Compressive Sensing in Clifford Analysis: Bicomplex and Quaternionic case

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Overview

What is it?

This field Compressive Sensing or Compressed Sensing (CS) started with Candes/Romberg/Tao 2006 which showed that the number of measurements can be small and still contain nearly all the useful information required for an effective reconstruction.

Main requirements

- a signal is expressible in terms of a linear combination of elements in a given basis (frame)
- it has a sparse representation

What is the main problem?

Suppose there is a signal $\mathbf{x} = (x_1, x_2, \ldots, x_N)^T$.

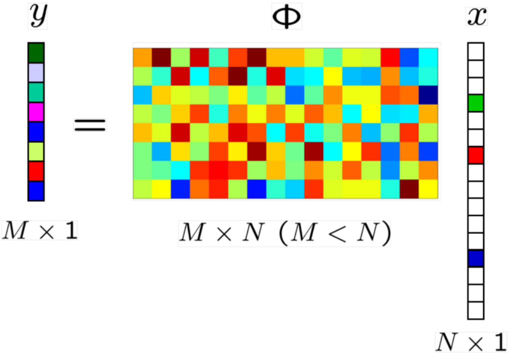
From this we measure just a few linear samples $\mathbf{y} = (y_1, y_2, \ldots, y_M)^T$, with $M \ll N$.

Can we reconstruct $\mathbf{x}$?

The answer is **yes** if $\mathbf{x}$ is sparse in a given basis or frame.

Graphic representation

Solve $y = \Phi x$ such that



Restricted Isometry Property - RIP

Definition

Φ has the RIP property if for each integer $k \in \mathbb{N}$ there exists an isometry constant $0 < \delta_k < 1$ that is, δ_k is the smallest positive real number such that it holds

$$(1 - \delta_k)\|x\|_{\ell_2}^2 \leq \|\Phi x\|_{\ell_2}^2 \leq (1 + \delta_k)\|x\|_{\ell_2}^2$$

for all $k-$sparse vectors x.

When is x reconstructible?

- if the signal x is $k-$sparse, then only $M \geq 2k$ data points are required for reconstruction; moreover,
- stability is attained if $M \geq Ck \ln(N/M)$.

Big Problem

RIP property is a too strong condition to demand.

SOLUTION: find a condition which can be fulfilled with a certain probability such that we have reconstruction of a signal by an

- ℓ_1-minimization procedure;
- uniformly distributed randomly chosen sampling points.

Fundamental Lemma

Let $c \in \ell_2(D)$ and $T := \mathrm{supp} c$. Assume

$$\mathcal{F}_{TX} : \ell_2(T) \to \ell_2(X)$$

to be injective. Suppose that there exists a vector $P \in \ell_2(D)$ with the following properties:

(i) $P_{\mathbf{k}} = \mathrm{sgn} c_{\mathbf{k}}$ for all $\mathbf{k} \in T$;

(ii) $|P_{\mathbf{k}}| < 1$ for all $\mathbf{k} \notin T$;

(iii) there exists a vector $\lambda \in \ell_2(X)$ such that $P = \mathcal{F}_X^* \lambda$ then c is unique minimizer to our ℓ_1-minimization problem

$$\min ||c_{\mathbf{k}}||_1 := \sum_{\mathbf{k} \in T} |c_{\mathbf{k}}|,$$

s.t.

$$f(x_j) := \sum_{\mathbf{k} \in T} c_{\mathbf{k}} \, \varphi(\mathbf{x_j}), \; j = 1, \ldots, N.$$

Which cases we already study?

