Summer School
"Singularities in Science and Engineering"
Ghent University, 22-31 Aug. 2022

THE GAME OF FRACTIONAL CALCULUS WITH SPECIAL FUNCTIONS

Virginia Kiryakova

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, Sofia

22 August 2022

In *Fractional Calculus* (FC), as in the (classical) *Calculus*, the notions of derivatives and integrals (of first, second, etc. or arbitrary, incl. non-integer order) are basic and co-related.
One of the most frequent approach in FC is to define first the integral of fractional order - say the Riemann-Liouville (R-L), and then by means of suitable integer-order differentiation operation, applied before or under its sign, a fractional derivative is defined - in the R-L sense, or in Caputo sense). The first mentioned FD of R-L type is closer to the theoretical studies in analysis, but has some shortages - from the point of view of interpretation of the initial conditions for Cauchy problems for fractional DEs, and also for the analysts' confusion that such a derivative of a constant is not zero in general. The Caputo (C-) FD (better to name Dzrbashjan-Caputo), helps to overcome these problems and to describe models of applied problems with physically consistent initial conditions.

In this lecture I try to give a short survey on a *Generalized Fractional Calculus* (GFC), as theory of a FC **based on the use of Special Functions (SF)**. It was presented in details in my (old) book:

V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman-J. Wiley, Harlow-N. York, 1994; (with first papers of 1986, etc. → Recent years works).

The notion of "generalized operators of fractional integration (and differentiation)" goes back to the ideas and works of Shyam L. Kalla (1970-1980), who allowed to have a kernel as some (arbitrary) SF. From these ideas, I was motivated to develop a detailed strong theory of GFC with many applications in different areas of Analysis: Integral Transforms, Operational Calculus, Special Functions, Differential Equations, Analytic Functions), etc.

However recently, the notion of "generalized fractional integral and derivatives" has been exploited also for other kinds of operators in FC, having in mind "generalizations" in various different aspects. Most popular now is the GFC in sense of Kochubei (2011) - Luchko (2021) et al., based on the so-called pairs of Sonine kernels, that also satisfy the fundamental theorems of the Calculus, and allow to describe non-locaility in a general form. (Refer to next LECTURES by Y. Luchko!, and very recent article in *Mathematics*, V. Tarasov on GFC ..., Vol.9 (2021)).

Therefore, it is necessary to specify what kind of "generalization" I mean. When I speak about GFC, need to emphasize that it is based on operators of fractional integration/ differentiation which involve in their kernels some SF: namely, specific classes of generalized hypergeometric functions. For our purposes, these SF are chosen to have singularities at the limits of integration! As shown, the singularity is indeed important so such a GFC theory can be build to satisfy the basic axioms of (classical) FC, to have enough reasonable long list of operational rules and particular cases, and to provide the mentioned wide scope of applications.

The core of this GFC is that these generalized integrals and derivatives of fractional multi-order are based on commuting m-tuple ($m = 1, 2, 3, \ldots$) compositions of operators of the classical FC with power weights (the so-called Erdélyi-Kober operators), but are represented in compact and explicit form by means of integral, integro-differential (R-L type) or differential-integral (C-type) operators, where *the kernels are special functions* of most general hypergeometric kind (Meijer G- and Fox H-functions). The use of the SF tools essentially simplifies the studies and the use of these GFC operators (instead of hardly to treat repeated integrals).

As particular cases of these GFC operators, they appear the known operators of classical FC, their generalizations introduced by other authors, the hyper-Bessel differential operators of higher integer order *m* as a multi-order $(1, 1, \dots, 1)$, the Gelfond-Leontiev generalized differentiation operators, many other integral and differential operators in Calculus that have been used in various topics, some of them not related to FC at all, or for fractional order models and differential equations.

1. Introduction to classical fractional calculus

Since 1695, for 327 years' period many known analysts and applied scientists contributed to the development of the "strange" Calculus where differentiations and integrations can be taken from arbitrary, including fractional (*non-integer*) orders, called as Fractional Calculus (*FC*). Nowadays, there are 48 years from two remarkable events in 1974: the appearance of the first book and the organizing the first conference dedicated specially to the topics of FC.

More details about the development and perspective of FC and its applications will be discussed in next coming Public Lecture.

The classical FC is based on several (equivalent or alternative) definitions for the operators of integration and differentiation of arbitrary (incl. real fractional or complex) order, for example, as a continuation of the classical integration and differentiation of integer order $n \in \mathbb{N}$: the n-fold integration

$$R^n f(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-2}} d\tau_{n-1} \int_0^{\tau_{n-1}} f(\tau_n) d\tau_n$$
$$= \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \qquad (1)$$

and the n-th order derivative $D^n f(t) = f^{(n)}(t)$.

