Global theory of subelliptic pseudo-differential operators and Fourier integral operators on compact Lie groups.

Duván Cardona

Ghent University

Methusalem Junior Seminar
KU Leuven
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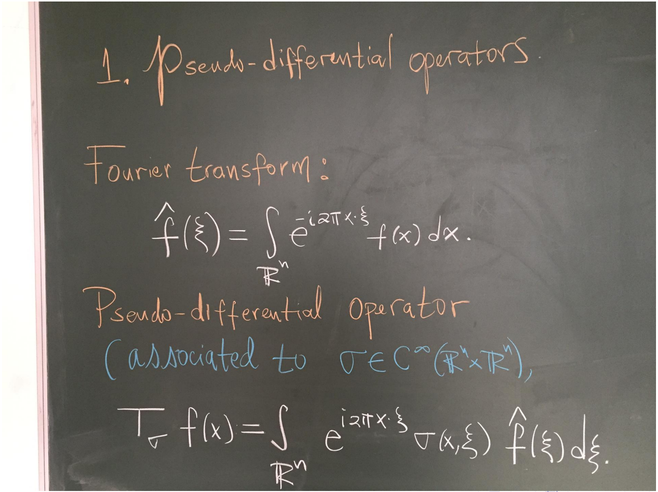
Joint work with Prof. Dr. Michael Ruzhansky.

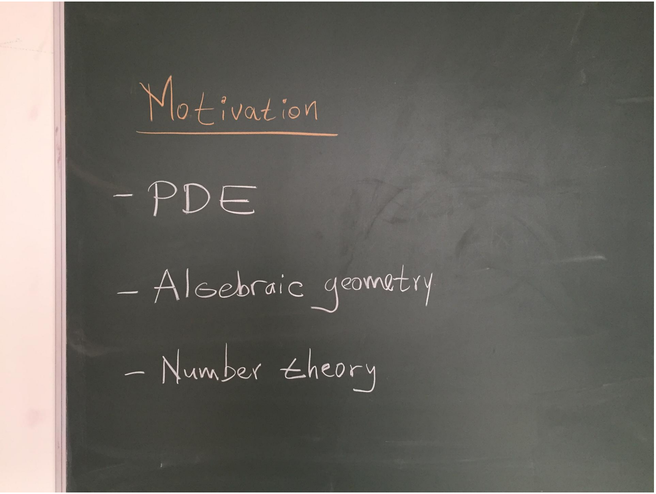
The setting...

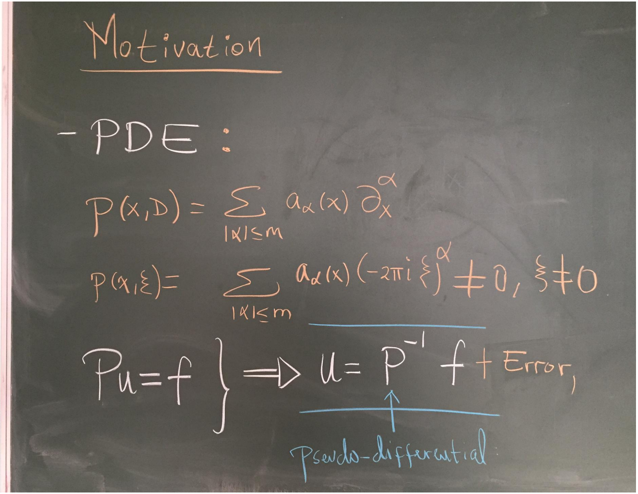
- Given a compact Lie group G, and the sub-Laplacian $\mathcal{L}$ associated to a system of vector fields $X = \{X_1, \cdots, X_k\}$ satisfying the Hörmander condition, in [CR20], we introduce a (subelliptic) pseudo-differential calculus associated to $\mathcal{L}$, based on the matrix-valued quantisation process developed previously by Michael Ruzhansky and Ville Turunen.
- [CR20]: Cardona, D., Ruzhansky, M. Subelliptic pseudo-differential operators and Fourier integral operators on compact Lie groups, submitted. arXiv:2008.09651.

