Nonlinear anisotropic Picone type identities for general vector fields and some applications

Abimbola Abolarinwa

Department of Mathematics, University of Lagos

Ghent Analysis & PDE Center, Ghent University

Summer School

Singularities in Science and Engineering

GHENT ANALYSIS PDE

22 – 31 August, 2022

Outlines

1. Introduction
   - Classical Picone identity
   - Anisotropic Euclidean p-Laplacian
   - General Vector Fields
2. Nonlinear Anisotropic Picone identities
3. Applications
   - Anisotropic functional spaces
   - Comparison principle -quasilinear system singular nonlinearities
   - Weighted anisotropic Hardy type inequalities

Introduction

Aim

- To present a generalization of nonlinear Picone type identities for anisotropic subelliptic p-Laplacian in the context of the general vector fields.
- To discuss some applications to comparison principle and hardy type inequalities

Introduction

Aim

- To present a generalization of nonlinear Picone type identities for anisotropic subelliptic p-Laplacian in the context of the general vector fields.
- To discuss some applications to comparison principle and hardy type inequalities

Picone identities -Historical background

- named after Mauro Picone (1885 -1977), is classical in the theory of homogeneous linear 2^o ODE.
- dated back to early 1900's, used successfully for Sturm comparison theorem and oscillation theory

Introduction

Aim

- To present a generalization of nonlinear Picone type identities for anisotropic subelliptic p-Laplacian in the context of the general vector fields.
- To discuss some applications to comparison principle and hardy type inequalities

Picone identities -Historical background

- named after Mauro Picone (1885 -1977), is classical in the theory of homogeneous linear 2^o ODE.
- dated back to early 1900's, used successfully for Sturm comparison theorem and oscillation theory
- generalized to PDE yielding several applications:
  - Hardy type and Rellich type inequalities to
  - Liouville type and Sturmian type comparison principles to
  - first eigenvalue monotonicity and Morse index of positive solutions.

- Mauro Picone (1910): Ann. Scuola Norm. Sup. Pisa. 11: 1–141

$$\frac{d}{dt}\left[\frac{u}{v}\left(f_1\frac{du}{dt}v - f_2u\frac{dv}{dt}\right)\right]$$
$$= (f_1 - f_2)\left(\frac{du}{dt}\right)^2 + f_2\left(\frac{du}{dt} - \frac{u}{v}\frac{dv}{dt}\right)^2 + (h_2 - h_1)u^2,$$

for $u, v \geq 0$ - differentiable $\forall\ t$, $v(t) \neq 0$,

- Mauro Picone (1910): Ann. Scuola Norm. Sup. Pisa. 11: 1–141

$$\frac{d}{dt}\left[\frac{u}{v}\left(f_1\frac{du}{dt}v - f_2 u\frac{dv}{dt}\right)\right]$$
$$= (f_1 - f_2)\left(\frac{du}{dt}\right)^2 + f_2\left(\frac{du}{dt} - \frac{u}{v}\frac{dv}{dt}\right)^2 + (h_2 - h_1)u^2,$$

for $u, v \geq 0$ - differentiable $\forall\ t$, $v(t) \neq 0$, & (u, v) solves 2^0 diff. system

$$\begin{cases} \dfrac{d}{dt}\left(f_1(t)\dfrac{du}{dt}\right) & +h_1(x)u = 0, \\ \dfrac{d}{dt}\left(f_2(t)\dfrac{dv}{dt}\right) & +h_2(x)v = 0. \end{cases}$$

- Mauro Picone (1910): Ann. Scuola Norm. Sup. Pisa. 11: 1–141

$$\frac{d}{dt}\left[\frac{u}{v}\left(f_1\frac{du}{dt}v - f_2 u\frac{dv}{dt}\right)\right]$$
$$= (f_1 - f_2)\left(\frac{du}{dt}\right)^2 + f_2\left(\frac{du}{dt} - \frac{u}{v}\frac{dv}{dt}\right)^2 + (h_2 - h_1)u^2,$$

for $u, v \geq 0$ - differentiable $\forall\ t$, $v(t) \neq 0$, & (u, v) solves 2^0 diff. system

$$\begin{cases} \dfrac{d}{dt}\left(f_1(t)\dfrac{du}{dt}\right) & +h_1(x)u = 0, \\ \dfrac{d}{dt}\left(f_2(t)\dfrac{dv}{dt}\right) & +h_2(x)v = 0. \end{cases}$$

Applied to:

- Sturmian comparison principle for $f_1(t) > f_2(t)$, $h_1(t) < h_2(t)$, and
- oscillation theorem of solutions to the system.

