A Crash Introduction to Non-Commutative Harmonic Oscillators (NCHOs)

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The **Non-Commutative Harmonic Oscillators (NCHOs)** are *a class of pseudo-differential operators* introduced by A. Parmeggiani and M. Wakyama in 2002-2003 and generalized in 2010 by Parmeggiani$^{(*)}$.

$^{(*)}$- A. Parmeggiani. *Spectral theory of Non-Commutative Harmonic Oscillators: An Introduction*. Lecture Notes in Mathematics, 1992. Springer-Verlag, Berlin, 2010

- A. Parmeggiani, M. Wakayama, *Oscillator representations and systems of ordinary differential equations*. Proc. Natl. Acad. Sci. USA 98, (2001), 26–30.

- A. Parmeggiani, M. Wakayama, *Non-commutative harmonic oscillators-I,-II*. Corrigenda and Remarks to I. Forum Math. 14 (2002), 539–604, 669–690, ibid. 15 (2003), 955–963.

Why study NCHOs?

- To study **vector-valued deformations** of the **scalar** (and fundamental) **harmonic oscillator**.

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- To study **vector-valued deformations** of the **scalar** (and fundamental) **harmonic oscillator**.

- To investigate **a-priori inequalities** like the ones of Hormander's, Fefferman's and Melin's.

Addressed Issues

- To define the **class of pseudo-differential operators named Non-Commutative Harmonic Oscillators (NCHOs)**.

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- To give one of the main tools in the study of this class: **the Decoupling Theorem**.

- To investigate **their spectral properties**, mainly **singularities of their Trace** and **precise results about the spectral zeta function** associated to them.

Contents

Weight and Admissible Metric

Definition

$$m(x,\xi) = (1+|x|^2+|\xi|^2)^{1/2},\ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

$$g_{x,\xi} = \frac{1}{m(x,\xi)^2}(|dx|^2+|d\xi|^2).$$

Scalar Symbols and Global Ellipticity

Definitions

The **class of symbols** $S(m^k, g)$, $k \in \mathbb{R}$, is the set of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ for which $\forall \alpha$, $\beta \in \mathbb{Z}_+^n$ there exists $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} m(x,\xi)^{k-|\alpha|-|\beta|}, \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

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$a \in S(m^k, g)$, $k \in \mathbb{R}$ is **globally elliptic** when there are $C, c > 0$ s.t.

$$|x| + |\xi| \geq c \Longrightarrow |a(x,\xi)| \geq C m(x,\xi)^k.$$

Classical Symbols (1)

Definition

The set $S_{\text{cl}}(m^k, g)$ of **classical symbols** consists of those $a \in S(m^k, g)$ for which there is a sequence $\{a_{k-2j}\}_{j>0} \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus (0,0))$, such that for any given $j > 0$ the function a_{k-2j} is *positively homogeneous of degree* $k - 2j$, i.e.

$$a_{k-2j}(tx, t\xi) = t^{k-2j} a_{k-2j}(x, \xi),\ \forall t > 0, \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$$

and

$$a(x, \xi) - \chi(x, \xi) \sum_{j=0}^{N} a_{k-2j}(x, \xi) \in S(m^{k-2(N+1)}, g), \quad \forall N \in \mathbb{Z}_+$$

where χ is an excision function.

Classical Symbols (2)

Remarks:

- If $a \in S_{\mathrm{cl}}(m^k, g)$, as above, we write

$$a(x,\xi) \sim \sum_{j\geq 0} a_{k-2j}(x,\xi).$$

Classical Symbols (2)

Remarks:

- If $a \in S_{\mathsf{cl}}(m^k, g)$, as above, we write

$$a(x,\xi) \sim \sum_{j\geq 0} a_{k-2j}(x,\xi).$$

- If $a \in S_{\mathsf{cl}}(m^k, g)$, as above, we have that a is **globally elliptic** iff

$$\min_{|x|^2+|\xi|^2=1} |a_k(x,\xi)| > 0.$$

Matrix-valued Symbols

Definition

$$S(m^k, g; \mathsf{M}_N) := \mathsf{M}_N \otimes S(m^k, g).$$

$$S_{\mathsf{cl}}(m^k, g; \mathsf{M}_N) := \mathsf{M}_N \otimes S_{\mathsf{cl}}(m^k, g).$$

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- $a \in S(m^k, g; \mathsf{M}_N)$ is **globally elliptic** when $\exists\, C,\ c > 0$ s.t.

$$|\det\, a(x,\xi)| \geq C m(x,\xi)^{Nk}, \text{ when } |x| + |\xi| \geq c.$$

- $a \in S_{\mathsf{cl}}(m^k, g; \mathsf{M}_N)$, $a \sim \sum_{j\geq 0} a_{k-2j}(x,\xi)$ is **globally elliptic** iff

$$\min_{|x|^2+|\xi|^2=1} |\det a_k(x,\xi)| > 0.$$

Weyl Quantization

Definition

With $a \in S(m^k, g; \mathsf{M}_N)$, its **Weyl quantization** (a pseudodifferential operator, ψ-do in the sequel) is defined by

$$a^{\mathsf{w}}(x, D)u(x) = (2\pi)^{-n} \int\int e^{i\langle x-y,\xi\rangle} a(\frac{x+y}{2}, \xi)u(y)\, dy\, d\xi, \; u \in \mathscr{S}(\mathbb{R}^n).$$

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Definition

$\mathrm{OP}S(m^\mu, g)$ is the **class of the ψ-do's** of the form

$$A = a^{\mathrm{w}}(x, D) + R$$

where

- $a \in S(m^k, g)$ is called the *symbol* of A
- R is *smoothing* (i.e. it is $\mathscr{S}' \longrightarrow \mathscr{S}$ continuous or, equivalently, it has Schwartz kernel in $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$).

Composition Formula

Theorem

Given $a_j \in S(m^{k_j}, g)$, $k_j \in \mathbb{R}$, $j = 1, 2$ we have

$$a_1^{\mathsf{w}}(x, D) \circ a_2^{\mathsf{w}}(x, D) = (a_1 \# a_2)^{\mathsf{w}}(x, D) \in OPS(m^{k_1+k_2}, g),$$

where

$$(a_1 \# a_2)(x, \xi) = \sum_{j \geq 0} \frac{1}{j!} \left(\frac{i}{2} (\langle D_\xi, D_y \rangle - \langle D_x, D_\eta \rangle) \right)^j a_1(x, \xi) a_2(y, \eta) \Big|_{x=y,\, \xi=\eta} .$$

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Remark: If $a \in S_{\mathsf{cl}}(m^k, g)$ and $b \in S_{\mathsf{cl}}(m^{k'}, g)$ are classical also $a \# b$ is classical and

$$(a \# b)_{k+k'} = a_k b_{k'},$$

$$(a \# b)_{k+k'-2} = a_k b_{k'-2} + a_{k-2} b_{k'} - \frac{i}{2}\{a_k, b_{k'}\}.$$

Semiclassical Symbols

Definition

We shall say that a function

$$a(X;h) = a(\cdot;h) \in C^\infty(\mathbb{R}^{2n}_X),$$

possibly depending on a parameter $h \in (0, h_0]$, $h_0 \in (0,1]$, *belongs to the symbol class*

$$S^k_\delta(m^\mu, g), \quad k, \mu \in \mathbb{R}, \ \delta \in [0, 1/2]$$

if $\forall \alpha \in \mathbb{Z}^{2n}_+$, $\exists C_\alpha > 0$ s.t.

$$|\partial^\alpha_X a(X;h)| \leq C_\alpha m(X)^{\mu-|\alpha|} h^{-k-|\alpha|\delta}, \quad \forall X \in \mathbb{R}^{2n}, \ \forall h \in (0, h_0].$$