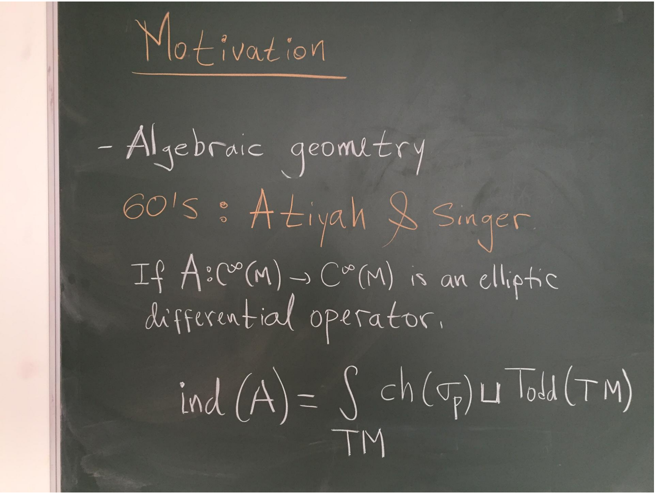
Outline

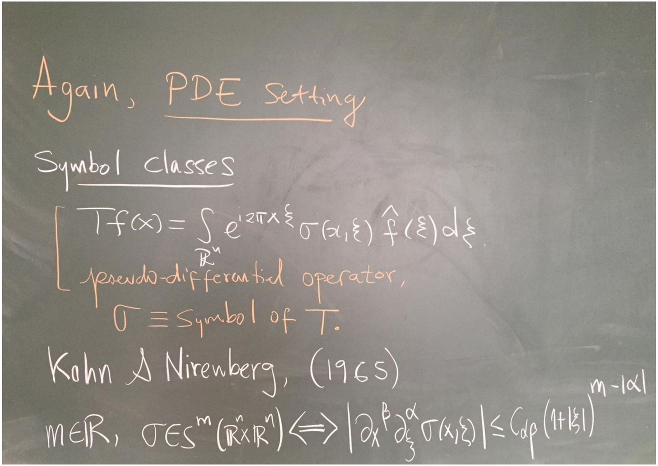
Fourier Transform, the main tool.

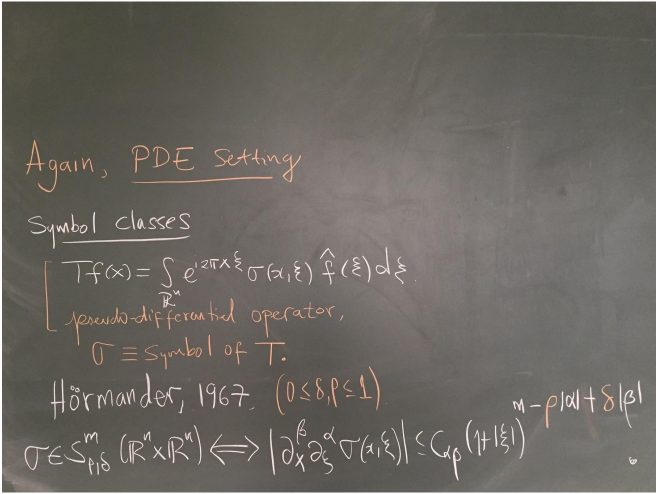



Motivation
- PDE
- Algebraic geometry
- Number theory


Motivation
- PDE :
$P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha$
$p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)(-2\pi i\xi)^\alpha \neq 0, \xi\neq 0$
$Pu=f \} \Longrightarrow u = P^{-1}f + \text{Error},$
pseudo-differential


Motivation
- Algebraic geometry
60's : Atiyah & Singer
If A: C^∞(M) → C^∞(M) is an elliptic differential operator,
ind(A) = ∫_TM ch(σ_p) ⊔ Todd(TM)


