolecular Chains

## The Structure of a Bent Macromolecular Chain






1) **A bent macromolecular chain consists of two intrinsically straight segments.**
2) **A bend or twist is a rotation at the separating point between the two segments with no translation.**

# A General Algorithm for Bent or Twisted Macromolecular Chains

## The PDF of the End-to-End Pose for a Bent Chain

**1) A convolution of 3 PDFs**

$$f(\mathbf{a},\mathbf{R}) = (f_1 * f_2 * f_3)(\mathbf{a},\mathbf{R})$$

**•$f_1(a,R)$ and $f_3(a,R)$ are obtained by solving the differential equation for nonbent polymer.**

**•$f_2(a,R)= \delta(a)\delta(R_b^{-1}R)$, where $R_b$ is the rotation made at the bend.**

**2) The convolution on SE(3)**

$$(f_i * f_j)(\mathbf{g}) = \int_{SE(3)} f_i(\mathbf{h}) f_j(\mathbf{h}^{-1} \circ \mathbf{g}) d(\mathbf{h})$$

# References


1) G. S. Chirikjian, ``Modeling Loop Entropy,'' Methods in Enzymology, 487, 2011
1) Y. Zhou, G. S. Chirikjian, ``Conformational Statistics of Semi flexible Macromolecular Chains with Internal Joints,'' Macromolecules. 39:1950-1960. 2006
1) Zhou, Y., Chirikjian, G.S., "Conformational Statistics of Bent Semi-flexible Polymers", Journal of Chemical Physics, vol.119, no.9, pp.4962-4970, 2003.
2) G. S. Chirikjian, Y. Wang, ``Conformational Statistics of Stiff Macromolecules as Solutions to PDEs on the Rotation and Motion Groups,'' Physical Review E. 62(1):880-892. 2000

# Finance (Revisited)

# The `ax+b' Group $Aff^{+}(1) = GL^{+}(1) \ltimes \mathbb{R}$

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \qquad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[Ad(g)] = \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix}$$

$$\begin{pmatrix} E_1^L f \\ E_2^L f \end{pmatrix} = \begin{pmatrix} a\,\partial\tilde{f}/\partial a + b\,\partial\tilde{f}/\partial b \\ \partial\tilde{f}/\partial b \end{pmatrix} \text{ and } \begin{pmatrix} E_1^R f \\ E_2^R f \end{pmatrix} = \begin{pmatrix} a\,\partial\tilde{f}/\partial a \\ a\,\partial\tilde{f}/\partial b \end{pmatrix}$$

**Promising to Write 2DOF Lagging Asset Equation as a diffusion**

$$\frac{\partial V}{\partial t} + ra\frac{\partial V}{\partial a} + \left[\left(\frac{\mu_2}{\mu_1} - \frac{\sigma_2}{\sigma_1}\right)\mu_1 + \frac{\sigma_2}{\sigma_1}r\right] a\frac{\partial V}{\partial b} + \frac{\sigma_1^2}{2}a^2\frac{\partial^2 V}{\partial a^2} + \sigma_1\sigma_2 a^2\frac{\partial^2 V}{\partial a\partial b} + \frac{\sigma_2^2}{2}a^2\frac{\partial^2 V}{\partial b^2} - rV = 0$$

**Only problem is that `ax_b' group is not unimodular. What to do ?**

# Bump Up to Tangent and Cotangent Bundle Groups

$$(g_1, X_1)\square(g_2, X_2) = (g_1 \circ g_2, Ad(g_1)X_2 + X_1)$$

$$G \ltimes \mathcal{G} \doteq \left\{ \begin{pmatrix} [Ad(g)] & \mathbf{x} \\ \mathbf{0}^T & 1 \end{pmatrix} \middle| g \in G \quad \text{and} \quad \mathbf{x} \in \mathcal{G}^\vee \right\}$$

$$(g_1, Y_1)\blacksquare(g_2, Y_2) = (g_1 \circ g_2, Ad(g_1)^{-T}Y_2 + Y_1)$$

$$G \ltimes \mathcal{G}^* \doteq \left\{ \begin{pmatrix} [Ad(g)]^{-T} & \mathbf{y} \\ \mathbf{0}^T & 1 \end{pmatrix} \middle| g \in G \quad \text{and} \quad \mathbf{y} \in (\mathcal{G}^*)^\vee \right\}$$