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$$|\partial^\alpha_X a(X;h)| \leq C_\alpha m(X)^{\mu - |\alpha|} h^{-k-|\alpha|\delta}, \quad \forall X \in \mathbb{R}^{2n}, \forall h \in (0, h_0].$$

Remark: As seen for *matrix-valued symbols*,

$$S^k_\delta(m^\mu, g; \mathsf{M}_N) := \mathsf{M}_N \otimes S^k_\delta(m^\mu, g)$$

and, more generally,

$$S^k_\delta(m^\mu, g; V) := V \otimes S^k_\delta(m^\mu, g)$$

for any given finite-dimensional complex vector space V.

h-Weyl Quantization

Definition

Given $a \in S^k_\delta(m^\mu, g; V)$, we define its **$h$-Weyl quantization** as

$$a^{\mathsf{w}}(x, hD)u(x) = (2\pi h)^{-n} \int\int e^{ih^{-1}\langle x-y,\xi\rangle} a(\frac{x+y}{2}, \xi; h)u(y)\, dy\, d\xi,$$

with $u \in \mathscr{S}(\mathbb{R}^n)$.

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with $u \in \mathscr{S}(\mathbb{R}^n)$.

Remark:
Note that the h-Weyl quantization is, in fact, the Weyl quantization of the symbol $a(x, h\xi; h)$.

Composition Lemma for Semiclassical Symbols

Lemma

Let $\mu_1, \mu_2, k_1, k_2 \in \mathbb{R}$, $\delta \in [0, 1/2)$. Given $a_j \in S_\delta^{k_j}(m^{\mu_j}, g)$, $k_j \in \mathbb{R}$, $j = 1, 2$ we have

$$a_1^{\mathsf{w}}(x, hD) \circ a_2^{\mathsf{w}}(x, hD) = (a_1 \# a_2)^{\mathsf{w}}(x, hD),$$

where for every $N_0 \in \mathbb{Z}_+$

$$(a_1 \# a_2)(x, \xi) = \sum_{j=0}^{N_0} \frac{1}{j!} \left(\frac{ih}{2} (\langle D_\xi, D_y \rangle - \langle D_x, D_\eta \rangle) \right)^j a_1(x, \xi; h) a_2(y, \eta; h) \Big|_{x=y,\, \xi=\eta} + h^{N_0+1} r_{N_0+1},$$

with $r_{N_0+1} \in S_\delta^{k_1+k_2+2(N_0+1)\delta}(m^{\mu_1+\mu_2-2(N_0+1)}, g; \mathsf{M}_N)$.

Semiclassical Classical Systems

Definition

We shall say that a **semiclassical symbol** $a \in S_0^k(m^\mu, g; \mathsf{M}_N)$ is **classical** and write $a \in S_{0,\mathsf{cl}}^k(m^\mu, g; \mathsf{M}_N)$ if

$$a(X;h) \sim h^{-k} \sum_{j \geq 0} h^j a_{\mu-2j}(X) \text{ in } S_0^k(m^\mu, g; \mathsf{M}_N), \ \forall X \in \mathbb{R}^{2n},$$

where the $a_{\mu-2j} \in S(m^{\mu-2j}, g; \mathsf{M}_N)$ are ***independent of*** h, $j \geq 0$.

Contents

GPD Symbols and Operators

Definitions

Let $\mu \in \mathbb{Z}_+$. A classical symbol $a \in S_{\mathrm{cl}}(m^k, g)$ is a **global polynomial differential** (**GPD** for short) **symbol of order** μ if

$$a = \sum_{j=0}^{[\mu/2]} a_{\mu-2j}$$

where the entries of the $a_{\mu-2j}$ are *homogeneous polynomials* in $X \in \mathbb{R}^{2n}$ of degree $\mu - 2j$.

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A **global polynomial differential operator** (**GPDO** for short) **of order** μ is the Weyl quantization of a **GPD symbol of order** μ.

Semiclassical GPD Systems and Operators

Definitions

We shall say that a *classical semiclassical symbol* $a \in S^0_{0,\mathrm{cl}}(m^k, g; \mathsf{M}_N)$ is a **semiclassical GPD system of order** μ if $\mu \in \mathbb{Z}_+$ and

$$a = \sum_{j=0}^{[\mu/2]} h^j a_{\mu-2j},\ a_{\mu-2j} \in S(m^{\mu-2j}, g; \mathsf{M}_N)$$

where the entries of the $a_{\mu-2j}$ are *homogeneous polynomials* in $X \in \mathbb{R}^{2n}$ of degree $\mu - 2j$.

We say that a semiclassical GPD system $a \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ of order μ is **elliptic** (resp. **positive elliptic**, when $a = a^*$) if the *principal part* a_μ is a **homogeneous globally elliptic** (resp. **globally positive elliptic**) **symbol**.

Semiclassical GPD Systems and Operators

Definitions

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A **semiclassical global polynomial differential operator of order** μ is the h-Weyl quantization of a **semiclassical GPD symbol of order** μ (i.e. a semiclassical GPD system with $N = 1$).

NCHO Definition

Definition

A **non-commutative harmonic oscillator** (NCHO for short) is a **system of GPDOs of order 2** i.e. it is the **Weyl quantization** of any given **2^{nd}-order** $N \times N$ **GPD system** $a \in S_{\mathrm{cl}}(m^2, g; \mathrm{M}_N)$.

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Remark: Hence,

$$a = a_2 + a_0,$$

where

a_2 is a *matrix* with **homogeneous quadratic forms entries** in $X = (x, \xi) \in \mathbb{R}^{2n}$,

and

a_0 is a **constant** *matrix*.

$Q_{(\alpha,\beta)}$ Definition

Definition

Let $\alpha,\ \beta \in \mathbb{C}$, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then we denote

$$Q_{(\alpha,\beta)}(x,\xi) := \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \frac{x^2+\xi^2}{2} + iJx\xi,\ x,\xi \in \mathbb{R}.$$

Hence, $Q_{(\alpha,\beta)} \in S_{\mathsf{cl}}(m^2, g; \mathsf{M}_2)$ and the NCHO $Q^{\mathsf{w}}_{(\alpha,\beta)}(x,D)$ is the system of GPDOs of order 2

$$Q^{\mathsf{w}}_{(\alpha,\beta)}(x,D) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \frac{x^2-\partial_x^2}{2} + J\left(x\partial_x + \frac{1}{2}\right).$$

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Remark:
A NCHO is *elliptic* (resp. *positive elliptic*) when it is elliptic (resp. positive elliptic) as a GPDO.
Hence if α, $\beta > 0$, $\alpha\beta > 1$, then $Q^{\mathrm{w}}_{(\alpha,\beta)}(x,D)$ is **positive elliptic** and **self-adjoint**.

C/A Relations (1)

Definition

In general, an $N \times N$ second-order partial differential systems with polynomial coefficients $P(X)$, $X \in \mathbb{R}^n \times \mathbb{R}^n$, admits **creation/annihilation relations** (**C/A relations**, for short) if one can find a matrix valued non-zero linear form

$$\mathbb{R}^n \times \mathbb{R}^n \ni X \longmapsto L(X) = \sum_{j=1}^{n} B_j \xi_j + \sum_{j=1}^{n} C_j x_j \in \mathsf{M}_N(\mathbb{C}),$$

s.t. there is $\mu = \mu(L) \in \mathbb{C} \setminus \{0\}$ with

$$[P^{\mathsf{w}}(x, D), L^{\mathsf{w}}(x, D)]\varphi = \mu L^{\mathsf{w}}(x, D)\varphi, \ \forall \varphi \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N).$$

We say that L is in C/A.