Again, PDE Setting
Symbol classes
$Tf(x) = \int_{\mathbb{R}^n} e^{i2\pi x\xi} \sigma(x,\xi) \hat{f}(\xi) d\xi$
pseudo-differential operator,
$\sigma \equiv$ Symbol of T.
Kohn & Nirenberg, (1965)
$m \in \mathbb{R}, \sigma \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \Longleftrightarrow |\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq C_{\alpha\beta} (1+|\xi|)^{m-|\alpha|}$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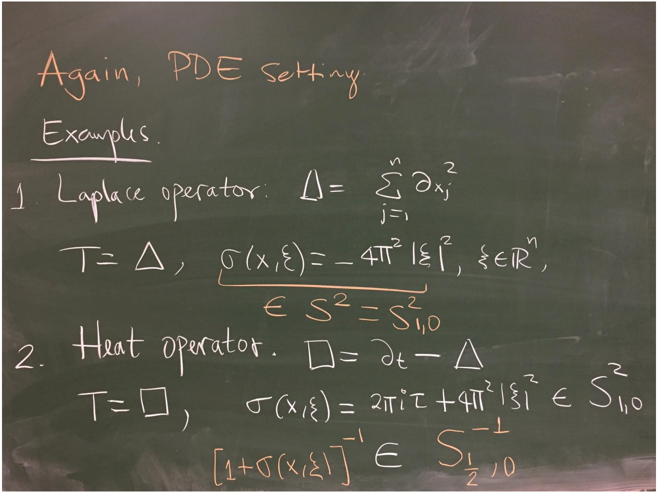
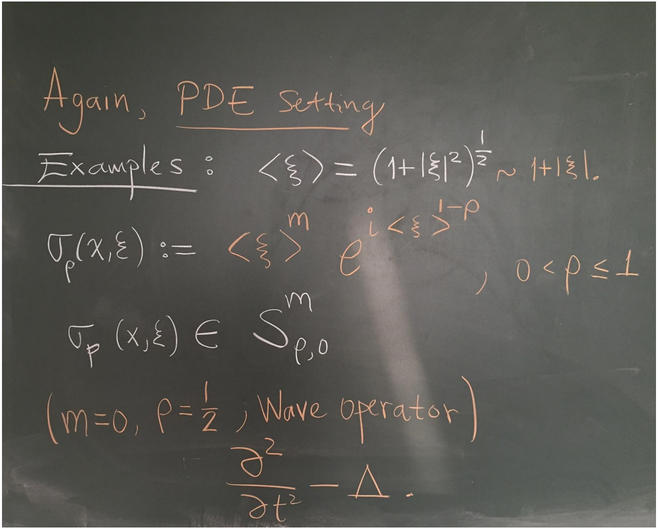
Hörmander, 1967 $(0\leq\delta,\rho\leq 1)$
$\sigma\in S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n) \Longleftrightarrow |\partial_x^\beta\partial_\xi^\alpha\sigma(x,\xi)|\leq C_{\alpha\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$


Again, PDE Setting.
Examples.
1. Laplace operator: $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$
$T = \Delta$, $\sigma(x,\xi) = -4\pi^2 |\xi|^2$, $\xi \in \mathbb{R}^n$,
$\in S^2 = S^2_{1,0}$
2. Heat operator. $\Box = \partial_t - \Delta$
$T = \Box$, $\sigma(x,\xi) = 2\pi i \tau + 4\pi^2 |\xi|^2 \in S^2_{1,0}$
$[1+\sigma(x,\xi)]^{-1} \in S^{-1}_{\frac{1}{2},0}$


Again, PDE Setting
Examples: $\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}} \sim 1+|\xi|.$
$\sigma_\rho(x,\xi) := \langle \xi \rangle^m e^{i\langle \xi \rangle^{1-\rho}}, \quad 0<\rho\leq 1$
$\sigma_\rho(x,\xi) \in S^m_{\rho,0}$
$(m=0,\ \rho=\frac{1}{2}$, Wave operator$)$
$\frac{\partial^2}{\partial t^2} - \Delta.$

Outline

1. There is a well-known formulation of pseudo-differential operators on compact manifolds, (and so on compact Lie groups) by using symbols defined by charts[1].
2. If $U \subset \mathbb{R}^n$ is open, the symbol $a : U \times \mathbb{R}^n \to \mathbb{C}$, belongs to the Hörmander class $S^m_{\rho,\delta}(U \times \mathbb{R}^n)$, $0 \leqslant \rho, \delta \leqslant 1,$ if for every compact subset $K \subset U$, the symbol inequalities,

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leqslant C_{\alpha,\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

hold true uniformly in $x \in K$ and $\xi \in \mathbb{R}^n$.