Classical Picone identity

- The classical Picone identity (multidimensional context):
- Allegretto (1986)[1]:

$$u, v \geq 0,\ v \neq 0 (\text{differentiable in } \mathbb{R}^n),$$

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0. \quad (1)$$

- applied extensively to the study of 2^o elliptic equations and systems involving Laplacian.

- Several extensions and generalization of Picone identity have been established in order to handle more general elliptic operators.

Classical Picone identity

- The classical Picone identity (multidimensional context):
- Allegretto (1986)[1]:

$$u, v \geq 0, \; v \neq 0 (\text{differentiable in } \mathbb{R}^n),$$

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0. \qquad (1)$$

- applied extensively to the study of 2^o elliptic equations and systems involving Laplacian.

- Several extensions and generalization of Picone identity have been established in order to handle more general elliptic operators.
- Allegretto and Huang (1998)[2] extended it in order to handle p-Laplace equations and eigenvalue problems involving p-Laplacian.

Picone identity for p-Laplacian

- The p-Laplacian version reads as follows, for $u \geq 0$, $v > 0$, then

$$\mathscr{L}_p(u,v) = \mathscr{R}_p(u,v) \tag{2}$$

where

$$\mathscr{L}_p(u,v) := |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla u,$$

$$\mathscr{R}_p(u,v) := |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v \geq 0.$$

Picone identity for p-Laplacian

- The p-Laplacian version reads as follows, for $u \geq 0$, $v > 0$, then

$$\mathscr{L}_p(u,v) = \mathscr{R}_p(u,v) \tag{2}$$

where

$$\mathscr{L}_p(u,v) := |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla u,$$

$$\mathscr{R}_p(u,v) := |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v \geq 0.$$

- Tyagi (2013) [13] - nonlinear versions of (1).
- Bal (2013) [3] - nonlinear versions of (2), (also (2016)Tirayaki [12] & Feng (2017)[5]) with several applications.
- Zographopoulos [14] studied properties of the principal eigenvalue of degenerate quasilinear elliptic system .

Anisotropic Euclidean p-Laplacian

- The anisotropic Euclidean p-Laplacian is defined for C^2-functions as

$$\mathscr{L}f := \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\left(\left|\frac{\partial f}{\partial x_k}\right|^{p_k-2}\frac{\partial f}{\partial x_k}\right) \tag{3}$$

$p_k > 1$, $k = 1, \cdots, N$.

Setting

- $p_k = 2 \implies$ the usual Laplacian,
- $p_k = p$ for all $k \implies$ the pseudo-p-Laplacian.

Applications in science and engineering

- anisotropic characteristics of some reinforced materials [11],
- dynamics of fluid in the anisotropic media having different conductivities in each direction [4].
- models involving (3) arise in image processing, computer vision [10], etc.

General Vector Fields & sub-Laplacian

- M - n-dimensional smooth manifold equipped with a volume form dx
- $\{X_k\}_{k=1}^N$, $n \geq N$, - a family of vector fields on M.
- Consider the operator

$$\mathcal{L} := \sum_{k=1}^{N} X_k^2,$$

known as canonical sub-Laplacian

- locally hypoelliptic if the commutators of $\{X_k\}_{k=1}^N$ generate the tangent space of M as the Lie algebra (Hörmander sums of squares theorem)
- also well studied under weaker assumption or without hypoellipticity
- The horizontal differential operator X_k identifies each vector field X with its derivative in the direction k.
- Examples of spaces for such vector fields
  -**Carnot groups, H-type groups,**
  - **Grushin plane**, $M = \mathbb{R}^2 := \{(x, y) : X_1 = \partial/\partial x,\ X_2 = x\partial/\partial y\}$.

M. Ruzhansky, D. Suragan, *Hardy inequalities on homogeneous groups*, Progress in Math. Vol. 327, Birkhäuser, (2019).

Anisotropic p-sub-Laplacian

- For $1 < p_k < \infty$, the anisotropic p-sub-Laplacian on M is defined by

$$\mathcal{L}_p f := \sum_{k=1}^{N} X_k(|X_k f|^{p_k - 2} X_k f),$$

subelliptic form of (3).

- If $M = \mathbb{R}^n$, (e.g. X_k are linearly independent and span the tangent space)

$$dx = \text{ Lebesgue measure,}$$

$$(X_1, \cdots, X_k) = \nabla \text{ is the Euclidean gradient,}$$

$$\mathcal{L} = \Delta \text{ is the Euclidean Laplacian.}$$

- By setting $p = 2$,
  the p-Laplacian becomes $\mathcal{L}_2 = \Delta$,.