Cases with results

1. Takenaka-Malmquist System Setting
2. Bicomplex Setting
3. Quaternionic Setting
4. Slice Monogenic Setting

Cases without results

1. Spherical harmonics Setting

In this presentation:

1. Bicomplex Setting
2. Quaternionic Setting

Takenaka-Malmquist System Setting

Takenaka-Malmquist System Setting

- Compressed Sensing with Nonlinear Fourier Atoms, Narciso Gomes, Stefan Hartmann and Paula Cerejeiras, Springer Birkhäuser, Cham (2016)
- https://link.springer.com/chapter/10.1007/978-3-319-42529-04

The Takenaka-Malmquist system

The Takenaka-Malmquist (TM) system with respect to a sequence $\{a_0, a_1, \ldots, a_n\}$ of points in the unit disk is given by

$$\mathcal{B}_0(z) = \frac{\sqrt{1-|a_0|^2}}{1-\overline{a_0}z},$$

$$\mathcal{B}_k(z) = \frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}z} \prod_{l=0}^{k-1} \frac{z-a_l}{1-\overline{a_l}z}, \quad k = 1, 2, \ldots .$$

Slice regular functions

Definition

Let U be an open set in $\mathbb{H}$ and let $f : U \to H$ be real differentiable. The function f is said to be (left) slice regular or (left) slice hyperholomorphic if for every $I \in \mathbb{S}$, its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ passing through origin and containing I and satisfies

$$\overline{\partial}_I f(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0$$

on $U \cap \mathbb{C}_I$. The class of (left) slice regular functions on U. Analogously, a function is said to be right slice regular in U if

$$(f_I\overline{\partial})f(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x+Iy) + \frac{\partial}{\partial y} f_I(x+Iy)I\right) = 0.$$

Representation Formula

Representation Formula

Given $s = x + Jy \in \mathbb{C}_J$, there are unique $\mathbb{C}_J$ such that

$$f(x + Jy) = (1 - JI)f(x + Iy) + (1 + JI)f(x - Iy)$$

$$\Leftrightarrow$$

$$e^{Jk\theta} = \frac{1}{2}\left[(1 - JI)e^{\mathbf{I}k\theta}(1 + JI)\right].$$

Bicomplex Setting

Bicomplex Setting

- Bicomplex signals with sparsity constraints, Narciso Gomes, Paula Cerejeiras, Wiley Library, (2018)
- https://onlinelibrary.wiley.com/doi/pdf/10.1002/mma.5059

Fourier basis

For the description of an image one uses the standard Fourier basis:

$$\left\{ e^{i\theta},\ \theta \in [0, 2\pi[\right\}. \quad (1)$$

The discrete Fourier transform (DFT) converts the sampled function from the time-domain to the frequency domain, that is to say, it converts a finite list of equally-spaced samples of a function into the list of coefficients for complex sinusoids.

For our setting we use the set of all automorphisms of the unit ball $\varphi : \mathbb{B}_1 \to \mathbb{B}_1$, ($\mathbb{B}_1$ the unit ball in $\mathbb{BC}$) restricted to the boundary. This set can be regarded as a generalization of the classical Fourier system (1).

Motivation

Bidimensional extension - image processing

- An approach for analyzing image is the concept of “bicomplex signal”.
- Embedding of a multi-channel signal into the bicomplex algebra.
- Representation of HSI-model ((hue-saturation-intensity)-model for human eye vision)
- The idempotent property allow us to use a similar approach to the one-dimensional case.

Bicomplex numbers

Definition

$$\mathbb{BC} := \{\zeta = \zeta_1 + \mathbf{j}\zeta_2 : \zeta_1 = a + \mathbf{i}b,\ \zeta_2 = c + \mathbf{i}d \in \mathbb{C}\},$$

where

$$\mathbf{j}^2 = \mathbf{i}^2 = -1 \ \text{ and } \ \mathbf{ij} = \mathbf{ji}.$$

Important: the imaginary units **i** and **j** are commuting!

Problem: generalization to high dimension implies loss of certain properties!

Zero Divisors

Existence of zero divisors:

$$\mathbf{e}^+ := \frac{1+\mathbf{ij}}{2} \quad \text{and} \quad \mathbf{e}^- := \frac{1-\mathbf{ij}}{2}.$$

Properties:

- Unitary sum: $\mathbf{e}^+ + \mathbf{e}^- = 1$;
- Difference property: $\mathbf{e}^+ - \mathbf{e}^- = \mathbf{ij}$;
- Idempotent property: $(\mathbf{e}^+)^2 = \mathbf{e}^+, \quad (\mathbf{e}^-)^2 = \mathbf{e}^-$.