The *Riemann-Liouville (R-L) integration of arbitrary order* $\delta > 0$ is defined by analogy with (1), replacing $(n-1)!$ by $\Gamma(\delta)$:

$$R^\delta f(t) = D^{-\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} f(\tau) d\tau = t^\delta \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(t\sigma) d\sigma, \qquad (2)$$

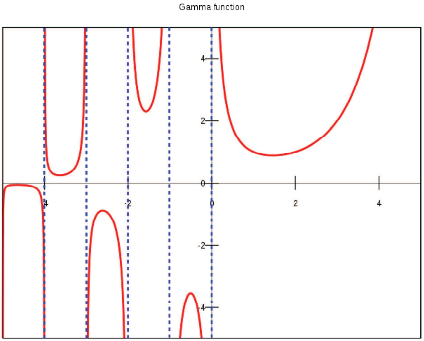
while $R^0 f(t) \equiv f(t)$ is the identity operator, and the *semigroup property* is satisfied:

$$R^{\delta_1} R^{\delta_2} = R^{\delta_2} R^{\delta_1} = R^{\delta_1+\delta_2}, \quad \text{for } \delta_1 \geq 0, \delta_2 \geq 0.$$

May be here is time to pay tribute to one of the basic functions in FC: the Euler Gamma function that generalizes the factorial $n!$ and allows n to take also non-integer and even complex values. Just to mention few of its well known properties, as

$$\Gamma(\sigma) = \int_0^\infty e^{-t} t^{\sigma-1} dt; \;\; \Gamma(\sigma+1) = \sigma\,\Gamma(\sigma); \;\; \Gamma(n+1) = n!\,.$$

Speaking about singularities, note that it has simple poles at the points $\sigma = -n, \;\; n = 0, 1, 2, ...$, see the plot.



B.T.W., there is an interesting old reading for the role of Γ-function, related to FC, as:

B. Ross, Serendipity in Mathematics or How One is Led to Discover that ..., *The American Mathematical Monthly*, Vol. 90, No 8 (Oct. 1983), 562-566; https://doi.org/10.2307/2322795 (now available at some cites).

SERENDIPITY IN MATHEMATICS
OR
HOW ONE IS LED TO DISCOVER THAT

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n2^n n!} = \frac{1}{2} + \frac{3}{16} + \frac{15}{144} + \cdots = \ln 4$$

BERTRAM ROSS
Department of Mathematics, University of New Haven, West Haven, CT 06516

This paper chronicles the course of a mathematical discovery. It is usually the case in (alleged) ccidental discovery that the starting point is far afield from the particular discovery. Our starting oint is the evaluation of the integral

1) $$\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\ln t\,dt, \quad \mathrm{Re}(\nu) \geqslant 0.$$

An integral of the form $\int_c^x (x-t)^{\nu-1} f(t)\,dt/\Gamma(\nu)$ is known as the Riemann-Liouville integral vhich is the cornerstone of the *fractional calculus*. More precisely stated, it is the integral which lefines integration and differentiation to an arbitrary order. The R-L integral can be denoted by $D_x^{-\nu} f(x)$, a notation devised by Harold T. Davis [**1**]. The subscripts on D denote the terminals of ntegration. The order of differintegration [**2**] can be rational, irrational or complex. When $\nu = n$, n integer, then ${}_0D_x^{-\nu} f(x)$ and ${}_0D_x^{\nu} f(x)$ are respectively ordinary integration and differentiation.

The idea of differentiation to an arbitrary order started in 1695 when L'Hôpital asked Leibniz vhat would happen with $d^n y/dx^n$ if $n = 1/2$. Hence, the topic started with the misnomer ractional calculus [**3**].

To return to R-L fractional integral:

This definition concerns integrations of (real part) nonnegative orders $\delta \geq 0$, but could not be used directly for the inverse operation, a differentiation as $D^{\delta} f(t) := R^{-\delta} f(t)$, $-\delta < 0$. However, a little trick is helpful for a suitable interpretation to avoid divergent integrals.