Motivating Literature

- We derive generalized nonlinear versions of Picone identities (2) for the anisotropic subelliptic p-Laplacian in the context of general vector fields.
- Generalization of Picone type identities to subelliptic context is growing rapidly owing to numerous applications in the analysis of PDE.
- Ruzhansky, Sabitbek and Suragan (2019 & 2021) [7, 8] - interesting extension to general vector fields and p-sub-Laplacian (with applications to Grushin plane, Heisenberg group, Stratified Lie groups).
- Suragan and Yessirkegenov (2021) [9] for nonlinear Picone's identities for anisotropic p-sub-Laplacian and p-biLaplacian (with applications to horizontal Hardy inequalities and weighted eigenvalue problem on Stratified Lie groups).

M. Ruzhansky, D. Suragan, *Hardy inequalities on homogeneous groups*, Progress in Math. Vol. 327, Birkhäuser, (2019).

Nonlinear Anisotropic Picone identities

Theorem

Let $\Omega \subset M$, u, v are nonconstant a.e. differentiable functions in Ω.

- Hypotheses

Nonlinear Anisotropic Picone identities

Theorem

Let $\Omega \subset M$, u, v are nonconstant a.e. differentiable functions in Ω.

- Hypotheses
  H1: $g_k, f_k : \mathbb{R}^+ \to \mathbb{R}^+$ are C^1 for $k = 1, \cdots, N$, satisfying the ffg:

$$
\begin{aligned}
g_k(u) > 0, \quad g_k'(u) > 0 \quad &\text{for} \quad u > 0, \quad x \in \Omega; \\
g_k(u) = 0, \quad g_k'(u) = 0 \quad &\text{for} \quad u = 0, \quad x \in \partial\Omega; \\
f_k(v) > 0, \quad f_k'(v) > 0 \quad &\text{for} \quad v > 0, \quad x \in \Omega;
\end{aligned}
$$

Nonlinear Anisotropic Picone identities

Theorem

Let $\Omega \subset M$, u, v are nonconstant a.e. differentiable functions in Ω.

- Hypotheses
  H1: $g_k, f_k : \mathbb{R}^+ \to \mathbb{R}^+$ are C^1 for $k = 1, \cdots, N$, satisfying the ffg:

$$\begin{aligned} g_k(u) > 0, \quad g_k'(u) > 0 \quad &\text{for} \quad u > 0, \quad x \in \Omega; \\ g_k(u) = 0, \quad g_k'(u) = 0 \quad &\text{for} \quad u = 0, \quad x \in \partial\Omega; \\ f_k(v) > 0, \quad f_k'(v) > 0 \quad &\text{for} \quad v > 0, \quad x \in \Omega; \end{aligned}$$

  H2: for $1 < p_k < \infty$

$$\sum_{k=1}^{N} \frac{g_k(u) f_k'(v)}{[f_k(v)]^2} |X_k v|^{p_k} \geq \sum_{k=1}^{N} (p_k - 1) \left[\frac{g_k'(u)}{p_k f_k(v)} |X_k v|^{p_k - 1} \right]^{\frac{p_k}{p_k - 1}}. \tag{4}$$

Theorem cont'd

Define the quantities $R(u, v)$ and $L(u, v)$ as follows ($p_k > 1$, $k = 1, \cdots, N$):

Theorem cont'd

Define the quantities $R(u, v)$ and $L(u, v)$ as follows ($p_k > 1$, $k = 1, \cdots, N$):

$$R(u, v) := \sum_{k=1}^{N} |X_k u|^{p_k} - \sum_{k=1}^{N} X_k \left(\frac{g_k(u)}{f_k(v)} \right) |X_k v|^{p_k - 2} X_k v, \qquad (5)$$

Theorem cont'd

Define the quantities $R(u,v)$ and $L(u,v)$ as follows ($p_k > 1$, $k = 1, \cdots, N$):

$$R(u,v) := \sum_{k=1}^{N} |X_k u|^{p_k} - \sum_{k=1}^{N} X_k \left(\frac{g_k(u)}{f_k(v)} \right) |X_k v|^{p_k - 2} X_k v, \tag{5}$$

$$L(u,v) := \sum_{k=1}^{N} |X_k u|^{p_k} - \sum_{k=1}^{N} \frac{g_k'(u)}{f_k(v)} |X_k v|^{p_k-2} X_k v X_k u + \sum_{k=1}^{N} \frac{g_k(u) f_k'(v)}{[f_k(v)]^2} |X_k v|^{p_k}. \tag{6}$$