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We say that L is in C/A.

Remark:

If L is in C/A relation with P, and P is formally self-adjoint, then L^* (the formal adjoint of L) is in C/A relation with P, with $\mu(L^*) = -\overline{\mu(L)}$.

C/A Relations (2)

Explication:

To clarify the previous definition, consider the case of the harmonic oscillator $p_0^{\mathrm{w}}(x, D)$ in one dimension. For

$$\ell(X) = ax + b\xi,$$

the relation

$$[p_0^{\mathrm{w}}, \ell^{\mathrm{w}}] = \mu \ell^{\mathrm{w}}$$

is equivalent to the condition on the symbols

$$-i\{p_0, \ell\} = \mu \ell.$$

Hence,

$$\mu = \pm 1 \text{ and } \ell(X) = \psi_{\mp}(X)$$

where, up to a scalar multiple,

$$\psi_{\pm}(X) := (\mp i\xi + x)/\sqrt{2}, \text{ **creation/annihilation operators**.}$$

C/A Relations (3)

Theorem (*)

Let $\alpha, \beta > 0$.

- For $\alpha \neq \beta$, the system $Q^{\mathrm{w}}_{(\alpha,\beta)}(x, D)$ does not admit *creation/annihilation relations*.
- For $\alpha = \beta$, the system $Q^{\mathrm{w}}_{(\alpha,\beta)}(x, D)$ admits *creation/annihilation relations*.

(*) A. Parmeggiani, *Non-commutative harmonic oscillators and related problems*. Milan J. Math. **82** (2014), no. 2, 343–387.

Contents

Decoupling Theorem Statement (1/2)

Theorem (1/2)

Let $\mu > 0$ and let

$$a = a^* \sim \sum_{j \geq 0} h^j a_{\mu - 2j} \in S^0_{0,\mathsf{cl}}(m^\mu, g; \mathsf{M}_N).$$

Suppose there is $e_0 \in S(1, g; \mathsf{M}_N)$ s.t.:

- $$e_0^* e_0 = e_0 e_0^* = I_N;$$
- $$e_0^* a_\mu e_0 = b_\mu = \begin{bmatrix} \lambda_{1,\mu} & 0 \\ 0 & \lambda_{2,\mu} \end{bmatrix},$$
  where $\lambda_{j,\mu} = \lambda^*_{j,\mu} \in S(m^\mu, g; \mathsf{M}_{N_j})$, $j = 1, 2, N_1 + N_2 = N$;
- $$d_{\lambda_1,\lambda_2}(X) \gtrsim m(X)^\mu, \ \forall X \in \mathbb{R}^{2n},$$
  where for each $X \in \mathbb{R}^{2n}$,

$$d_{\lambda_1,\lambda_2}(X) = \inf \left\{ |\mu_1 - \mu_2|; \ \mu_1 \in \operatorname{Spec}(\lambda_{1,\mu}), \ \mu_2 \in \operatorname{Spec}(\lambda_{2,\mu}(X)) \right\}. \qquad (1)$$

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Decoupling Theorem Statement (2/2)

Theorem (2/2)

$\longrightarrow$

Then there exists $e \in S^0_{0,\mathsf{cl}}(1, g; \mathsf{M}_N)$ with principal symbol e_0 such that:

$$e^{\mathsf{w}}(x, hD)^* e^{\mathsf{w}}(x, hD) - I,\ e^{\mathsf{w}}(x, hD) e^{\mathsf{w}}(x, hD)^* - I \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N),$$

$$e^{\mathsf{w}}(x, hD)^* a^{\mathsf{w}}(x, hD) e^{\mathsf{w}}(x, hD) - b^{\mathsf{w}}(x, hD) \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N),$$

with $b \sim \sum_{j \geq 0} h^j b_{\mu-2j} \in S^0_{0,\mathsf{cl}}(m^\mu, g; \mathsf{M}_N)$ **blockwise diagonal**:

$$b_{\mu-2j} = \begin{bmatrix} b_{1,\mu-2j} & 0 \\ 0 & b_{2,\mu-2j} \end{bmatrix},\ \forall j \geq 0,$$

where

$$b_{k,\mu-2j} \in (m^{\mu-2j}, g; \mathsf{M}_{N_k}), \quad N_1 + N_2 = N, \quad b_{1,\mu} = \lambda_{1,\mu}, b_{2,\mu} = \lambda_{2,\mu}.$$

Decoupling Theorem Proof (1)

Proof

Finding $e^{\mathsf{w}}(x, hD)$ s.t. the principal symbol is e_0 and

$$e^{\mathsf{w}}(x, hD)e^{\mathsf{w}}(x, hD)^* = I + r^{\mathsf{w}}(x, hD), \text{ with } r \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N),$$

then, by existence of a two-sided parametrix for an elliptical operator, we have that

$$e^{\mathsf{w}}(x, hD)^* e^{\mathsf{w}}(x, hD) = I + s^{\mathsf{w}}(x, hD), \text{ with } s \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N).$$

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$\Rightarrow$ ***It suffices to prove the existence of***
$e^{\mathsf{w}}(x,hD)$ ***and*** b
with the required properties.

$\longrightarrow$

Decoupling Theorem Proof (2)

We will **proceed by induction** showing that $\forall k \in \mathbb{Z}^+$ there exist

$$e_{-2k} \in S(m^{-2k}, g; \mathsf{M}_N), \quad b_{j,\mu-2k} \in S(m^{\mu-2k}, g; \mathsf{M}_{N_j}),\ j = 1, 2.$$

such that, with $E_{N_0}(X) := \sum_{k=0}^{N_0} h^k e_{-2k}(X)$,

$$E_{N_0} \#_h E_{N_0} = I + h^{N_0+1} S^0_{0,\mathsf{cl}}(m^{-2(N_0+1)}, g; \mathsf{M}_N), \tag{2}$$

and

$$E_{N_0} \#_h a \#_h E_{N_0} = \sum_{k=0}^{N_0} h^k b_{\mu-2k} + h^{N_0+1} S^0_{0,\mathsf{cl}}(m^{\mu-2(N_0+1)}, g; \mathsf{M}_N), \tag{3}$$

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where the $b_{\mu-2k} = \begin{bmatrix} b_{1,\mu-2k} & 0 \\ 0 & b_{2,\mu-2k} \end{bmatrix}$ are in block-diagonal form. We shall then take $e \sim \sum_{k\geq 0} h^k e_{-2k}$.

Decoupling Theorem Proof (3)

$\longrightarrow$

- *Base case*: follows from the hypothesis.