For noninteger $\delta > 0$ we take $n := [\delta] + 1$ (the smallest integer greater than δ), then we can define properly the two popular kinds of fractional order derivatives: by means of compositions of differentiation of integer order n and integration of nonnegative fractional order $n - \delta \geq 0$:

the *R-L fractional derivative* - by the differ-integral expression

$$D^{\delta} f(t) := D^n D^{\delta-n} f(t) = \left(\frac{d}{dt}\right)^n R^{n-\delta} f(t)$$
$$= \left(\frac{d}{dt}\right)^n \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\tau)^{n-\delta-1} f(\tau) d\tau \right\}, \quad (3)$$

and

the *Caputo* (Djrbashjan) *fractional derivative* - by means of the integro-differential expression

$${}^*D^{\delta} f(t) := D^{\delta-n} D^n f(t) = R^{n-\delta} f^{(n)}(t)$$
$$= \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\tau)^{n-\delta-1} f^{(n)}(\tau) d\tau \right\}. \quad (4)$$

In suitable functional spaces, both FDs are left inverse to the R-L integral:

$$D^{\delta} R^{\delta} f(t) = {}^*D^{\delta} R^{\delta} f(t) = f(t).$$

From the formula

$$D^{\delta}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\delta)}\, t^{\alpha-\delta}, \quad \delta > 0,\ \alpha > -1,$$

it comes the interesting situation, going in *conflict* with the classical Calculus: for $\alpha = 0$, the R-L fractional derivative of a constant is zero *only* for positive integer values $\delta = n = 1, 2, 3, \dots$, but not for arbitrary $\delta \notin \mathbb{N}_+$:

$$D^{\delta}\{c\} = c\,\frac{t^{-\delta}}{\Gamma(1-\delta)}, \quad \text{while Caputo's der.:}\ \ {}^{*}D^{\delta}\{c\} = 0,\ \text{always.}$$

The R-L definition is preferred for the theoretical developments and their applications in pure mathematics, but the Caputo derivative is more suitable for mathematical models of real phenomena also for more important reasons: to be able to consider problems where the initial conditions are given by limit values of integer order derivatives at the lower terminal ($t = 0$), instead of fractional order integrals or derivatives, that can hardly be interpreted physically.

For example, the Cauchy problem for a simple fractional order differential equation *with R-L derivative* has the form

$$\begin{cases} D^{\delta} y(t) - \lambda y(t) = f(t)\ , \quad t > 0, \\ D^{\delta-j} y(t)|_{t=0} = b_j\ ,\ j = 1, 2, ..., n, \end{cases} \qquad n-1 < \delta \leq n,$$

and has its solution in terms of Mittag-Leffler function (Examples 42.1, 42.2 in Samko-Kilbas-Marichev book):

$$y(t) = \sum_{j=1}^{n} b_j\ t^{\delta-j} E_{\delta,1+\delta-j}(\lambda t^{\delta}) + \int_0^t (t-\tau)^{\delta-1}\ E_{\delta,\delta}[\lambda(t-\tau)^{\delta}]\ f(\tau) d\tau.$$

And for the same differential equation *with Caputo fractional derivative*, the IVP is stated in more comprehensive form

$$\begin{cases} {}^{*}D^{\delta} y(t) - \lambda y(t) = f(t)\ , \quad t > 0, \\ y^{i-1}(0) = b_i\ ,\ i = 1, 2, ..., n, \end{cases} \qquad n-1 < \delta \leq n.$$

Its solution, again in term of M-L functions, have same structure (Podlubny; Srivastava-Kilbas-Trujillo) but with different parameters

$$y(t) = \sum_{i=1}^{n} c_i\ t^{i-1} E_{\delta,i}(\lambda t^{\delta}) + \int_0^t (t-\tau)^{\delta-1}\ E_{\delta,\delta}[\lambda(t-\tau)^{\delta}]\ f(\tau) d\tau.$$

In both above solutions, the *Mittag-Leffler functions* take part.

To underline the difference in the form of the initial conditions which accompany FDEs in terms of R-L and C- derivatives, recall also the corresponding Laplace transform formulas:

The *Laplace transform of the Caputo derivative* is:

$$\mathcal{L}\{{}^*D^\delta f(t); s\} = \int_0^\infty \exp(-st)\, {}^*D^\delta f(t) dt$$

$$= s^\delta \mathcal{L}\{f(t); s\} - \sum_{k=0}^{n-1} s^{\delta-k-1} f^{(k)}(+0), \quad n-1 < \delta < n,$$

and allows utilization of initial values of classical integer-order derivatives with known physical meanings. While the *Laplace transform of the R-L derivative* is:

$$\mathcal{L}\{D^\delta f(t); s\} = \int_0^\infty \exp(-st)\, D^\delta f(t) dt$$

$$= s^\delta \mathcal{L}\{f(t); s\} - \sum_{k=0}^{n-1} s^k D^{(\delta-k-1)}(t)|_{t=0}, \quad n-1 < \delta < n,$$

and the initial conditions cause problems with their interpretation.