Theorem cont'd

Define the quantities $R(u,v)$ and $L(u,v)$ as follows ($p_k > 1$, $k = 1, \cdots, N$):

$$R(u,v) := \sum_{k=1}^{N} |X_k u|^{p_k} - \sum_{k=1}^{N} X_k \left(\frac{g_k(u)}{f_k(v)} \right) |X_k v|^{p_k - 2} X_k v, \qquad (5)$$

$$L(u,v) := \sum_{k=1}^{N} |X_k u|^{p_k} - \sum_{k=1}^{N} \frac{g_k'(u)}{f_k(v)} |X_k v|^{p_k - 2} X_k v X_k u + \sum_{k=1}^{N} \frac{g_k(u) f_k'(v)}{[f_k(v)]^2} |X_k v|^{p_k}. \qquad (6)$$

Then

$$R(u,v) = L(u,v) \geq 0. \qquad (7)$$

Theorem cont'd

Moreover,

$$L(u, v) = 0 \tag{8}$$

a.e. in Ω if and only if

Theorem cont'd

Moreover,

$$L(u,v) = 0 \tag{8}$$

a.e. in Ω if and only if

$$u = \alpha v \quad \text{a.e in } \Omega \quad \text{for} \quad \alpha \in \mathbb{R}, \tag{9}$$

$$|X_k u|^{p_k} = \left[\frac{g_k'(u)}{p_k f_k(v)}|X_k v|^{p_k - 1}\right]^{\frac{p_k}{p_k-1}}, \quad k = 1, \cdots N, \tag{10}$$

and

$$\frac{g_k(u) f_k'(v)}{[f_k(v)]^2}|X_k v|^{p_k} = (p_k - 1)\left[\frac{g_k'(u)}{p_k f_k(v)}|X_k v|^{p_k - 1}\right]^{\frac{p_k}{p_k-1}}, \tag{11}$$

(12)

$$k = 1, \cdots, N.$$

Remark

1. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ and $f_k(y) = y$ in (5) and (6) $\Longrightarrow$ the classical Picone identity (1).

Remark

1. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ and $f_k(y) = y$ in (5) and (6) $\Longrightarrow$ the classical Picone identity (1).

2. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ in (5) and (6) $\Longrightarrow$ Tyagi's result [13] which is the nonlinear analogue of (1).

Remark

1. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ and $f_k(y) = y$ in (5) and (6) $\Longrightarrow$ the classical Picone identity (1).

2. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ in (5) and (6) $\Longrightarrow$ Tyagi's result [13] which is the nonlinear analogue of (1).

3. allow $p_k = p$, $g_k = u^p$ for each k in (5) and (6), $\Longrightarrow$ a special case proved in [3] (Euclidean space) and a special case proved in [9] (stratified Lie groups).

Remark

1. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ and $f_k(y) = y$ in (5) and (6) $\implies$ the classical Picone identity (1).

2. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ in (5) and (6) $\implies$ Tyagi's result [13] which is the nonlinear analogue of (1).

3. allow $p_k = p$, $g_k = u^p$ for each k in (5) and (6), $\implies$ a special case proved in [3] (Euclidean space) and a special case proved in [9] (stratified Lie groups).

4. The case $p_k = p$, $g_k = u^p$ and $f_k(y) = y^{p-1}$ for each k in (5) and (6) was proved in [2] ($M = \mathbb{R}^n$ and Euclidean p-Laplacian), in [6] (for the Heisenberg group and horizontal p-sub-Laplacian) and in [7] (for general vector fields).

Remark

1. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ and $f_k(y) = y$ in (5) and (6) $\Longrightarrow$ the classical Picone identity (1).

2. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ in (5) and (6) $\Longrightarrow$ Tyagi's result [13] which is the nonlinear analogue of (1).

3. allow $p_k = p$, $g_k = u^p$ for each k in (5) and (6), $\Longrightarrow$ a special case proved in [3] (Euclidean space) and a special case proved in [9] (stratified Lie groups).

4. The case $p_k = p$, $g_k = u^p$ and $f_k(y) = y^{p-1}$ for each k in (5) and (6) was proved in [2] ($M = \mathbb{R}^n$ and Euclidean p-Laplacian), in [6] (for the Heisenberg group and horizontal p-sub-Laplacian) and in [7] (for general vector fields).