[1] Hörmander, L. The Analysis of the linear partial differential operators Vol. III. Springer-Verlag, (1985).

Outline

3. Then, a continuous linear operator $A : C_0^\infty(U) \rightarrow C^\infty(U)$ is a pseudo-differential operator of order m, of (ρ, δ)-type, if there exists a function $a \in S^m_{\rho,\delta}(U \times \mathbb{R}^n)$, satisfying

$$Af(x) = \int\limits_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} a(x,\xi)(\mathscr{F}_{\mathbb{R}^n} f)(\xi) d\xi,$$

for all $f \in C_0^\infty(U)$, where

$$(\mathscr{F}_{\mathbb{R}^n} f)(\xi) := \int\limits_{U} e^{-i2\pi x\cdot\xi} f(x) dx,$$

is the Euclidean Fourier transform of f at $\xi \in \mathbb{R}^n$.

Let M be a closed manifold.

4. The class $S^m_{\rho,\delta}(U\times\mathbb{R}^n)$ on the phase space $U\times\mathbb{R}^n$, is invariant under coordinate changes only if $\rho\geqslant 1-\delta$, while a symbolic calculus (closed for products, adjoints, parametrices, etc.) is only possible for $\delta<\rho$ and $\rho\geqslant 1-\delta$.
5. $A: C^\infty_0(M)\to C^\infty(M)$ is a pseudo-differential operator of order m, of (ρ,δ)-type, $\rho\geqslant 1-\delta$, if for every local coordinate patch $\omega: M_\omega\subset M\to U\subset\mathbb{R}^n$, and for every $\phi,\psi\in C^\infty_0(U)$, the operator

$$Tu:=\psi(\omega^{-1})^*A\omega^*(\phi u),\ u\in C^\infty(U),$$ [2]

is a pseudo-differential operator with symbol in $S^m_{\rho,\delta}(U\times\mathbb{R}^n)$.

[2] As usually, ω^* and $(\omega^{-1})^*$ are the pullbacks induced by the maps ω and ω^{-1}, respectively.

Conclusion:

6. In this case we write that $A \in \Psi^m_{\rho,\delta}(M, \text{loc})$, $\delta < \rho$, $\rho \geqslant 1-\delta$.
5. To $A \in \Psi^m_{\rho,\delta}(M; \text{loc})$ one associates a (principal) symbol $a \in S^m_{\rho,\delta}(T^*M)$,[3] which is uniquely determined, only as an element of the quotient algebra $S^m_{\rho,\delta}(T^*M)/S^{m-1}_{\rho,\delta}(T^*M)$.

(Q): When, is it possible to define a notion of a global symbol (without using local coordinate systems) allowing a global quantisation formula for the Hörmander class $\Psi^m_{\rho,\delta}(M; \text{loc})$?

[3] which is a section of the cotangent bundle T^*M.

Let $M = G$ be a compact Lie group. (Ex: $G = \mathrm{SU}(2)$).

- Let $\widehat{G}$ be the family of all equivalence classes of continuous and irreducible unitary representations of G.

There is a global definition of symbols on the phase space $G \times \widehat{G}$, that provides global Hörmander classes of symbols $S^m_{\rho,\delta}(G \times \widehat{G})$, $0 \leqslant \rho, \delta \leqslant 1,$ such that:[4]

- $\Psi^m_{\rho,\delta}(G \times \widehat{G}) := \mathrm{Op}(S^m_{\rho,\delta}(G \times \widehat{G})) = \Psi^m_{\rho,\delta}(G; \mathrm{loc})$, for $\delta < \rho,$ and $\rho \geqslant 1 - \delta,$ (this implies that $\rho > \frac{1}{2}$).
- New classes for $\delta \leq \rho,$ allowing to study the borderline case $\rho = \delta$.