Along with the R-L definition of the operator of fractional integration, several modifications and their generalizations are also widely used. The most useful classical fractional integrals however are the *Erdélyi-Kober (E-K) operators*, allowing wider applications with the freedom of choice of 3 parameters:

$$I_{\beta}^{\gamma,\delta} f(t) = t^{-\beta(\gamma+\delta)} \int_0^t \frac{(t^{\beta} - \tau^{\beta})^{\delta-1}}{\Gamma(\delta)} \tau^{\beta\gamma} f(\tau)\, d(\tau^{\beta})$$

$$= \int_0^1 \frac{(1-\sigma)^{\delta-1}\,\sigma^{\gamma}}{\Gamma(\delta)} f(t\sigma^{1/\beta})\, d\sigma, \quad \delta \geq 0, \gamma \in \mathbb{R}, \beta > 0, \qquad (5)$$

are *used essentially in our works on generalized FC*.

See e.g.: I.N. Sneddon, The use in mathematical analysis of E-K operators and some of their applications. In: *"Intern. Conf. FC and its Appl., 1974" (New Haven)* = *L.N.M.* **457** (1975).

For our GFC, we consider *compositions of commutable E-K operators* but written *in form of single integrals involving special functions*, instead of by repeated integrals. And call them generalized fractional integrals (multiple E-K integrals), then introduce the corresponding generalized fractional derivative of R-L and C-type.

2. Introduction to the generalized fractional calculus (GFC)

Several authors, mainly in 60's-70's, studied and used different modifications of the R-L fractional integral, replacing the singular kernel $(t-\tau)^{\delta-1}/\Gamma(\delta)$ (at t) or $(1-\sigma)^{\delta-1}/\Gamma(\delta)$ (at upper limit 1), by some special functions (of arguments τ/t or $(t-\tau)$, if integrals are from 0 to t).

The so-called *hypergeometric operators of fractional integration*

$$\mathcal{H}f(t) = \frac{\mu t^{-\gamma-1}}{\Gamma(1-\delta)} \int_0^t {}_2F_1\left(\delta, \beta+m;\ \eta; a\left(\frac{\tau}{t}\right)^{\mu}\right) \tau^{\gamma} f(\tau) d\tau, \quad (6)$$

are defined by means of the Gauss hypergeometric function in the kernel. See works by Love (1967), Saxena, Kalla and Saxena, Saigo, McBride, also Tricomi, Sprinkhuizen-Kuiper, Koornwinder.

Example of *fractional integration operators involving other special functions* (Bessel), is given by Lowndes (1985):

$$I_\lambda(\eta,\nu+1)f(t) = c\, t^{-(\nu+\eta+1)} \int_0^t \tau^{2\eta+1}(t^2-\tau^2)^{\frac{\nu}{2}} J_\nu(\lambda\sqrt{t^2-\tau^2}) f(\tau) d\tau, \quad (7)$$

related to the second order Bessel type differential operator $B_\eta = t^{-2\eta-1}(d/dt)t^{2\eta+1}(d/dz)$.

One of the most general fractional integration operators of R-L type can be obtained when the *kernel-function is an arbitrary Meijer G-function*, as in Kalla (1970), also in Parashar, Rooney:

$$\mathcal{I}_G f(t) = t^{-\gamma-1} \int_0^t G_{p,q}^{m,n} \left[a(\frac{\tau}{t})^r \middle| \begin{array}{c} (a_j)_1^p \\ (b_k)_1^q \end{array} \right] t^{\gamma} f(\tau) d\tau, \qquad (8)$$

or its further generalization, the *Fox H-function*, as in Kalla (1969), also in Srivastava and Buschman (1973), and others:

$$\mathcal{I}_H f(t) = t^{-\gamma-1} \int_0^t H_{p,q}^{m,n} \left[a(\frac{\tau}{t})^r \middle| \begin{array}{c} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{array} \right] \tau^{\gamma} f(\tau) d\tau. \qquad (9)$$

However, in his papers of years 1970-1979, Kalla suggested that all the above operators of R-L type can be considered as "*generalized operators of fractional integration*" of the more general form:

$$\mathcal{I} f(t) = t^{-\gamma-1} \int_0^t \Phi(\frac{\tau}{t}) \tau^{\gamma} f(\tau) d\tau = \int_0^1 \Phi(\sigma) \sigma^{\gamma} f(t\sigma) d\sigma, \qquad (10)$$

where the kernel $\Phi(\sigma)$ can be an arbitrary continuous function s.t. the integral makes sense in some functional spaces.