The property of idempotent allows to separate the space of bicomplex in terms of $\mathbf{e}^+$, $\mathbf{e}^-$ as projectors.

Decomposition

Decomposition

Given $\zeta = \zeta_1 + \mathbf{j}\zeta_2 \in \mathbb{BC}, (\zeta_1, \zeta_2 \in \mathbb{C})$, there are unique $z, w \in \mathbb{C}$ such that

$$\zeta = z\mathbf{e}^+ + w\mathbf{e}^-,$$

with $z = \zeta_1 - \mathbf{i}\zeta_2$ and $w = \zeta_1 + \mathbf{i}\zeta_2$.

Conjugation

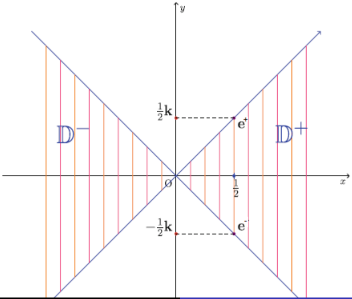
The correspondent conjugation of a given bicomplex number $\zeta = z\mathbf{e}^+ + w\mathbf{e}^-$ is defined as

$$\overline{\zeta} = \overline{z}\mathbf{e}^+ + \overline{w}\mathbf{e}^-.$$

Hyperbolic positive numbers

The idempotent representation of a bicomplex number $\zeta = a + b\mathbf{ij} = a + b\mathbf{k}$, with $a, b \in \mathbb{R}$, is $\zeta = \nu\mathbf{e}^+ + \mu\mathbf{e}^-$ such that $\nu, \mu \in \mathbb{R}$, with $\nu = b + a$ and $\mu = b - a$.
Then, we define the set of positive hyperbolic numbers as

$$\mathbb{D}^+ := \{\nu\mathbf{e}^+ + \mu\mathbf{e}^- \mid \nu, \mu \geq 0\}.$$



Inner product and related norms

- The "inner product" of two bicomplex numbers, $\zeta = z_1\mathbf{e}^+ + w_1\mathbf{e}^-$ and $\eta = z_2\mathbf{e}^+ + w_2\mathbf{e}^-$, is defined as

$$\langle \zeta, \eta \rangle := z_1\overline{z}_2\mathbf{e}^+ + w_1\overline{w}_2\mathbf{e}^- = \langle z_1, z_2 \rangle_{\mathbb{C}}\mathbf{e}^+ + \langle w_1, w_2 \rangle_{\mathbb{C}}\mathbf{e}^-.$$

- The "bicomplex norm" is defined as

$$|\zeta|_{\mathbf{k}} := |z_1|\mathbf{e}^+ + |z_2|\mathbf{e}^- \in \mathbb{D}^+.$$

Then the induced (euclidean) norm by this inner product is

$$\|\zeta\| := 2\mathrm{Re}\left(|\zeta|_{\mathbf{k}}\right).$$

Inverse

An invertible bicomplex number $\zeta = z\mathbf{e}^{+} + w\mathbf{e}^{-}$ has its inverse given by

$$\zeta^{-1} = z^{-1}\mathbf{e}^{+} + w^{-1}\mathbf{e}^{-}$$

where z^{-1} and w^{-1} are the complex multiplicative inverses of z and w, respectively.

High dimension extension

In $\mathbb{R}^2$ the classical discrete Fourier transform of a signal f is given by

$$F(x_1, x_2) = \sum_{m=t_1}^{M-1} \sum_{t_2=0}^{N-1} f(t_1, t_2) e^{i2\pi\left(\frac{t_1 x_1}{M} + \frac{t_2 x_2}{N}\right)}$$

Definition ($\mathbb{BC}-$Discrete Fourier Transform)

Given a signal f in $\mathbb{R}^2$, we define its $\mathbb{BC}-$discrete Fourier transform as

$$F(x_j^1, x_j^2) = \sum_{k_1, k_2 \in T} f(k_1, k_2) \left[e^{\mathbf{i} k_1 x_j^1} \mathbf{e}^+ + e^{\mathbf{i} k_2 x_j^2} \mathbf{e}^- \right]$$

Our setting

Denote by $\prod_q$ the space of functions spanned by $d = (2q+1)^2$ elements of the bicomplex system.