5. The case $g_k = u^{p_k}$ for $k = 1, \cdots, N$ in (5) and (6) was presented by in [9].

Remark

1. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ and $f_k(y) = y$ in (5) and (6) $\Longrightarrow$ the classical Picone identity (1).

2. $M = \mathbb{R}^n$, $p_k = 2$, $g_k = u^2$ in (5) and (6) $\Longrightarrow$ Tyagi's result [13] which is the nonlinear analogue of (1).

3. allow $p_k = p$, $g_k = u^p$ for each k in (5) and (6), $\Longrightarrow$ a special case proved in [3] (Euclidean space) and a special case proved in [9] (stratified Lie groups).

4. The case $p_k = p$, $g_k = u^p$ and $f_k(y) = y^{p-1}$ for each k in (5) and (6) was proved in [2] ($M = \mathbb{R}^n$ and Euclidean p-Laplacian), in [6] (for the Heisenberg group and horizontal p-sub-Laplacian) and in [7] (for general vector fields).

5. The case $g_k = u^{p_k}$ for $k = 1, \cdots, N$ in (5) and (6) was presented by in [9].

6. The case $g_k = u^{p_k}$ and $f_k(y) = y^{p_k - 1}$ for $k = 1, \cdots, N$ in (5) and (6) was proved by [8] for stratified Lie groups.

Anisotropic Sobolev spaces

- $\Omega \subset M$ be an open domain. Define the anisotropic functional spaces $\mathcal{D}^{1,p_k}(\Omega)$, $1 < p_k < \infty$, $k = 1, \cdots, N$, as

$$\mathcal{D}^{1,p_k}(\Omega) := \{u \in \mathcal{D}^{1,1}(\Omega) : \ |X_k u| \in L^{p_k}(\Omega)\}$$

w.r.t the norm

$$\|u\|_{\mathcal{D}^{1,p_k}(\Omega)} := \int_\Omega |u| dx + \sum_{k=1}^{N} \left(\int_\Omega |X_k u|^{p_k} dx \right)^{\frac{1}{p_k}}.$$

- Consider the anisotropic functional

$$\mathbb{J}_{p_k}(u) := \sum_{k=1}^{N} \left(\int_\Omega |X_k u|^{p_k} dx \right)^{\frac{1}{p_k}},$$

then, define the anisotropic functional class $\mathcal{D}_0^{1,p_k}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ w.r.t. to the norm generated by $\mathbb{J}_{p_k}(u)$.

- $\mathcal{D}^{1,p_k}(\Omega)$ and $\mathcal{D}_0^{1,p_k}(\Omega)$ are both separable and reflexive Banach spaces.

Definition -weak solutions

- Consider the nonlinear anisotropic p-sub-Laplacian equation

$$\begin{aligned} -\sum_{k=1}^{N} X_k(|X_k u|^{p_k-2} X_k u) &= \sum_{k=1}^{N} F_k(u), && x \in \Omega, \\ u &> 0, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned} \tag{13}$$

Definition -weak solutions

- Consider the nonlinear anisotropic p-sub-Laplacian equation

$$\begin{aligned} -\sum_{k=1}^{N} X_k(|X_k u|^{p_k-2} X_k u) &= \sum_{k=1}^{N} F_k(u), & x \in \Omega, \\ u &> 0, & x \in \Omega, \\ u &= 0, & x \in \partial\Omega. \end{aligned} \tag{13}$$

Definition- Weak Solution

By the (weak) solution of (13), we refer to a positive function $u \in \mathcal{D}_0^{1,p_k}(\Omega)$ satisfying

$$\sum_{k=1}^{N} \int_{\Omega} |X_k u|^{p_k-2} X_k u X_k \phi dx = \sum_{k=1}^{N} \int_{\Omega} F_k(u) \phi dx \tag{14}$$

for all $\phi \in C_0^{\infty}(\Omega)$.