Decoupling Theorem Proof (3)

$\longrightarrow$

- *Base case*: follows from the hypothesis.
- *Inductive step*: look for $e_{-2(N_0+1)} \in S(m^{-2(N_0+1)}, g; \mathsf{M}_N)$ so to satisfy the unitarity condition (2)

$$\left(E^{\mathsf{w}}_{N_0} + h^{N_0+1} e^{\mathsf{w}}_{-2(N_0+1)}\right)\left((E^{\mathsf{w}}_{N_0})^* + h^{N_0+1}\left(e^{\mathsf{w}}_{-2(N_0+1)}\right)^*\right) - I = h^{N_0+2} r^{\mathsf{w}}(x, hD),$$

with $r \in S^0_{0,\mathsf{cl}}(m^{-2(N_0+1)}, g; \mathsf{M}_N)$, i.e., look for $e_{-2(N_0+1)}$ s.t.

$$S^{\mathsf{w}}_{N_0} + h^{N_0+1}\left(e^{\mathsf{w}}_0 (e^{\mathsf{w}}_{-2(N_0+1)})^* + e^{\mathsf{w}}_{-2(N_0+1)}(e^{\mathsf{w}}_0)^*\right) = h^{N_0+2}\tilde{r}^{\mathsf{w}}(x, hD),$$

with $\tilde{r} \in S^0_{0,\mathsf{cl}}(m^{-2(N_0+1)}, g; \mathsf{M}_N)$.
By the composition formula, look at the coefficient of h^{N_0+1} and require it to be zero, obtaining the equation

$$s_{-2(N_0+1)} + e_0 (e_{-2(N_0+1)})^* + e_{-2(N_0+1)} e_0^* = 0.$$

$\longrightarrow$

Decoupling Theorem Proof (4)

⟶

The solution of the equation is

$$e_{-2(N_0+1)} = -\frac{1}{2}s_{-2(N_0+1)}e_0 + \alpha_{-2(N_0+1)}e_0 \text{ with } \alpha_{-2(N_0+1)} + \alpha^*_{-2(N_0+1)} = 0. \quad (4)$$

▸ 32

Decoupling Theorem Proof (4)

$\longrightarrow$

The solution of the equation is

$$e_{-2(N_0+1)} = -\frac{1}{2} s_{-2(N_0+1)} e_0 + \alpha_{-2(N_0+1)} e_0 \text{ with } \alpha_{-2(N_0+1)} + \alpha^*_{-2(N_0+1)} = 0. \quad (4)$$

▸ 32

Now we impose the diagonalization condition (3) of a^{w}:

$$(E^{\mathsf{w}}_{N_0+1})\, a^{\mathsf{w}} E^{\mathsf{w}}_{N_0+1} = \sum_{k=0}^{N_0+1} h^k b^{\mathsf{w}}_{\mu-2k} + h^{N_0+1} S^0_{0,\mathsf{cl}}(m^{\mu-2(N_0+1)}, g; \mathsf{M}_N).$$

Besides,

$$(E^{\mathsf{w}}_{N_0+1})\, a^{\mathsf{w}} E^{\mathsf{w}}_{N_0+1} = (E^{\mathsf{w}}_{N_0})\, a^{\mathsf{w}} E^{\mathsf{w}}_{N_0} + h^{N_0+1} \Big((e^{\mathsf{w}}_{-2(N_0+1)})^* a^{\mathsf{w}} e^{\mathsf{w}}_0 + (e^{\mathsf{w}}_0)^* a^{\mathsf{w}} e^{\mathsf{w}}_{-2(N_0+1)} \Big) + h^{N_0+2} S^0_{0,\mathsf{cl}}(m^{\mu-2(N_0+1)}, g; \mathsf{M}_N)$$

$\longrightarrow$

Decoupling Theorem Proof (5)

$\longrightarrow$

The condition for the blocks $b_{j,\mu-2k}$ are already satisfied for $0 \leq k \leq N_0$, *independently* of $e_{-2(N_0+1)}$.

Let $q_{\mu-2(N_0+1)}$ be the coefficient of h^{N_0+1} in

$$E^*_{N_0} \#_h a \#_h E_{N_0},$$

then the coefficient of h^{N_0+1} in $E^*_{N_0+1} \#_h a \#_h E_{N_0+1}$ is

$$\begin{aligned} & q_{\mu-2(N_0+1)} + e^*_{-2(N_0+1)} a_\mu e_0 + e^*_0 a_\mu e_{-2(N_0+1)} \\ = & q_{\mu-2(N_0+1)} + (e^*_{-2(N_0+1)} e_0) e^*_0 a_\mu e_0 + e^*_0 a_\mu e_0 (e^*_0 e_{-2(N_0+1)}). \end{aligned}$$

Decoupling Theorem Proof (5)

$\longrightarrow$

The condition for the blocks $b_{j,\mu-2k}$ are already satisfied for $0 \leq k \leq N_0$, *independently* of $e_{-2(N_0+1)}$.

Let $q_{\mu-2(N_0+1)}$ be the coefficient of h^{N_0+1} in

$$E^*_{N_0} \#_h a \#_h E_{N_0},$$

then the coefficient of h^{N_0+1} in $E^*_{N_0+1} \#_h a \#_h E_{N_0+1}$ is

$$q_{\mu-2(N_0+1)} + e^*_{-2(N_0+1)} a_\mu e_0 + e^*_0 a_\mu e_{-2(N_0+1)}$$
$$= q_{\mu-2(N_0+1)} + (e^*_{-2(N_0+1)} e_0) e^*_0 a_\mu e_0 + e^*_0 a_\mu e_0 (e^*_0 e_{-2(N_0+1)}).$$

Hence

$$\left.\begin{array}{r} \text{denoting: } \tau = -\frac{1}{2} e^*_0 s_{-2(N_0+1)} e_0 \overset{(4)}{=} \tau^* \\ \text{denoting: } \beta = e^*_0 \alpha_{-2(N_0+1)} e_0 = -\beta^* \end{array}\right\} \Rightarrow e^*_{-2(N_0+1)} e_0 = \tau - \beta$$

$$\Rightarrow q_{\mu-2(N_0+1)} + (e^*_0 a_\mu e_0)\tau + \tau(e^*_0 a_\mu e_0) + (e^*_0 a_\mu e_0)\beta - \beta(e^*_0 a_\mu e_0).$$

$\longrightarrow$

Decoupling Theorem Proof (6)

Now, we want to kill the off-diagonal terms in the previous expression by the choice of β and so $\alpha_{-2(N_0+1)}$. In fact, upon writing

$$q_{\mu-2(N_0+1)} + (e_0^* a_\mu e_0)\tau + \tau(e_0^* a_\mu e_0) = \begin{bmatrix} u_1 & \gamma \\ \gamma^* & u_2 \end{bmatrix}, \tag{5}$$

where the $u_j = u_j^*$ are $N_j \times N_j$ blocks, $j = 1, 2$, we look for β in the form

$$\begin{bmatrix} 0 & \delta \\ -\delta^* & 0 \end{bmatrix},$$

and using

$$e_0^* a_\mu e_0 = b_\mu = \begin{bmatrix} \lambda_{1,\mu} & 0 \\ 0 & \lambda_{2,\mu} \end{bmatrix}$$

we are led to the matrix equation

$$\lambda_{1,\mu}\delta - \delta\lambda_{2,\mu} = -\gamma. \tag{6}$$

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Decoupling Theorem Proof (7)

$\longrightarrow$

Lemma:

Let $E = E^* \in S(m^\mu, g; \mathsf{M}_{N_1})$ and $F = F^* \in S(m^\mu, g; \mathsf{M}_{N_2})$ be such that

$$d_{E,F}(X) \geq c_0 m(X)^\mu, \; \forall X \in \mathbb{R}^{2n}.$$

Then for each $X \in \mathbb{R}^{2n}$ the map

$$\begin{cases} \Phi_{E,F}(X) : \mathsf{Mat}_{N_1\times N_2}(\mathbb{C}) \longrightarrow \mathsf{Mat}_{N_1\times N_2}(\mathbb{C}), \\ \Phi_{E,F}(X)T = E(X)T - TF(X), \end{cases}$$

is an isomorphism. Moreover,

$$\left\|\Phi_{E,F}(X)^{-1}\right\| \leq \frac{C}{m(X)^\mu}, \; \forall X \in \mathbb{R}^{2n},$$

for a universal constant $C > 0$. Hence, if $S \in S(m^{\mu-2k}, g; \mathsf{Mat}_{N_1\times N_2}(\mathbb{C}))$, for some $k \in \mathbb{Z}^+$, we have that

$$X \longmapsto T(X) := \Phi_{E,F}(X)^{-1}(S(X)) \in S(m^{-2k}, g; \mathsf{Mat}_{N_1\times N_2}(\mathbb{C})).$$

$\longrightarrow$

Decoupling Theorem Proof (8)

$\longrightarrow$

By this Lemma, the hypothesis (1) yields that the (6) has a unique smooth $N_1 \times N_2$ matrix-valued solution

$$\delta \in S(m^{-2(N_0+1)}, g; \mathrm{Mat}_{N_1\times N_2}(\mathbb{C})).$$

Since this fixes β, and hence $\alpha_{-2(N0+1)}$,

the terms $b_{j,\mu-2(N_0+1)}$ are the block-diagonal terms in (5).