Additionally, we consider the signal f to have its support on a set T satisfying to $|T| \leq M \ll d$, that is to say,

$$f(x) = \sum_{(k,\tilde{k})\in T} c_{k\tilde{k}} \left(e^{ix^1}\mathbf{e}^+ + e^{ix^2}\mathbf{e}^- \right),$$

with the sequence $\left(c_{k\tilde{k}} := c_k\mathbf{e}^+ + c_{\tilde{k}}\mathbf{e}^-\right)_{k\tilde{k}} \in \mathbb{BC}$ having support on T and $x = (x^1, x^2) \in \mathbb{R}^2$.

Main Goal

Given a sampling set

$$X := \{x_1, x_2, \ldots, x_N\}$$

of linearly independent random variables (l.i.r.v.) having uniform distribution on $[0, 2\pi]^2$ our goal is to reconstruct the signal f from the sparse samples

$$\{f(x_j), j = 1, 2, \ldots, N\},$$

and this with a certain (known) probability.

Main theorem

Theorem

Assume $f \in \prod_q$ *with some sparsity* $M \in \mathbb{N}$*, and consider* $X := \{x_1, x_2 \ldots, x_N\}$ *a sampling set of l.i.r.v. with uniform distribution on* $[0, 2\pi]^2$.
Choose $n \in \mathbb{N}$, $\beta > 0$, $\kappa > 0$ *and* $K_1, K_2, \ldots, K_n \in \mathbb{N}$ *s. t.* $a := \sum_{m=1}^{n} \beta^{n/K_m} < 1 \quad \text{and} \quad \frac{\kappa}{1-\kappa} \leq \frac{1-a}{1+a} M^{-3/2}$.
For $\theta := N/M$ *we have that f can be reconstructed with probability at least*

$$1 - \left(D\beta^{-2n} \sum_{m=1}^{n} G_{2mK_m}(\theta) + M\kappa^{-2} G_{2n}(\theta) \right)$$

from its sample values $f(x_1), \ldots, f(x_N)$ *by solving the following* ℓ^1*-minimization problem (2).*

ℓ^1-minimization problem

$$\min ||(c_{k\tilde{k}})||_1 := \sum_{k \in T} |c_{k\tilde{k}}|, \tag{2}$$

subject to

$$g(x_j) := \sum_{(k,\tilde{k}) \in T} c_{k\tilde{k}} \left(e^{ix_j^1} \mathbf{e}^+ + e^{ix_j^2} \mathbf{e}^- \right), \; j = 1, \ldots, N.$$

Hereby,

$$G_n(\theta) = \theta^{-n} \sum_{k=1}^{\lfloor n/2 \rfloor} S_2(n,k) \theta^k$$

where $S_2(n,k)$ denotes the Stirling numbers of second kind.

Proof - notations

Some **auxiliary notations**:

$\ell^2(D)$, $\ell^2(T)$, $\ell^2(X)$ will denote the ℓ^2-spaces of sequences in those spaces endowed with the usual Euclidean norm.

The operator $\mathcal{F}_X : \ell^2(D) \to \ell^2(X)$.

We also consider the operator $\mathcal{F}_{TX}$ as the restriction of $\mathcal{F}_X$ which denote the sequences with support only on T acting from $\ell^2(T)$ in $\ell^2(X)$, and the their adjoint operators $\mathcal{F}^*_{TX}$,

$$\mathcal{F}^*_X : \ell^2(X) \to \ell^2([-q,q]^2) \text{ and } \mathcal{F}^*_{TX} : \ell^2(X) \to \ell^2(T).$$

Sampling matrix

We consider the sampling matrix

$$\mathcal{F}_X := \left[e^{ikx_\ell^1} \mathbf{e}^+ + e^{i\tilde{k}x_\ell^2} \mathbf{e}^- \right]_{k,\tilde{k}=1,\dots,d,\ \ell=1,\dots,N.}$$