Application to comparison principle

Anisotropic Picone identities yields a linear relation between u and v of:

Anisotropic quasilinear system with singular nonlinearities

$$\begin{cases} -\sum_{k=1}^{N} X_k(|X_k u|^{p_k-2} X_k u) = \sum_{k=1}^{N} f_k(v), & x \in \Omega, \\ -\sum_{k=1}^{N} X_k(|X_k v|^{p_k-2} X_k v) = \sum_{k=1}^{N} \frac{[f_k(v)]^2 u}{g_k(u)}, & x \in \Omega, \\ g_k(u) > 0,\ f_k(v) > 0,\ u > 0,\ v > 0, & x \in \Omega, \\ g_k(u) = 0,\ f_k(v) = 0,\ u = 0,\ v = 0, & x \in \partial\Omega. \end{cases} \tag{15}$$

Proposition 1

Let $(u, v) \in \mathcal{D}_0^{1,p_k}(\Omega) \times \mathcal{D}_0^{1,p_k}(\Omega)$ be a pair of (weak)-solutions to (15). Then $u = \alpha v$ a.e. in Ω, where $\alpha \in \mathbb{R}$ is a constant.

Proof of Proposition 1

Since $(u, v) \in \mathcal{D}_0^{1,p_k} \times \mathcal{D}_0^{1,p_k}$ a pair of solutions to (15), it follows that

Proof of Proposition 1

Since $(u, v) \in \mathcal{D}_0^{1,p_k} \times \mathcal{D}_0^{1,p_k}$ a pair of solutions to (15), it follows that

$$\sum_{k=1}^{N} \int_\Omega |X_k u|^{p_k - 2} X_k u X_k \phi_1 dx = \sum_{k=1}^{N} \int_\Omega f_k(v) \phi_1 dx, \tag{16}$$

$$\sum_{k=1}^{N} \int_\Omega |X_k v|^{p_k - 2} X_k v X_k \phi_2 dx = \sum_{k=1}^{N} \int_\Omega \frac{[f_k(v)]^2 u}{g_k(u)} \phi_2 dx, \tag{17}$$

for any pair of functions $(\phi_1, \phi_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$.

Proof of Proposition 1

Since $(u,v) \in \mathcal{D}_0^{1,p_k} \times \mathcal{D}_0^{1,p_k}$ a pair of solutions to (15), it follows that

$$\sum_{k=1}^{N} \int_\Omega |X_k u|^{p_k-2} X_k u X_k \phi_1 dx = \sum_{k=1}^{N} \int_\Omega f_k(v)\phi_1 dx, \tag{16}$$

$$\sum_{k=1}^{N} \int_\Omega |X_k v|^{p_k-2} X_k v X_k \phi_2 dx = \sum_{k=1}^{N} \int_\Omega \frac{[f_k(v)]^2 u}{g_k(u)} \phi_2 dx, \tag{17}$$

for any pair of functions $(\phi_1, \phi_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$.
Letting $\phi_1 \to u$ and $\phi_2 = \frac{g_k(u)}{f_k(v)}$ in (16) and (17), respectively, we obtain

$$\sum_{k=1}^{N} \int_\Omega |X_k u|^{p_k} dx = \sum_{k=1}^{N} \int_\Omega |X_k v|^{p_k-2} X_k v X_k \left(\frac{g_k(u)}{f_k(v)}\right) dx.$$

Proof of Proposition 1

Since $(u, v) \in \mathcal{D}_0^{1,p_k} \times \mathcal{D}_0^{1,p_k}$ a pair of solutions to (15), it follows that

$$\sum_{k=1}^{N} \int_\Omega |X_k u|^{p_k-2} X_k u X_k \phi_1 dx = \sum_{k=1}^{N} \int_\Omega f_k(v) \phi_1 dx, \tag{16}$$

$$\sum_{k=1}^{N} \int_\Omega |X_k v|^{p_k-2} X_k v X_k \phi_2 dx = \sum_{k=1}^{N} \int_\Omega \frac{[f_k(v)]^2 u}{g_k(u)} \phi_2 dx, \tag{17}$$

for any pair of functions $(\phi_1, \phi_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$.
Letting $\phi_1 \to u$ and $\phi_2 = \frac{g_k(u)}{f_k(v)}$ in (16) and (17), respectively, we obtain

$$\sum_{k=1}^{N} \int_\Omega |X_k u|^{p_k} dx = \sum_{k=1}^{N} \int_\Omega |X_k v|^{p_k-2} X_k v X_k \left(\frac{g_k(u)}{f_k(v)} \right) dx.$$

Hence, $\int_\Omega R(u, v) dx = 0 \implies R(u, v) = 0.$ □

Weighted anisotropic Hardy inequalities

Application of the Theorem to derivation of a generalized anisotropic Hardy inequality for the general vector fields.