[4] Ruzhansky, M., Turunen, V. Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics Birkhäuser-Verlag, Basel, 2010

Representations on a compact Lie Group.

- Unitary Representation: A unitary representation is a continuous mapping

$$\xi \in \mathrm{HOM}(G, U(H_\xi)), \;\; \xi(x)\xi(y) = \xi(xy), \, \xi(x)^* = \xi(x)^{-1},$$

for some (finite-dimensional) vector space $H = H_\xi$. We define by $d_\xi = dim(H_\xi)$ the dimension of ξ.

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- Equivalent Representations: Two representations

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are equivalent, if there exist a linear bijection $\phi : H_\xi \to H_\eta$, such that $\forall x \in G, \ \xi(x) = \phi^{-1}\eta(x)\phi$.

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- $\widehat{G}$ consists of all equivalence classes of continuous irreducible unitary representations of G.

Fourier Analysis on a compact Lie group G. Peter-Weyl Theorem, 1927.

Let us consider a compact Lie group G with discrete unitary dual $\widehat{G}$ that is, the set of equivalence classes of all continuous irreducible unitary representations of G. We identify $H_\xi \cong \mathbb{C}^{d_\xi}$, and $\mathrm{Hom}(H_\xi) \cong \mathbb{C}^{d_\xi \times d_\xi}$.

- Fourier transform of $f \in C^\infty(G)$,

$$(\mathscr{F} f)(\xi) \equiv \widehat{f}(\xi) := \int_G f(x)\xi(x)^* dx \in \mathbb{C}^{d_\xi \times d_\xi}, \quad [\xi] \in \widehat{G}. \tag{1.1}$$

- Fourier inversion formula,

$$f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)\widehat{f}(\xi)). \tag{1.2}$$

Fourier Analysis on a compact Lie group G. Peter-Weyl Theorem, 1927.

Let $\sigma := \mathscr{F}(k)$, for some distribution $k \in \mathscr{D}'(G)$. We denote $\mathscr{D}'(\widehat{G}) := \mathscr{F}(\mathscr{D}'(G))$.

- The Inverse Fourier transform of σ, at $x \in \widehat{G}$, is defined via

$$(\mathscr{F}^{-1}\sigma)(x) := \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)\sigma(\xi)). \tag{1.3}$$

Continuous Linear operators on G and the Fourier Transform. We write $\xi(x)=[\xi_{ij}(x)]_{i,j=1}^{d_\xi}\in\mathbb{C}^{d_\xi\times d_\xi}$.

Theorem

Let $A:C^\infty(G)\to C^\infty(G)$ *be a continuous linear operator. Then:*

$$Af(x)=\sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma(x,\xi)(\mathscr{F}f)(\xi)]\quad f\in C^\infty(G),$$

where

$$\sigma(x,\xi):=\xi(x)^*A\xi(x):=\xi(x)^*[A\xi_{ij}(x)]_{i,j=1}^{d_\xi},$$

Global Pseudo-differential operators on G

- A global pseudo-differential operator T_σ associated to a function/distribution $\sigma \in C^\infty((G \times \widehat{G}), \cup_{[\xi]\in\widehat{G}} \mathbb{C}^{d_\xi \times d_\xi})$[5] is formally defined by

$$T_\sigma f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)\sigma(x,\xi)\widehat{f}(\xi)), \quad x \in G. \tag{1.4}$$

[5] Observe that for every $(x,[\xi]) \in G \times \widehat{G}$, $\sigma(x,\xi) : H_\xi \to H_\xi$, $H_\xi \cong \mathbb{C}^{d_\xi}$.

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- The function σ is called the global symbol of the pseudo-differential operator T_σ.

[5] Observe that for every $(x,[\xi]) \in G \times \widehat{G}$, $\sigma(x,\xi) : H_\xi \to H_\xi$, $H_\xi \cong \mathbb{C}^{d_\xi}$.