This conclude the inductive step and the proof of the theorem.

$\alpha = \beta > 1$ Case (1)

Lemma

$Q^{\mathrm{w}}_{(\alpha,\alpha)}$ is *unitarily equivalent* to a *scalar* harmonic oscillator.

$\alpha = \beta > 1$ Case (1)

Lemma

$Q^{\mathrm{w}}_{(\alpha,\alpha)}$ is *unitarily equivalent* to a *scalar* harmonic oscillator.

Proof

Let

$$(U_\alpha f)(x) = \frac{1}{\alpha^{1/4}} f\left(\frac{x}{\alpha^{1/2}}\right), \text{ and } (U_\pm f)(x) = e^{\pm i x^2} f(x),\ f \in \mathscr{S}'(\mathbb{R}),$$

be the unitary operators associated, respectively, with the symplectic transformations

$$\chi_\alpha : (x,\xi) \longmapsto (\alpha^{1/2} x, \frac{1}{\alpha^{1/2}}\xi),\ \chi_\pm : (x,\xi) \longmapsto (x, \xi \pm x).$$

$\alpha = \beta > 1$ Case (2)

Put

$$L_\alpha(x,\xi) := \frac{1}{2}(\xi^2 + (\alpha^2 - 1)x^2),$$

by Hormander (65) we get that:

$$U_\alpha^{-1} Q^{\mathrm{w}}_{(\alpha,\alpha)}(x,D) U_\alpha = e^{x^2 J/2} L^{\mathrm{w}}_\alpha(x,D) I_2 e^{-x^2 J/2}. \tag{7}$$

Let $\{\nu_+, \nu_-\}$ be the *unitary* basis of $\mathbb{C}^2$ made of the eigenvectors of J, where

$$J\nu_\pm = \pm i \nu_\pm.$$

Write $f \in \mathscr{S}'(\mathbb{R}, \mathbb{C}^2)$ as

$$f = f_+\nu_+ + f_-\nu_-, \quad \text{where } f_\pm \in \mathscr{S}'(\mathbb{R}).$$

In the basis $\{\nu_+, \nu_-\}$, $Q^{\mathrm{w}}_{(\alpha,\alpha)}(x,D)$ is represented by

$$\begin{bmatrix} \alpha h(x,D) - (xD + Dx)/2 & 0 \\ 0 & \alpha h(x,D) + (xD + Dx)/2 \end{bmatrix}$$

$$= \begin{bmatrix} (L_\alpha \circ (\chi_- \circ \chi_\alpha^{-1}))^{\mathrm{w}}(x,D) & 0 \\ 0 & (L_\alpha \circ (\chi_+ \circ \chi_\alpha^{-1}))^{\mathrm{w}}(x,D) \end{bmatrix}$$

where

$$(\chi_- \circ \chi_\alpha^{-1})(x,\xi) = (\tfrac{1}{\alpha^{1/2}}x, \alpha^{1/2}\xi \pm \tfrac{1}{\alpha^{1/2}}x).$$

$Q^{\mathrm{w}}_{(\alpha,\alpha)}$ Spectrum

By equation (7) and by spectrum conservation in case of unitary transformations, the previous Lemma leads to the following result:

Theorem

When $\alpha = \beta > 1$ one has

$$\mathrm{Spec}(Q^{\mathrm{w}}_{(\alpha,\alpha)}) = \left\{ \sqrt{\alpha^2 - 1} \left(N + \frac{1}{2} \right) ; N \in \mathbb{Z}_+ = \{0, 1, ...\} \right\},$$

with *multiplicity of the eigenvalues* always equal to 2.

$\alpha = \beta = 1$ and $0 < \alpha = \beta < 1$ Cases

Remarks: In a similar way we can prove that:

- If $\alpha = \beta = 1$, then $Q^{\mathsf{w}}_{(\alpha,\alpha)}$ is unitary equivalent to $-\frac{\partial_x^2}{2} I_2$. Hence we have:

$$\mathrm{Spec}(Q^{\mathsf{w}}_{(\alpha,\alpha)}) = \mathrm{Spec}_{\mathsf{ess}}(Q^{\mathsf{w}}_{(\alpha,\alpha)}) = [0, +\infty).$$

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- If $0 < \alpha = \beta < 1$, then $Q^{\mathrm{w}}_{(\alpha,\alpha)}$ is unitary equivalent to $\sqrt{1-\alpha^2}\left(-\frac{\partial_x^2}{2} - \frac{x^2}{2}\right)$.
  Hence, we have:

$$\mathrm{Spec}(Q^{\mathrm{w}}_{(\alpha,\alpha)}) = \mathrm{Spec}_{\mathrm{ess}}(Q^{\mathrm{w}}_{(\alpha,\alpha)}) = \mathbb{R}.$$

Normal Forms (1)

- A. Parmeggiani, *Non-Commutative Harmonic Oscillators and Related Problems*, Milan J. Math.

Definitions

- $p^{\mathsf{w}}(s; x, D) := (D_x^2 + sx^2)/2.$
- $0 < \delta := \sqrt{\alpha\beta}.$
- $\epsilon := \sqrt{|\alpha\beta - 1|}.$
- $s := \operatorname{sgn}(\alpha\beta - 1)$, with $\operatorname{sgn}(0) = 0$.
- $W_0 := [v_+|v_-]$, with $v_\pm$ the eigenvectors of J related to eigenvalues $\pm i$.
- $$U_0 := \begin{bmatrix} U_- U_\delta^* & 0 \\ 0 & U_+ U_\delta \end{bmatrix} \mathsf{W}_0^*, \; U_\epsilon := \begin{bmatrix} U_{1/\epsilon}^* & 0 \\ 0 & U_{1/\epsilon}^* \end{bmatrix} U_0,$$
  with $U_\delta : f(x) \longmapsto \delta^{-1/4} f(x/\delta^{1/2}), \quad U_\pm : f(x) \longmapsto e^{\pm ix^2/2} f(x).$
- $$V_\epsilon(x) := V(x/\epsilon^{1/2}), \text{ with } V(x) := \begin{bmatrix} \omega_+ & -\omega_- e^{-ix^2} \\ -\omega_- e^{ix^2} & \omega_+ \end{bmatrix}.$$

Normal Forms (2)

Theorem $(^*)$

For all $\lambda \in \mathbb{C}$ and α, $\beta > 0$ one has the following factorization, which is valid in $\mathscr{S}'(\mathbb{R};\mathbb{C}^2)$:

- when $\epsilon = 0$ (i.e. $\alpha\beta = 1$) we have

$$Q^{\mathrm{w}}_{(\alpha,\beta)}(x,D) - \lambda = A^{1/2}\mathsf{U}_0^*\left(\frac{1}{2}D_x^2 - \lambda V(x)\right)\mathsf{U}_0 A^{1/2};$$

- when $\epsilon > 0$ (i.e. $\alpha\beta \neq 1$) we have

$$Q^{\mathrm{w}}_{(\alpha,\beta)}(x,D) - \lambda = \frac{1}{\delta}A^{1/2}A^{1/2}\mathsf{U}_\epsilon^*\left(\epsilon p^{\mathrm{w}}(s;x,D) - \frac{\lambda}{\delta}V_\epsilon(x)\right)\mathsf{U}_\epsilon A^{1/2}.$$

$(^*)$ A. Parmeggiani, *Non-commutative harmonic oscillators and related problems*. Milan J. Math. **82** (2014), no. 2, 343–387.