Proposition 2

Let Ω -open bounded in M. $v \in \mathcal{D}_0^{1,p_k}(\Omega)$, $p_k > 1$, $k = 1, \cdots, N$, satisfying

$$\begin{aligned} -X_k(|X_k v|^{p_k-2} X_k v) &\geq H_k(x) f_k(v), && x \in \Omega, \\ f_k(v) > 0,\ v > 0, & && x \in \Omega, \\ f_k(v) = 0,\ v = 0, & && x \in \partial\Omega, \end{aligned} \tag{18}$$

where $f_k : \mathbb{R}^+ \to \mathbb{R}^+$ is C^1 and $H_k(x) \geq 0$ - weight function. Then for any nonnegative $u \in C_0^1(\Omega)$ with $g_k(u) > 0$, $g_k(u)$ is C^1, there holds

$$\sum_{k=1}^{N} \int_\Omega |X_k u|^{p_k} dx \geq \sum_{k=1}^{N} \int_\Omega H_k(x) g_k(u) dx. \tag{19}$$

Proof of Proposition 2

Application of Picone identity, the divergence theorem and (18) leads to

$$
\begin{aligned}
0 &\le \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx \\
&= \sum_{k=1}^{N} \int_\Omega |X_k u|^{p_k} dx - \sum_{k=1}^{N} \int_\Omega X_k \left(\frac{g_k(u)}{f_k(v)} \right) |X_k v|^{p_k - 2} X_k v dx \\
&\le \sum_{k=1}^{N} \int_\Omega |X_k u|^{p_k} dx - \sum_{k=1}^{N} \int_\Omega H_k(x) g_k(u) dx,
\end{aligned}
$$

which is the desired inequality (19). □

Proof of Proposition 2

Application of Picone identity, the divergence theorem and (18) leads to

$$\begin{aligned} 0 &\leq \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx \\ &= \sum_{k=1}^N \int_\Omega |X_k u|^{p_k} dx - \sum_{k=1}^N \int_\Omega X_k \left(\frac{g_k(u)}{f_k(v)} \right) |X_k v|^{p_k - 2} X_k v dx \\ &\leq \sum_{k=1}^N \int_\Omega |X_k u|^{p_k} dx - \sum_{k=1}^N \int_\Omega H_k(x) g_k(u) dx, \end{aligned}$$

which is the desired inequality (19). □

Remark

Putting $p_k = p$ (constant) and $g_k(u) = u^p$ for each k, then we have isotropic Hardy inequality for the general vector fields

$$\int_\Omega |\nabla_X u|^p dx \geq C_p \int_\Omega H(x)|u|^p dx.$$

Examples - anisotropic weighted Hardy inequalities

Corollary 1

Let $\Omega \subset M$ be any domain. Let $\beta > 2$, $1 < p_k < \beta$, $k = 1, \cdots, N$. Then for all $u \in C_0^1(\Omega)$, we have

$$\sum_{k=1}^{N} \int_{\Omega} |X_k u|^{p_k} dx \geq \sum_{k=1}^{N} \left(\frac{\beta - p_k}{p_k} \right)^{p_k} \int_{\Omega} \frac{|X_k \rho|^{p_k}}{\rho^{p_k}} |u|^{p_k} dx, \qquad (20)$$

where ρ is a homogeneous norm on M.

Examples - anisotropic weighted Hardy inequalities

Corollary 1

Let $\Omega \subset M$ be any domain. Let $\beta > 2$, $1 < p_k < \beta$, $k = 1, \cdots, N$. Then for all $u \in C_0^1(\Omega)$, we have

$$\sum_{k=1}^{N} \int_{\Omega} |X_k u|^{p_k} dx \geq \sum_{k=1}^{N} \left(\frac{\beta - p_k}{p_k} \right)^{p_k} \int_{\Omega} \frac{|X_k \rho|^{p_k}}{\rho^{p_k}} |u|^{p_k} dx, \qquad (20)$$

where ρ is a homogeneous norm on M.

To prove :

- Assume $\mathcal{L}$ possesses a fundamental solution. in an open set of M, say Φ
- The function $\rho(x) = \Phi^{\frac{1}{2-\beta}}$, $x \neq 0$, is a homogeneous quasi norm on M
- a continuous function, $\rho : M \to (0, \infty)$, smooth away from the origin.
- In the case of general homogeneous Carnot group, $\beta = Q \geq 3$ is precisely the homogeneous dimension of the group.
- To prove , choose $v = \Phi_y^{\frac{\varphi}{2-\beta}} = \rho^{\varphi}$, where $\varphi = -\left(\frac{\beta - p_k}{p_k}\right) < 0$ into (18).