We define the sign function in bicomplex case as

$$\operatorname{sgn} c := \left(\frac{c_k}{|c_k|} \mathbf{e}^+ + \frac{c_{\tilde{k}}}{|c_{\tilde{k}}|} \mathbf{e}^- \right).$$

We consider the following **necessary lemmas**:

Proof - necessary lemma

Lemma

Let f and g be two bicomplex functions. It holds

1. $|\mathrm{sgn} f|_{\mathbf{k}} = 1$;
2. $f\,\overline{\mathrm{sgn} f} = |f|_{\mathbf{k}}$;
3. $|g|_{\mathbf{k}}\,|\mathrm{sgn} f|_{\mathbf{k}} = |g\,\mathrm{sgn} f|_{\mathbf{k}}$.

Lemma

For a non-singular matrix $\mathcal{F}_{TX}^{*}\mathcal{F}_{TX}$ we have

$$(\mathcal{F}_{TX}^{*}\mathcal{F}_{TX})^{-1} = [\mathcal{F}_{TX}^{*}\mathcal{F}_{TX}]_1^{-1}\,\mathbf{e}^{+} + [\mathcal{F}_{TX}^{*}\mathcal{F}_{TX}]_2^{-1}\,\mathbf{e}^{-}.$$

Proof - necessary lemma

Lemma

Let $c_k \mathbf{e}^+ + c_{\tilde{k}} \mathbf{e}^- =: c \in \ell^2(D)$ and $T := \mathrm{supp} c$. Assume

$$\mathcal{F}_{TX} : \ell^2(T) \to \ell^2(X)$$

to be injective. Suppose that there exists a vector $P \in \ell^2(D)$ with the following properties:

(i) $P_{k\tilde{k}} = \mathrm{sgn}\ (c_{k\tilde{k}})_{k\tilde{k}}$ for all $(k, \tilde{k}) \in T$,

(ii) $|P_{k\tilde{k}}| < 1$ for all $(k, \tilde{k}) \notin T$,

(iii) there exists a $(\lambda_1, \lambda_2) \in \ell^2(X)$ such that
$P = \left([\mathcal{F}_X^*]_1 \mathbf{e}^+ + [\mathcal{F}_X^*]_2 \mathbf{e}^-\right)(\lambda_1 \mathbf{e}^+ + \lambda_2 \mathbf{e}^-)$
$= [\mathcal{F}_X^*]_1 \lambda_1 \mathbf{e}^+ + [\mathcal{F}_X^*]_2 \lambda_2 \mathbf{e}^-$.
Then c is unique minimizer to the problem (2).

Lemma

If $N \geq |T|$ then $\mathcal{F}_{TX}$ is injective almost surely.

Moreover, we introduce the restriction operator
$R_T : \ell^2([-q,q]^2) \to \ell^2(T)$, $R_T c_{k\tilde{k}} = c_{k\tilde{k}}$ for $(k,\tilde{k}) \in T$.

Its adjoint $R_T^* = E_T : \ell^2(T) \to \ell^2([-q,q]^2)$ is the operator that extends a vector outside T by zero, i.e., $(E_T d)_{k\tilde{k}} = d_{k\tilde{k}}$ for $(k,\tilde{k}) \in T$ and $(E_T d)_{k\tilde{k}} = 0$ otherwise.

Assuming that $\mathcal{F}_{TX}^* \mathcal{F}_{TX} : \ell^2(T) \to \ell^2(T)$ is invertible, we construct, P explicitly by

$$P := \mathcal{F}_X^* \mathcal{F}_{TX} (\mathcal{F}_{TX}^* \mathcal{F}_{TX})^{-1} R_T \mathrm{sgn}(c), \text{ with}$$

$$\mathcal{F}_X^* \mathcal{F}_{TX} := \mathbf{e}^+ \left[\mathcal{F}_X^* \mathcal{F}_{TX}\right]_1 + \mathbf{e}^- \left[\mathcal{F}_X^* \mathcal{F}_{TX}\right]_2 \text{ and}$$

$$\mathcal{F}_{TX}^* \mathcal{F}_{TX} := \mathbf{e}^+ \left[\mathcal{F}_{TX}^* \mathcal{F}_{TX}\right]_1 + \mathbf{e}^- \left[\mathcal{F}_{TX}^* \mathcal{F}_{TX}\right]_2.$$