$\alpha \neq \beta,\ \alpha\beta > 1$ Case

Remark:

$$\lambda_+(x,\xi) \neq \lambda_-(x,\xi), \quad \forall (x,\xi) \neq (0,0)$$

$$\Downarrow \leftarrow \text{Decoupling Theorem}$$

Lemma

There is $E \in S_{\mathrm{cl}}(1, g; \mathscr{L}(\mathbb{C}))$ with $E \sim \sum_{j\geq 0} E_{-2j}$, s.t.

$$E^{\mathrm{w}}(E^{\mathrm{w}})^* - I,\ (E^{\mathrm{w}})^* E^{\mathrm{w}} - I \text{ is smoothing}$$

and $\Lambda = \begin{bmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{bmatrix}$, with $\Lambda_\pm \in S_{\mathrm{cl}}(m^2, g)$, s.t.

$$(E^{\mathrm{w}})^* Q^{\mathrm{w}}_{(\alpha,\beta)} E^{\mathrm{w}} - \Lambda^{\mathrm{w}} \text{ is smoothing}$$

where $\Lambda \sim \sum_{j\geq 0} \Lambda_{2-2j}$, $\Lambda_{2-2j} = \begin{bmatrix} \Lambda^+_{2-2j} & 0 \\ 0 & \Lambda^-_{2-2j} \end{bmatrix}$ with $\Lambda_2 = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$ and $\Lambda_0 = \begin{bmatrix} \Lambda^+_0 & 0 \\ 0 & \Lambda^-_0 \end{bmatrix} = 0$.

U_Q Diagonalization

Lemma

One has

$$U_{Q_{(\alpha,\beta)}}(t) = E^{\mathrm{w}} U_\Lambda(t) (E^{\mathrm{w}})^* + S(t),$$

where:

- $U_{Q_{(\alpha,\beta)}}(t) = e^{-itQ_{(\alpha,\beta)}}$,
- $U_\Lambda(t) = e^{-it\Lambda}$,

and the Schwartz kernel $\mathsf{K}_{s(\cdot)}$ of $S(\cdot)$ belongs to

$$C^\infty(\mathbb{R}_t; \mathscr{S}(\mathbb{R}^2_{x,y})) =: C^\infty_t \mathscr{S}_{x,y}.$$

Hence, an FIO-approximation of $U_\Lambda(t)$ yields an FIO-approximation of $U_{Q_{(\alpha,\beta)}}(t)$, for every $t \in \mathbb{R}$.

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U_Q Diagonalization Proof (1)

Proof

Note in the first place that, as proved by Helffer *B.* [4],

$$U_Q, U_\Lambda \in C^\infty(\mathbb{R}_t; \mathscr{L}(\mathscr{S}(\mathbb{R}, \mathbb{C}^2))),$$

and, by duality,

$$U_Q, U_\Lambda \in C^\infty(\mathbb{R}_t; \mathscr{L}(\mathscr{S}'(\mathbb{R}, \mathbb{C}^2))).$$

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and, by duality,

$$U_Q, U_\Lambda \in C^\infty(\mathbb{R}_t; \mathscr{L}(\mathscr{S}'(\mathbb{R}, \mathbb{C}^2))).$$

Consider

$$V_Q(t) := (E^{\mathrm{w}})^* U_Q(t) E^{\mathrm{w}},$$

By the Decoupling Theorem and by remembering that U_Λ is the operator solution of the Schröedinger equation with potential Λ^{w} we have

$$(D_t + \Lambda^{\mathrm{w}}) V_Q(t) = S_1(t),\ V_Q(0) = I + S_0,$$

with $\mathsf{K}_{S_1(\cdot)} \in C_t^\infty \mathscr{S}_{x,y}$ and $\mathsf{K}_{S_0} \in \mathscr{S}_{x,y}$.

U_Q Diagonalization Proof (2)

Let $\tilde{V}_Q(t) := V_Q(t) - S_0$ and $\tilde{S}_1(t) := S_1(t) - \Lambda^{\mathsf{w}} S_0$ ($\mathsf{K}_{\tilde{S}_1(\cdot)} \in C_t^\infty \mathscr{S}_{x,y}$). Then the Schrödinger equation gives

$$(D_t + \Lambda^{\mathsf{w}})\tilde{V}_Q(t) = \tilde{S}_1(t),\ \tilde{V}_Q(0) = V_Q(0) - S_0 = I.$$

Hence

$$\tilde{V}_Q(t) = U_\Lambda(t) + i\int_0^t U_\Lambda(t-s)\tilde{S}_1(s)\,ds,$$

and

$$V_Q(t) = U_\Lambda(t) + i\int_0^t U_\Lambda(t-s)\tilde{S}_1(s)\,ds + S_0.$$

It thus follows that

$$U_Q(t) = E^{\mathsf{w}}\left(U_\Lambda(t) + i\int_0^t U_\Lambda(t-s)\tilde{S}_1(s)\,ds + S_0\right)(E^{\mathsf{w}})^* + S_2(t),$$

with $K_{S_2(\cdot)} \in C_t^\infty \mathscr{S}_{x,y}$.

U_Q Diagonalization Proof (3)

$\longrightarrow$

By putting

$$S(t) := E^{\mathrm{w}} \left(i \int_0^t U_\Lambda(t-s)\tilde{S}_1(s)\, ds + S_0 \right) (E^{\mathrm{w}})^* + S_2(t),$$

we thus obtain $\mathsf{K}_{S(\cdot)} \in C_t^\infty \mathscr{S}_{x,y}$ and

$$U_Q(t) = E^{\mathrm{w}} U_\Lambda(t)(E^{\mathrm{w}})^* + S(t).$$

This concludes the proof of the lemma.

S_Q Singular Support (1)

Definitions

$$S_Q(t) : \mathscr{S}(\mathbb{R}) \ni \phi \longrightarrow \mathrm{Tr}\,(U_\phi) \in \mathbb{C}$$

where $t \in \mathbb{R}$ and

$$\mathrm{Tr}\,(U_\phi) := \int \mathrm{Tr}(U_\phi(x,x))\,dx$$

with $U_\phi(x,y) \in \mathscr{S}(\mathbb{R}\times\mathbb{R}, \mathscr{L}(\mathbb{C}^2))$ is the Schwartz kernel of

$$\int \phi(t) e^{-itQ^{\mathbf{w}}_{(\alpha,\beta)}}\,dt, \quad \phi \in \mathscr{S}(\mathbb{R}).$$

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$$I_\varphi(\tau) := \mathrm{Tr}\left(\int \varphi(t) e^{it\tau} e^{-itQ_{(\alpha,\beta)}}\, dt \right)$$

S_Q Singular Support (1)

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$$\int \phi(t) e^{-itQ^{\mathrm{w}}_{(\alpha,\beta)}}\,dt, \quad \phi \in \mathscr{S}(\mathbb{R}).$$

$$I_\varphi(\tau) := \mathrm{Tr}\left(\int \varphi(t) e^{it\tau} e^{-itQ_{(\alpha,\beta)}}\,dt\right)$$