The global symbol of the Laplacian

- (Eigenvalues of the Laplacian on compact Lie groups) There exists a non-negative real number $\lambda_{[\xi]}$ depending only on the equivalence class $[\xi] \in \hat{G}$, but not on the representation ξ, such that

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$$\mathcal{L}_G \xi(x) = \lambda_{[\xi]} \xi(x),$$

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$$\langle \xi \rangle = (1 + \lambda_{[\xi]})^{\frac{1}{2}}.$$

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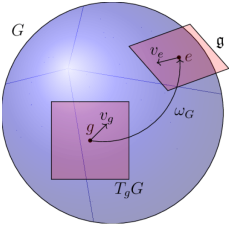
  where $\mathcal{L}_G$ is the positive Laplacian on the group G. The symbol of the positive Laplacian is given by
  $$\sigma_{\mathcal{L}_G}(x,\xi) = \lambda_{[\xi]} I_{H_\xi}.$$
- Definition: (Scalar-valued elliptic weight)
  $$\langle \xi \rangle := (1+\lambda_{[\xi]})^{\frac{1}{2}}.$$
- For every $z \in \mathbb{C}$, $B_z := (1+\mathcal{L}_G)^{\frac{z}{2}} \in \Psi^{\mathrm{Re}(z)}_{1,0}(G, \mathrm{loc})$, and its matrix-valued symbol is given by
  $$\sigma_{(1+\mathcal{L}_G)^{\frac{z}{2}}}(x,\xi) = \langle \xi \rangle^z I_{H_\xi}.$$

The global symbol of a sub-Laplacians on a compact Lie group.

Let $\mathbb{X} = \{X_1, \cdots, X_n\}$ be an o.n.b. of the Lie algebra $\mathfrak{g} \cong T_eG$.



- Laplacian: $\mathcal{L}_G = -X_1^2 - \cdots - X_n^2$.
- sub-Laplacian: $\mathcal{L}_X = -X_1^2 - \cdots - X_k^2$, where $X = \{X_1, \cdots, X_k\}$ satisfies the Hörmander condition at step κ.

The global symbol of the sub-Laplacian

- The matrix-valued function

$$\widehat{\mathcal{L}}(\xi) = [\mathcal{L}(\xi_{ij})]_{i,j=1}^{d_\xi}$$

is the global symbol of the sub-Laplacian $\mathcal{L}$.

- Definition: (Matrix-valued subelliptic weight)

$$\mathcal{M}(\xi) := (1 + \widehat{\mathcal{L}}(\xi))^{\frac{1}{2}}.$$

How to define the class $S^m_{\rho,\delta}(G\times\widehat{G})$? of global symbols on $G\times\widehat{G}$? We use difference operators.

If $[\xi_0]\in\widehat{G}$, consider the matrix

$$\xi_0(g)-I_{d_{\xi_0}}=[\xi_0(g)_{ij}-\delta_{ij}]_{i,j=1}^{d_\xi},\quad g\in G. \tag{1.5}$$

Then, we associated to the function
$q_{ij}(g):=\xi_0(g)_{ij}-\delta_{ij},\quad g\in G$, a difference operator via

$$\mathbb{D}_{\xi_0,i,j}:=\mathscr{F}(\xi_0(g)_{ij}-\delta_{ij})\mathscr{F}^{-1}:\mathscr{D}'(\widehat{G})\rightarrow\mathscr{D}'(\widehat{G}). \tag{1.6}$$

From a sequence $\mathbb{D}_1=\mathbb{D}_{\xi_0,j_1,i_1},\cdots,\mathbb{D}_n=\mathbb{D}_{\xi_0,j_n,i_n}$ of operators of this type we define $\mathbb{D}^\alpha=\mathbb{D}_1^{\alpha_1}\cdots\mathbb{D}_n^{\alpha_n}$, where $\alpha\in\mathbb{N}^n$.

Elliptic and subelliptic pseudo-differential operators on compact Lie groups. (Recall that $\mathcal{M}(\xi) := (1 + \widehat{\mathcal{L}(\xi)})^{\frac{1}{2}}$).