Auxiliary operators and expectation value

We start with some auxiliary operators:
$H := \ell^2(T) \to \ell^2([-q,q]^2)$ and $H_0 := \ell^2(T) \to \ell^2(T)$,
$I := \mathbf{e}^+ I + \mathbf{e}^- I$ and $E := \mathbf{e}^+ E + \mathbf{e}^- E$.

$$H_0 = \mathbf{e}^+ \left(NI - [\mathcal{F}^*_{TX}\mathcal{F}_{TX}]_1\right) + \mathbf{e}^- \left(NI - [\mathcal{F}^*_{TX}\mathcal{F}_{TX}]_2\right),$$

$$H := \mathbf{e}^+ \left(NE - [\mathcal{F}^*_{X}\mathcal{F}_{TX}]_1\right) + \mathbf{e}^- \left(NE - [\mathcal{F}^*_{X}\mathcal{F}_{TX}]_2\right).$$

The expectation value of a bicomplex random variable function are represented as

$$\mathbb{E}\left[f(\zeta_1 + \mathbf{j}\zeta_2)\right] := \mathrm{Sc}\left[\int_{C_1} f_1(z)\,\frac{dz}{z}\mathbf{e}^+ + \int_{C_2} f_2(w)\,\frac{dw}{w}\mathbf{e}^-\right]$$
$$= \frac{1}{4\pi\mathbf{i}}\int_0^{2\pi} f_1\left(e^{\mathbf{i}x^1}\right)dx^1 + \frac{1}{4\pi\mathbf{i}}\int_0^{2\pi} f_2\left(e^{\mathbf{i}x^2}\right)dx^2.$$

Proof - probability results 1

The study of the powers of H leads to

$$\mathbb{E}_X[|||H_0^n|||]_F^2 = \sum_{k=1}^{\min\{n,N\}} \frac{N!}{(N-k)!} \sum_{\mathcal{A}\in P(2n,k)} \mathcal{C}(\mathcal{A},T),$$

where $\mathcal{C}(\mathcal{A},T)$ denotes

$$\sum_{\substack{k_1,k_2,\ldots,k_{2n}\in T \\ k_{r+1}^1\neq k_r^1,\, k_{s+1}^2\neq k_s^2}} \times \prod_{A\subset\mathcal{A}}\left[\delta\left(\sum_{r\in A}(k_{r+1}^1-k_r^1)\right)+\delta\left(\sum_{s\in A}(k_{s+1}^2-k_s^2)\right)\right].$$

Proof - probability results

Since $||(N^{-1}H_0)^n||_F \leq \kappa$ (< 1) means $\mathcal{F}_{TX}$ is injective

$$\begin{aligned} P(||(N^{-1}H_0)^n||_F \geq \kappa) &= P(||(H_0)^n||_F^2 \geq N^{2n}\kappa^2) \\ &\leq N^{-2n}\kappa^{-2n}\mathbb{E}_X[||(H_0)^n||_F^2] \end{aligned}$$

Using the argument that the probability of failure can be bounded by

- the probability of $\mathcal{F}_{TX}$ **not** being injective, and
- $\sup_{(k,\tilde{k})\in T^c}|P_{k\tilde{k}}| \geq 1$

we get the desired result.

- There is much similarity between complex and bicomplex signals; however, there is no equivalent to FFT up to now (unless one uses two independent complex FFTs).
- Here, we show that compressive sensing algorithms can be applied with a high probability of success.
- In this work we show that a fast reconstruction scheme can be obtained by compressive sensing principles.

Applications - Bicomplex

Figure: The original image - $N^2 = 262144$ pixels (512×512)

The computations were made on a computer with Intel(R) Core(TM) i7-4790U CPU 3.60 GHz, RAM 16 GB, Windows 8.1, OS 64-bit(win64), and running Matlab R2020b.