Remark: $\mathscr{F}_{t\longrightarrow\tau}(\varphi S_Q)(\tau) = \left\langle S_Q(t), \varphi(t)e^{-it\tau}\right\rangle = I_\varphi(-\tau)$. Hence,

$$\varphi \in C_0^\infty(\mathbb{R}),\ \mathrm{supp}\varphi \cap \mathrm{sing\ supp}(S_Q) \Longleftrightarrow I_\varphi(\tau) = O(|\tau|^{-\infty}),\ \text{as } |\tau| \longrightarrow +\infty$$

S_Q Singular Support (2)

Theorem

Let $\mathscr{L} := \mathscr{L}_+ \cup \mathscr{L}_-$, where

$$\mathscr{L}_\pm = \{kT_\pm;\, k \in \mathbb{Z}\}$$

is the set of periods, and their opposites, of the periodic trajectories of the $H_\pm$ of energy 1 (i.e. $\lambda_\pm = 1$) and where 0 is thought of as a trivial period. Then

$$\text{sing supp}(S_Q) \subset \mathscr{L}.$$

sing supp$(S_Q) \subset \mathscr{L}$ Proof (1)

Proof

Note that 0 is an isolated point of $\mathscr{L}$ which is a closed set.

Take $\varphi \in C_c^\infty(\mathbb{R})$, with supp $\varphi \cap \mathscr{L} = \emptyset$.

By using the Diagonalization Lemma, we may rewrite $I_\varphi(\tau)$ as

$$I_\varphi(\tau) = \mathrm{Tr}\left(\int \varphi(t) e^{it\tau} E^{\mathsf{w}} U_\Lambda(t) (E^{\mathsf{w}})^* \, dt\right) + O(|\tau|^{-\infty}), \text{ as } |\tau| \longrightarrow +\infty,$$

Replacing $U_\Lambda(t)$ by its FIO-approximation

$$\begin{bmatrix} F_+(t) & 0 \\ 0 & F_-(t) \end{bmatrix}$$

where $F_\pm(t) \in I^0_{\mathsf{cl}}(a_\pm, \phi_\pm)$ is the FIO-approximations of the $e^{-it\Lambda^{\mathsf{w}}_\pm}$.
($I^0_{\mathsf{cl}}(a_\pm, \phi_\pm)$ denotes the class of classical FIO with phase-function ϕ and 0th-order amplitude a).

sing supp$(S_Q) \subset \mathscr{L}$ Proof (2)

We must study the operator

$$E_\varphi(\tau) := E^{\mathrm{w}} \left(\int \varphi(t) e^{it\tau} \begin{bmatrix} F_+(t) & 0 \\ 0 & F_-(t) \end{bmatrix} dt \right) (E^{\mathrm{w}})^*. \tag{8}$$

▸ 53

Putting

$$E^{\mathrm{w}} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

we have

$$E_{ij} F_\pm(t) E^*_{i'\,j'} \in I^0_{\mathsf{cl}}(\tilde{a}_\pm, \phi_\pm),$$

i.e. the FIOs have the same phase-functions as those of the $F_\pm(t)$.

sing supp$(S_Q) \subset \mathscr{L}$ Proof (3)

$\longrightarrow$

Since the *principal part* of

$$E_{ij} F_{\pm}(t) E^*_{i'\,j'}$$

is just the product of the respective principal symbols, we obtain by:

– a stationary-phase argument,

– the assumption supp $\varphi \cap \mathscr{L} = \emptyset$,

that

$$I_{\varphi}(\tau) = \mathrm{Tr} E_{\varphi}(\tau) = O(|\tau|^{-\infty}) \quad \text{as } |\tau| \longrightarrow +\infty.$$

The fact that one has to consider *periodic trajectories with energy* $\lambda_{\pm} = 1$ follows by the homogeneity of the phase-functions.

NCHO Weyl Law

Theorem

Let $\varphi \in C_c^\infty(\mathbb{R})$ be even, real-valued, such that $\varphi(0) = 1$, and with so small a support that

$$\text{supp}\varphi \cap \mathscr{L} = \{0\}.$$

Then,

$$I_\varphi(\tau) = O(|\tau|^{-\infty}) \quad (9)$$

$$\text{as } \tau \longrightarrow -\infty,$$

$$I_\varphi(\tau) = \int_{\lambda_+=1} \frac{ds}{|\nabla\lambda_+|} + \int_{\lambda_-=1} \frac{ds}{|\nabla\lambda_-|} + O(\tau^{-1}) = T_+ + T_- + O(\tau^{-1}) \quad (10)$$

$$\text{as } \tau \longrightarrow +\infty$$

▸ 54

$$N_{Q^{\mathbf{w}}_{(\alpha,\beta)}}(\lambda) = (2\pi)^{-1}\left(\int_{\lambda_+=1} \frac{ds}{|\nabla\lambda_+|} + \int_{\lambda_-=1} \frac{ds}{|\nabla\lambda_-|}\right)\lambda^{\frac{2n}{\mu}} + O(\lambda^{\frac{2(n-1)}{\mu}})$$

$$\text{as } \lambda \longrightarrow +\infty$$

NCHO Weyl Law Proof (1)

Proof

Consider the operator $E_\varphi(\tau)$ defined in (8). By the same arguments used in the proof of sing supp$(S_Q) \subset \mathscr{L}$, one gets

$$\begin{aligned} I_\varphi(\tau) =& \mathrm{Tr}\left(\int \varphi(t)e^{it\tau}\left[E_{11}F_+(t)E_{11}^* + E_{12}F_-(t)E_{12}^*\right]dt\right) + \\ &+ \mathrm{Tr}\left(\int \varphi(t)e^{it\tau}\left[E_{21}F_+(t)E_{21}^* + E_{22}F_-(t)E_{22}^*\right]dt\right) + O(|\tau|^{-\infty}). \end{aligned}$$

The conclusion (9) follows by the result for the scalar case (Helffer B.[4]).
As regards (10), upon denoting by e_{ij} the principal symbol of E_{ij}, we have

$$I_\varphi(\tau) = \int_{\lambda_+=1}(|e_{11}|^2+|e_{21}|^2)\frac{ds}{|\nabla\lambda_+|} + \int_{\lambda_-=1}(|e_{12}|^2+|e_{22}|^2)\frac{ds}{|\nabla\lambda_-|} + O(|\tau|^{-1}).$$

$\longrightarrow$

NCHO Weyl Law Proof (2)

$\longrightarrow$

Since the matrix

$$[v_0^+|v_0^-] = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix},$$

is a *unitary* matrix, we obtain

$$|e_{11}|^2 + |e_{21}|^2 = 1 = |e_{12}|^2 + |e_{22}|^2,$$

whence formula (10) follows.
The Weyl asymptotics is finally obtained since under our hypothesis has already been proved for the scalar case to which we have traced (Helffer [4]). This concludes the proof of the theorem.

Contents

Definition

Definiton

Given a *self-adjoint operator* $0 < A = A^*$ with a *discrete spectrum* $\{\lambda_j\}_{j\geq 1}$, the **spectral zeta function** associated with A is by definition the series

$$\zeta_A(s) = \sum_{j\geq 1} \frac{1}{\lambda_j^s}, \quad s \in \mathbb{C} \text{ with } \mathrm{Re}\, s \text{ sufficiently large.}$$

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Remark:
It is well-known that for the spectral zeta function of the *harmonic oscillator*

$$H = p_0^{\mathrm{w}}(x, D) \quad (n = 1)$$

one has

$$\zeta_H(s) = \sum_{j\geq 0} \frac{1}{(j + 1/2)^s} = (2^s - 1)\zeta(s) = (2^s - 1)\sum_{j\geq 1} \frac{1}{j^s},$$

where $\zeta(s)$ is the *Riemann zeta function*.