# Lie Algebras for these Groups

$$\tilde{E}_i = \left\{ \begin{pmatrix} [ad(E_i)] & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} \text{ for } i = 1, \cdots, N \text{ and } \begin{pmatrix} \mathbb{O} & \mathbf{e}_{i-N} \\ \mathbf{0}^T & 0 \end{pmatrix} \text{ for } i = N+1, \cdots, 2N \right\}$$

$$\begin{pmatrix} [ad(X)] & \mathbf{y} \\ \mathbf{0}^T & 0 \end{pmatrix}^{\vee} = \begin{pmatrix} X^{\vee} \\ \mathbf{y} \end{pmatrix}$$

$$\tilde{E}_i = \left\{ \begin{pmatrix} -[ad(E_i)]^T & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} \text{ for } i = 1, \cdots, N \text{ and } \begin{pmatrix} \mathbb{O} & \mathbf{e}^*_{i-N} \\ \mathbf{0}^T & 0 \end{pmatrix} \text{ for } i = N+1, \cdots, 2N \right\}$$

$$\begin{pmatrix} -[ad(X)]^T & \mathbf{y} \\ \mathbf{0}^T & 0 \end{pmatrix}^{\vee} = \begin{pmatrix} X^{\vee} \\ \mathbf{y} \end{pmatrix}$$

**Theorem 1.** *The tangent bundle group* $(TG, \square) = G \ltimes \mathcal{G}$ *of an N-dimensional Lie group with a trivial center is a 2N-dimensional unimodular Lie group if and only if the group G is unimodular.*

**Theorem 2.** *The cotangent bundle group* $(TG^*, \blacksquare) = G \ltimes \mathcal{G}^*$ *of an N-dimensional Lie group with a trivial center is always a 2N-dimensional unimodular Lie group independent of the unimodularity of G.*

$$Ad(\breve{Aff}^+(1))^{-T} \ltimes \mathbb{R}^2$$

$$h = \left(\begin{array}{cc|c} 1 & b/a & x \\ 0 & 1/a & y \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\tilde{E}_1 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \ \tilde{E}_2 = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \ \tilde{E}_3 = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \text{ and } \tilde{E}_4 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} \tilde{E}_1^L f \\ \tilde{E}_2^L f \\ \tilde{E}_3^L f \\ \tilde{E}_4^L f \end{pmatrix} = \begin{pmatrix} a\,\partial\tilde{f}/\partial a + b\,\partial\tilde{f}/\partial b - y\,\partial\tilde{f}/\partial y \\ \partial\tilde{f}/\partial b + y\,\partial\tilde{f}/\partial x \\ \partial\tilde{f}/\partial x \\ \partial\tilde{f}/\partial y \end{pmatrix} \text{ and } \begin{pmatrix} \tilde{E}_1^R f \\ \tilde{E}_2^R f \\ \tilde{E}_3^R f \\ \tilde{E}_4^R f \end{pmatrix} = \begin{pmatrix} a\,\partial\tilde{f}/\partial a \\ a\,\partial\tilde{f}/\partial b \\ \partial\tilde{f}/\partial x \\ (b/a)\,\partial\tilde{f}/\partial x + (1/a)\,\partial\tilde{f}/\partial y \end{pmatrix}$$

$$\frac{\partial u}{\partial t'} = \left(r - \frac{\sigma_1^2}{2}\right)\tilde{E}_1^R u + \left[\left(\frac{\mu_2}{\mu_1} - \frac{\sigma_2}{\sigma_1}\right)\mu_1 + \frac{\sigma_2}{\sigma_1} r - \frac{\sigma_1\sigma_2}{2}\right]\tilde{E}_2^R u + \frac{1}{2}\left(\sigma_1^2(\tilde{E}_1^R)^2 + \sigma_1\sigma_2(\tilde{E}_2^R\tilde{E}_1^R + \tilde{E}_1^R\tilde{E}_2^R) + \sigma_2^2(\tilde{E}_2^R)^2\right)u$$

# For more on this, see

Jayaraman, A.S., Campolo, D. and Chirikjian, G.S., Black-Scholes Theory and Diffusion Processes on the Cotangent Bundle of the Affine Group. *Entropy*, *22*(4), p.455, 2020

# Conclusions

**Using a combination of Gaussian and Fourier solutions, we can solve a number of PDEs in Finance, Robotics, and Molecular Statistical Mechanics**