Applications - Bicomplex

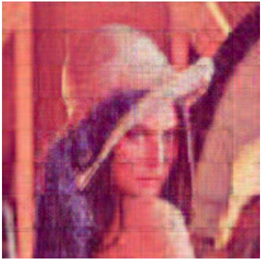


Figure: The reconstructed image with 15% of the total information

Figure: The reconstructed image with 30% of the total information

Applications - Bicomplex

Figure: The reconstructed image with 50% of the total information

Figure: The reconstructed image with 75% of the total information

Quaternionic Setting

Quaternionic Setting

- Compressed Sensing for Quaternionic Signals, Narciso Gomes, Uwe Kähler and Stefan Hartmann, Complex Analysis and Operator Theory. 11, 417-455(2017)
- https://link.springer.com/article/10.1007/s11785-016-0607-7

Quaternionic algebra

The quaternionic algebra is an extension of complex numbers to a 4D algebra. Every element of $\mathbb{H}$ is a linear combination of a real and three orthogonal imaginary units (denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$) with real coefficients

$$\mathbb{H} = \{q \ : \ q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where the elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ obey the Hamilton's multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

The vectorial part is denoted as $Vec(q) = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$. The scalar part is denoted as $Sc(q) = q_0$.

$\mathbb{H}$-valued function space

$\mathbb{H}$-**conjugation** of a given $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 = Sc(q) + Vec(q)$ is

$$\overline{q} = Sc(q) - Vec(q).$$

Consider the quaternion-valued left-Hilbert module $L_2(\mathbb{R}^2; \mathbb{H})$ equipped with the quaternionic-valued inner product

$$(f, g) := \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x}.$$

Additionally, consider also the real-valued inner product

$$\langle f, g \rangle := Sc(f, g) = \int_{\mathbb{R}^2} Sc[f(\mathbf{x})\overline{g(\mathbf{x})}]d\mathbf{x}.$$

Our setting and goal

Denote by $\prod_q$ the space of functions spanned by trigonometric quaternionic polynomials P of degree at most q .

Now, we consider the signal

$$P(x,y) = \sum_{(k,m)\in T} \left(e^{\mathbf{i}kx} e^{\mathbf{j}my}\right) c_{k,m},$$

with the sequence $(c_{k,m})$, $c_{k,m} \in \mathbb{H}$ having support on T and $(x,y) \in \mathbb{R}^2$.

Given the samples $P(x_j, y_j), j = 1, \ldots, N$, we want to reconstruct P by means of an ℓ_1-minimization process.

Main theorem

Theorem

Assume $f \in \prod_q$ *with some sparsity* $M \in \mathbb{N}$, *and consider* $X := \{x_1, x_2 \ldots, x_N\}$ *a sampling set of l.i.r.v. with uniform distribution on* $[0, 2\pi]^2$.
Choose $n \in \mathbb{N}$, $\beta > 0$, $\kappa > 0$ *and* $K_1, K_2, \ldots, K_n \in \mathbb{N}$ *s. t.*

$$a := \sum_{m=1}^{n} \beta^{n/K_m} < 1 \quad \text{and} \quad \frac{\kappa}{1-\kappa} \leq \frac{1-a}{1+a} M^{-3/2}.$$

For $\theta := N/M$ *we have that* f *can be reconstructed with probability at least*

$$1 - \left(\beta^{-2n} \sum_{m=1}^{n} G_{2mK_m}(\theta) + M\kappa^{-2} G_{2n}(\theta) \right)$$

from its sample values $f(x_1), \ldots, f(x_N)$ *by* ℓ^1*-minimization.*

How do we apply our fundamental lemma?

- Sampling operator

$$\mathcal{F}_X c(x_i, y_i) = \sum_{k,m \in D} e^{\mathbf{i} k x_i} e^{\mathbf{j} m y_i} c_{k,m}.$$

- Auxiliary operators:

$$H := \ell^2(T) \to \ell^2(D) \quad \text{and} \quad H_0 := \ell^2(T) \to \ell^2(T)$$
$$H := NE_T - \mathcal{F}_X^* \mathcal{F}_{TX} \quad \text{and} \quad H_0 = NI - \mathcal{F}_{TX}^* \mathcal{F}_{TX},$$

with

- $\mathcal{F}_{TX}$ the restriction of $\mathcal{F}_X$ supported only on T.
- $(E_T d)_k = d_k$ for $k \in T$ and $(E_T d)_k = 0$ for $k \notin T$.