Corollary 2

Let $M = \mathbb{G} \setminus \{x' = 0\}$ be a stratified Lie group, and N, the dimension of the first stratum. There holds for all $u \in C_0^1(\Omega)$, $1 < p_k < N$, $k = 1, \cdots, N$

$$\sum_{k=1}^{N} \int_\Omega |X_k u|^{p_k} dx \geq \sum_{k=1}^{N} \left(\frac{p_k - 1}{p_k} \right)^{p_k} \int_\Omega \frac{|u|^{p_k}}{|x'_k|^{p_k}} dx, \tag{21}$$

where $|x'| = (x_1'^2 + \cdots + x_N'^2)^{\frac{1}{2}}$ (Euclidean norm).

Corollary 2

Let $M = \mathbb{G} \setminus \{x' = 0\}$ be a stratified Lie group, and N, the dimension of the first stratum. There holds for all $u \in C_0^1(\Omega)$, $1 < p_k < N$, $k = 1, \cdots, N$

$$\sum_{k=1}^{N} \int_{\Omega} |X_k u|^{p_k} dx \geq \sum_{k=1}^{N} \left(\frac{p_k - 1}{p_k} \right)^{p_k} \int_{\Omega} \frac{|u|^{p_k}}{|x'_k|^{p_k}} dx, \tag{21}$$

where $|x'| = (x_1'^2 + \cdots + x_N'^2)^{\frac{1}{2}}$ (Euclidean norm).

- Here we take M to be a stratified Lie group.
- the volume measure dx is the Haar measure.
- The left invariant vector field X_k has an explicit form

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', \cdots, x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}},$$

  where $x = (x', x^{(2)}, \cdots, , x^{(r)})$, and $x^{(l)} = (x_1^{(l)}, \cdots, x_{N_l}^{(l)})$ are the variables in the l^{th}-stratum.
- Choose auxiliary function $v = \prod_{k=1}^{N} |x'_k|^{\varphi_k}$, $\varphi_k = \frac{p_k - 1}{p_k}$ into (18).

Acknowledgement

- CDC of IMU-SIMONS
- CDC of EMS-SIMONS
- Ghent Analysis and PDE Center
- MAARG - UNILAG
- Prof. Michael Ruzhansky

References

W. Allegretto, *Positive solutions and spectral properties of weakly coupled elliptic systems*, . Math. Anal. Appl. 120, 2 (1986), 723–729.

W. Allegretto, Y. X. Huang, *A Picone's identity for the p-Laplacian and applications*, Nonl. Anal. Th. Meth. Appl., 32(7) (1998), pp. 819–830.

K. Bal, *Generalized Picone's identity and its applications*, Electron. J. Diff. Equations., 2013(243), (2013), pp. 1–6.

J. Bear, Dynamics of Fluids in Porous Media, Elsevier, New York, 1972.

T. Feng, *A new nonlinear Picone identity and applications*, Mathematica Applicata, 30(2), (2017), 278–283.

P. Niu, H. Zhang, Y. Wang, *Hardy Type and Rellich Type Inequalities on the Heisenberg Group*, Proc. Amer. Math. Soc., 129(12) (2001), 3623–3630.

M. Ruzhansky, B. Sabitbek, D. Suragan, *Principal frequency of p-versions of sub-Laplacians for general vector fields*, Z. Anal. Anwend. 40 (2021), 97–109.

M. Ruzhansky, B. Sabitbek, D. Suragan, *Weighted anisotropic Hardy and Rellich type inequalities for general vector fields*, Nonlinear Differ. Eqn. Appl. 26, 13 (2019).

D. Surugan, N. Yessirkegenov, *Generalised nonlinear Picone identities for p-sub-Laplacians and p- biharmonic operators and applications*, Adv. Op. Th. 6(53) (2021), 1–17.

G. Sapiro, Geometric partial differential equations and image analysis, Cambridge University Press (2001).

Q. Tang, Regularity of minimizer of nonisotropic integrals of the calculus of variations, Ann. Mat. Pura Appl. 164 (1993), 77–87.

A. Tirayaki, *Generalized nonlinear Picone's identity for the p-Laplacian and its applications*, Electron. J. Diff. Equations, 2016(269), (2016), 1–7.

J. Tyagi, *A nonlinear picone's identity and its applications*, Applied Mathematics Letters, 26, (2013), 624–626.

N. B. Zographopoulos, *On the principal eigenvalue of degenerate quasilinear elliptic systems*, Math. Nachr. 281 (9), (2008), 1351–1365.

Thank you for your attention !