- Elliptic Hörmander classes: $\sigma \in S^m_{\rho,\delta}(G \times \widehat{G})$, if

$$\|\partial_X^{(\beta)} \mathbb{D}_\xi^\alpha \sigma(x,\xi)\|_{\text{op}} \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\gamma|+\delta|\beta|} \tag{1.7}$$

[6] Ruzhansky, M., Turunen, V. Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics Birkhäuser-Verlag, Basel, 2010

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- Solution for (Q)[6]

$$\text{Op}(S^m_{\rho,\delta}(G\times\widehat{G})) = \Psi^m_{\rho,\delta}(G,\text{loc}),\ \ 0\leqslant\delta<\rho\leqslant 1,\ \rho\geqslant 1-\delta.$$

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- Solution for (Q)[6]

$$\text{Op}(S^m_{\rho,\delta}(G \times \widehat{G})) = \Psi^m_{\rho,\delta}(G, \text{loc}), \ \ 0 \leqslant \delta < \rho \leqslant 1, \ \rho \geqslant 1 - \delta.$$

- Subelliptic Hörmander classes: $\sigma \in S^{m,\mathcal{L}}_{\rho,\delta}(G \times \widehat{G})$, if,

$$\|\widehat{\mathcal{M}}(\xi)^{(\rho|\alpha|-\delta|\beta|-m)} \partial_X^{(\beta)} \mathbb{D}_\xi^\alpha a(x,\xi)\|_{\text{op}} \leq C_{\alpha,\beta}.$$

[6] Ruzhansky, M., Turunen, V. Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics Birkhäuser-Verlag, Basel, 2010

Example: Powers of the subelliptic Bessel potential.

Let us denote

$$\Psi^m_{\rho,\delta}(G\times\widehat{G}) := \mathrm{Op}(S^m_{\rho,\delta}(G\times\widehat{G})),\ m\in\mathbb{R}.$$

- For every $s\in\mathbb{R}$, $\mathcal{B}_s := (1+\mathcal{L})^{\frac{s}{2}} \in \Psi^s_{1,0}(G\times\widehat{G})$, if $s>0$, and $\mathcal{B}_{-s} := (1+\mathcal{L})^{-\frac{s}{2}} \in \Psi^{-s/\kappa}_{1/\kappa,0}(G\times\widehat{G})$. Here, κ is the step of the Hörmander system $X=\{X_1,\cdots,X_k\}$, and $\mathcal{L}=-\sum_{j=1}^k X_k^2$.
- For every $z\in\mathbb{C}$, $\mathcal{B}_z := (1+\mathcal{L})^{\frac{z}{2}} \in \Psi^{s,\mathcal{L}}_{1,0}(G\times\widehat{G})$, $s=\mathfrak{Re}(z)$.

Outline

The subelliptic pseudo-differential calculus on G.

- Define $\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G}):=\{T_\sigma:\sigma\in S^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})\}$, for $0\leq\delta\leq\rho\leq 1$. Then:

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  - If $T_\sigma\in\Psi^{m_1,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$ and $T_\tau\in\Psi^{m_2,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$, then, $T_\sigma\circ T_\tau\in T_\sigma\in\Psi^{m_1+m_2,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$,

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  - If $T_\sigma\in\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$ then $T^*_\sigma\in\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$.

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  - If $T_\sigma\in\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$ then $T_\sigma^*\in\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$.
  - (Calderón-Vaillancourt Theorem). $T_\sigma:L^2(G)\to L^2(G)$ is bounded if $m=0$ and $1\leq\delta\leq\rho\leq 1$, $\delta<1/\kappa$.

The subelliptic pseudo-differential calculus on G.

- Define $\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G}):=\{T_\sigma:\sigma\in S^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})\}$, for $0\leq\delta\leq\rho\leq 1$. Then:
  - If $T_\sigma\in\Psi^{m_1,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$ and $T_\tau\in\Psi^{m_2,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$, then, $T_\sigma\circ T_\tau\in T_\sigma\in\Psi^{m_1+m_2,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$,
  - If $T_\sigma\in\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$ then $T_\sigma^*\in\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$.
  - (Calderón-Vaillancourt Theorem). $T_\sigma:L^2(G)\to L^2(G)$ is bounded if $m=0$ and $1\leq\delta\leq\rho\leq 1$, $\delta<1/\kappa$.
  - (Fefferman L^p-Theorem). Let $1\leq\delta<\rho\leq 1$, and let $1<p<\infty$. $T_\sigma:L^p(G)\to L^p(G)$ is bounded, for all $T_\sigma\in S^{-m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})\}$, if $m\geq Q(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right|$.