ζ_A Holomorphicity

Property

If A is *elliptic*, *self-adjoint* and *positive*, then ζ_A is *holomorphic* for $\mathrm{Re} s > 2n/\mu$.

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$$\lambda_j(A) \approx j^{\mu/2n}$$

that is true for any *elliptic*, *self-adjoin*t, *positive* operator.

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Remark:

If $A = Q^{\mathsf{w}}_{(\alpha,\beta)}$ and α, $\beta > 0$ and $\alpha\beta > 1$, then, by the property above, we have

$$\zeta_{Q^{\mathsf{w}}_{(\alpha,\beta)}} \text{ is } \textbf{holomorphic} \text{ for Re} s > 1$$

since here $n = 1$ and $\mu = 2$.

Goal

A more detailed study of the **holomorphy** of the **zeta function** of an *elliptic* $N \times N$ *system* of GPDOs in $\mathbb{R}^n$ of order $\mu \in \mathbb{N}$.

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Method adopted:

Construction by ***Robert's approach*** (which follows by the **Shubin's one)** of the zeta function of an elliptic operator regardless its spectrum.

Robert's Construction of ζ_A (1)

Definition

Let $0 < A = A^* \in OPS_{\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ elliptic $N \times N$ system of GPDOs in $\mathbb{R}^n$ of order $\mu \in \mathbb{N}$.
We define

$$m_\lambda(X) := (1 + |X|^2 + |\lambda|^{2/\mu})^{1/2}$$

and the *metric*

$$g_{\lambda,X} := m_\lambda(X)^{-2}|dX|^2.$$

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Remark:
With this weight and metric we have:

$$A - \lambda \in OPS_{\mathsf{cl}}(m_\lambda^\mu, g_\lambda; \lambda \in \Lambda; \mathsf{M}_N)$$

and $A - \lambda$ is **elliptic** for $\lambda \in \Lambda$.

Robert's Construction of ζ_A (2)

Procedure

1. ▶ Construction of a *parametrix*

$$B_\lambda \in OPS_{\mathsf{cl}}(m_\lambda^{-\mu}, g_\lambda; \lambda \in \Lambda; \mathsf{M}_N)$$

of

$$A_\lambda := A - \lambda$$

such that

$$B_\lambda A_\lambda = I + R_\lambda,\ A_\lambda B_\lambda = I + R_\lambda^{'},\ R_\lambda, R_\lambda^{'} \in OPS_{\mathsf{cl}}(m_\lambda^{-\infty}, g_\lambda; \lambda \in \Lambda; \mathsf{M}_N).$$

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2 ▶ Take the symbol of B_λ

$$b(X;\lambda) \sim \sum_{j\geq 0} b_{-\mu-2j}(X;\lambda)$$

⟶

Robert's Construction of ζ_A (3)

$\longrightarrow$

3 ▶ Take

$$a_z(X) \sim \frac{1}{2\pi i}\int_\gamma \lambda^z b(X;\lambda)\, d\lambda$$

where $\gamma \subset \mathbb{C}$ is the curve

$$\{z \in \mathbb{C};\ |z| = c,\ |\arg z| \in [\theta^{'}, 2\pi - \theta^{'}]\} \cup \{z \in \mathbb{C};\ \arg z = \theta^{'},\ |z| \geq c\}$$
$$\cup\{z \in \mathbb{C};\ \arg z = \theta^{'},\ |z| \geq c\},$$

for some fixed $\theta \in (\theta, \pi/4)$ and $c > 0$, oriented s.t. the circle-part is clockwise oriented.

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for some fixed $\theta \in (\theta, \pi/4)$ and $c > 0$, oriented s.t. the circle-part is clockwise oriented.

4 ▶ If $a_{z,N}$ is any truncation of the asymptotics of a_z, with remainder $r_{z,N}$, one defines

$$A^z := a^{\mathsf{w}}_{z,N}(x, D) + r^{\mathsf{w}}_{z,N}(x, D), \text{ when } \mathrm{Re} z < 0,$$

and for $\mathrm{Re} z < \ell$, $\ell \in \mathbb{Z}_+$,

$$A^z = A^{z-\ell} A^\ell,$$

so that $A^z \in OPS_{\mathrm{cl}}(m^{\mu \mathrm{Re} z}, g; \mathsf{M}_N)$.

Robert's Theorem on ζ_A (1)

Theorem

Let $\mathsf{K}^{(z)}$ be the Schwartz kernel of A^z. Then,

$$\zeta_A(s) = \mathrm{Tr} A^{-s} = \int_{\mathbb{R}^n} \mathrm{Tr}\, \mathsf{K}^{(-s)}(x,x)\, dx,$$

is holomorphic in $\{s \in \mathbb{C};\, \mathrm{Re} s > 2n/\mu\}$, and can be extended as a meromorphic function in $\mathbb{C}$, with at most **simple poles** belonging to the sequence

$$s_j = \frac{2n}{\mu} - \frac{2j}{\mu}, j \in \mathbb{Z}_+,$$

with residue

$$\mathrm{Res}(\zeta_A, s_j) = \frac{\mu}{i(2\pi)^{n+1}} \int_{\mathbb{S}^{2n-1}} \int_\gamma \lambda^{-s_j} \mathrm{Tr}\, b_{-\mu-2j}(\omega,\lambda)\, d\lambda\, d\omega$$

The function ζ_A is holomorphic in 0 with value

$$\zeta_A(0) = \frac{1}{\mu(2\pi)^n} \int_{\mathbb{S}^{2n-1}} \int_\gamma \mathrm{Tr}\, b_{-\mu-2n}(\omega,-\lambda)\, d\lambda\, d\omega.$$

Robert's Theorem on ζ_A (2)

Remarks:

Since in our case A is a system of GPDOs, we have that ζ_A is holomorphic in $-j$ ($j \in \mathbb{N}$), with value

$$\zeta_A(-j) = \frac{(-1)^j}{\mu(2\pi)^n} \int_{\mathbb{S}^{2n-1}} \int_0^{+\infty} \operatorname{Tr} b_{-\mu-2n-j\mu}(\omega, -\lambda)\, d\lambda\, d\omega,$$

hence it is surely 0 when $j\mu$ is *not* even, for in this case $b_{-\mu-2n-j\mu} = 0$.

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hence it is surely 0 when $j\mu$ is *not* even, for in this case $b_{-\mu-2n-j\mu} = 0$.

Note:

$$\zeta_A(-j) = \mathrm{Tr} A^j,$$

so that, since in our case A is a system of GPDOs, the *value* $\zeta_A(-j)$ *is the trace of a* ***local*** *operator.*

The Ichinose-Wakayama Theorem

Theorem

There exist constants $C_{Q,j}$, $j \in \mathbb{N}$, such that $\zeta_Q(s)$ is represented, for every integer $\nu \in \mathbb{N}$, as

$$\zeta_Q(s) = \frac{1}{\Gamma(s)} \left[\frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} \frac{1}{s-1} + \sum_{j=1}^{\nu} \frac{C_{Q,j}}{s+2j-1} + H_{Q,\nu}(s) \right],$$

with:

– $\Gamma(s)$ the *Euler gamma function*,
– $H_{Q,\nu}$ holomorphic in Re$s > -2\nu$.

Hence, the *spectral zeta function* $\zeta_Q(s)$

- is *meromorphic* in the whole complex plane $\mathbb{C}$ with a simple pole at $s = 1$,
- has zeros (the so-called "trivial zeros") for $0, -2, -4, \ldots$ (the non-positive even integers $2\mathbb{Z}_-$).

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Thank you very much for your attention!