Proof - probability results 1

The study of the powers of H_0 leads to

$$\mathbb{E}_X[|||H_0^n|||]_F^2 = \sum_{k=1}^{\min\{n,N\}} \frac{N!}{(N-k)!} \sum_{\mathcal{A}\in P(2n,k)} \mathcal{C}_{\mathbb{H}}(\mathcal{A}, T),$$

where $\mathcal{C}_{\mathbb{H}}(\mathcal{A}, T)$ is bounded by

$$\mathcal{C}_{\mathbb{H}}(\mathcal{A}, T) \leq |T|^{2n-k+1} \leq M^{2n-k+1}.$$

Proof - probability results

Since $||(N^{-1}H_0)^n||_F \leq \kappa$ (< 1) means $\mathcal{F}_{TX}$ is injective

$$\begin{aligned} P(||(N^{-1}H_0)^n||_F \geq \kappa) &= P(||(H_0)^n||_F^2 \geq N^{2n}\kappa^2) \\ &\leq N^{-2n}\kappa^{-2n}\mathbb{E}_X[||(H_0)^n||_F^2] \end{aligned}$$

Using the argument that the probability of failure can be bounded by

- the probability of $\mathcal{F}_{TX}$ **not** being injective, and
- $\sup_{(k,\tilde{k})\in T^c}|P_{k\tilde{k}}| \geq 1$

we get the desired result.

Applications - Quaternions

Figure: The original image

Figure: The reconstructed image.

The original and reconstructed images from Lena with $N^2 = 262144$ pixels (512×512). For the reconstruction we use 40000 pixels ($M = 625$ samples in each block) which corresponds to $\approx 15.26\%$ of the total information.

Applications - Quaternions

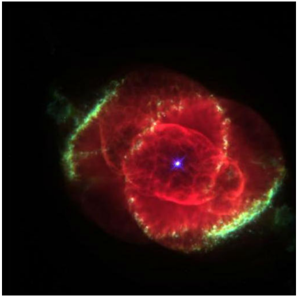


Figure: The original image

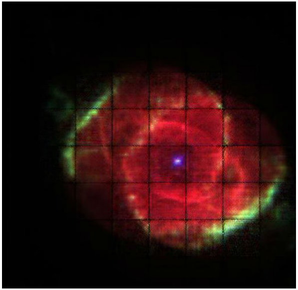


Figure: The reconstructed image.

The original and reconstructed images from Galaxy with $N^2 = 262144$ pixels (512×512). For the reconstruction we use 40000 pixels ($M = 625$ samples in each block) which corresponds to $\approx 15.26\%$ of the total information.

Applications - Quaternions

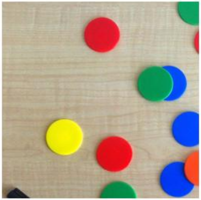


Figure: The original image

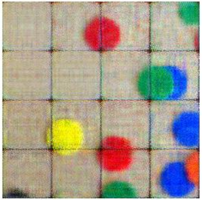


Figure: The reconstructed image.

The original and reconstructed images from Coloured chips with $N^2 = 65536$ pixels (256×256). For the reconstruction we use 10000 pixels ($M = 625$ samples in each block) which corresponds to $\approx 15.26\%$ of the total information.

Conclusion

Further challenges

- Extension to higher dimensions is more difficult since we loose the norm property
    - $\|X^n\| \leq \|X\|^n$
    - $\|XY\| \leq \|X\|\|Y\|$
    - $\|\lambda X\| \leq |\lambda| \, \|X\|$

Results:

- Here, we show that in the Bicomplex and Quaterninic settings compressive sensing algorithms can be applied with a high probability of success.

Further challenges:

- This approach can be extended to:
  - Compressed sensing principles and monogenic wavelets with applications to Deep Learning, $>$ Tufts University, School of Arts and Science, Department of Mathematics, MA

Thank you!

https://unicv.edu.cv/pt/