The subelliptic functional calculus on G.

- The subelliptic calculus is stable under the spectral functional calculus of the sub-Laplacian:
  - Let $f \in S^{\frac{m}{2}}(\mathbb{R}_0^+)$, $m \in \mathbb{R}$. Then, for all $t > 0$, $f(t\mathcal{L}) \in S^{m,\mathcal{L}}_{1,0}(G \times \widehat{G})$.
- The subelliptic calculus is stable under the action of the complex functional calculus.

$$F(A) := -\frac{1}{2\pi i} \oint\limits_{\partial\Lambda_\varepsilon} F(z)(A - zI)^{-1} dz. \qquad (2.1)$$

  - Let $m > 0$, and let $0 \leqslant \delta < \rho \leqslant 1$. Let $a \in S^{m,\mathcal{L}}_{\rho,\delta}(G \times \widehat{G})$ be a parameter $\mathcal{L}$-elliptic symbol with respect to Λ. Let us assume that F satisfies the estimate $|F(\lambda)| \leqslant C|\lambda|^s$ uniformly in λ, for some $s \in \mathbb{R}$. Then $\sigma_{F(A)} \in S^{ms,\mathcal{L}}_{\rho,\delta}(G \times \widehat{G})$, for F satisfying some suitable conditions.

Suitable conditions mean:

(CI). $\Lambda_{\varepsilon} := \Lambda \cup \{z : |z| \leqslant \varepsilon\}$, $\varepsilon > 0$, and $\Gamma = \partial\Lambda_{\varepsilon} \subset$ Resolv(A) is a positively oriented curve in the complex plane $\mathbb{C}$.

(CII). F is an holomorphic function in $\mathbb{C} \setminus \Lambda_{\varepsilon}$, and continuous on its closure.

(CIII). We will assume decay of F along $\partial\Lambda_{\varepsilon}$ in order that the operator (2.1) will be densely defined on $C^{\infty}(G)$ in the strong sense of the topology on $L^2(G)$.

Fefferman L^p-theorem on G.

- (Elliptic Fefferman L^p-Theorem: Delgado and Ruzhansky). Let $1 \leq \delta < \rho \leq 1$, and let $1 < p < \infty$. $T_\sigma : L^p(G) \to L^p(G)$ is bounded, for all $T_\sigma \in S^{-m}_{\rho,\delta}(G \times \widehat{G})\}$, if and only if,

$$m \geq n(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right|.$$

- (Subelliptic Fefferman L^p-Theorem). Let $1 \leq \delta < \rho \leq 1$, and let $1 < p < \infty$. $T_\sigma : L^p(G) \to L^p(G)$ is bounded, for all $T_\sigma \in S^{-m,\mathcal{L}}_{\rho,\delta}(G \times \widehat{G})\}$, if,

$$m \geq Q(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right|.$$

Applications:

- (subelliptic Garding Inequality).

$$\mathrm{Re}(a(x,D)u,u) \geqslant C_1\|u\|_{L^{2,\mathcal{L}}_{\frac{m}{2}}(G)} - C_2\|u\|^2_{L^2(G)}.$$

- Well-posedness for the Cauchy problem

$$\text{(PVI)}: \begin{cases} \frac{\partial v}{\partial t} = K(t,x,D)v + f, \\ v(0) = u_0, v \in \mathscr{D}'((0,T)\times G) \end{cases} \quad (2.2)$$

- Asymptotic expansions in spectral geometry

$$\textbf{Tr}(A\psi(tE)) = t^{-\frac{Q+m}{q}}\left(\sum_{k=0}^{\infty} a_k t^k\right) + \frac{c_Q}{q}\int\limits_0^\infty \psi(s)\times\frac{ds}{s}.$$

- Classification in Dixmier ideals, Sharp-L^p-estimates for oscillatory Fourier multipliers.

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