At that time, Kalla established some general properties of (11), analogous to those of the classical fractional integrals, and studied some special cases. By suitable choices of the kernel-function Φ, operators (11) can be shown to include the other known fractional integrals (classical or generalized) as particular cases.

See e.g., S.L. Kalla, Operators of fractional integration, In: *Proc. Complex Analysis and Appls, Kozubnik 1979 = L.N.M.* **798** (1980), 258-280; doi: 10.1007/bfb0097270.

However, taking an arbitrary G- or H-function in the kernel of (11) did not allow to develop some detailed theory of a generalized fractional calculus and to think about possible applications. Indeed, a very particular, or a very general choice of the kernel special function, prevented the other authors to develop further their operators' theory and the results (publications containing formal manipulations only) were *just to announce possibilities to use the SF in the contentext of FC*. It happened that the kernel-functions (even if too general) should be chosen suitably (and with necessary singularities) so to allow a detailed GFC theory and with applications.

Here we mention in brief the basic definitions of the mentioned two generalized hypergeometric functions, G- and H-, and will focus again on their properties and numerous particular cases in the next Lecture 2 (Friday).

The *Fox's H-function* is the generalized hypergeometric function

$$H^{m,n}_{p,q}(\sigma) = H^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{array}{c}(a_1, A_1), \dots, (a_p, A_p)\\ (b_1, B_1), \dots, (b_q, B_q)\end{array}\right] \\ = H^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{array}{c}(a_j, A_j)_1^p\\ (b_k, B_k)_1^q\end{array}\right] = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q}(s)\sigma^s ds, \tag{11}$$

where the integrand (i.e. the Mellin transform) has the form

$$\mathcal{H}^{m,n}_{p,q}(s) = \frac{\prod\limits_{k=1}^{m}\Gamma(b_k - B_k s)\prod\limits_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod\limits_{k=m+1}^{q}\Gamma(1 - b_k + B_k s)\prod\limits_{j=n+1}^{p}\Gamma(a_j - A_j s)},$$

$\mathcal{L}$ is a suitable contour in $\mathbb{C}$; the orders $(m, n; p, q)$ are integers s.t. $0 \leq m \leq q$, $0 \leq n \leq q$; the parameters $A_j, j = 1, ..., p$ and $B_k, k = 1, ..., q$ are *positive* and $a_j, j = 1, ..., p$, $b_k, k = 1, ..., q$ are complex numbers s.t. $A_j(b_k + l) \neq B_k(a_j - l' - 1)$; l, $l' = 0, 1, 2, ...$; $j = 1, ..., p$, $k = 1, ..., q$.

In particular, when all $A_j = B_k = 1$, we obtain the *Meijer's G-function* (Bateman-Erdélyi, HTF, Vol.1):

$$H^{m,n}_{p,q}\left[\sigma \left| \begin{array}{c} (a_j,1)_1^p \\ (b_k,1)_1^q \end{array} \right.\right] = G^{m,n}_{p,q}\left[\sigma \left| \begin{array}{c} (a_j)_1^p \\ (b_k)_1^q \end{array} \right.\right], \qquad (12)$$

that is,

$$G^{m,n}_{p,q}\left[\sigma \left| \begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \right.\right]$$

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{k=1}^{m}\Gamma(b_k - s)\ \prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+s)\ \prod_{j=n+1}^{p}\Gamma(a_j-s)}\ \sigma^s ds.$$

The lucky hint for myself was to make a very proper choice of peculiar Meijer's G-functions or Fox's H-functions as kernel-functions $\Phi(\sigma)$ in (11), namely of the type

$$\Phi(\sigma) = G^{m,n}_{n,m+n}[\sigma],\ \Phi(\sigma) = H^{m,n}_{n,m+n}[\sigma],\ m = 1,2,3,...;\ n = 0,1,2,3,...,$$

with *singularities* at the low and upper limits of integration.

Next, for simplicity, we consider our *generalized operators of fractional integration* of R-L type, by such kernel-functions G or H-, when $n=0$: namely, $G^{m,0}_{m,m}$ and $H^{m,0}_{m,m}$. For such a choice, a full theory (GFC) has been developed and various applications to different areas of analysis, differential equations, problems of mathematical physics, etc. have been demonstrated.

The wide applicability of this GFC theory is hidden in the fact that *our operators (with above mentioned SF) happen to be compositions of finite number of commutable (classical) E-K operators*:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) = \prod_{k=1}^{m} I^{\gamma_k,\delta_k}_{\beta_k} f(t) \tag{13}$$

$$= \int_0^1 \dots \int_0^1 \frac{(1-\sigma_1)^{\delta_1-1} \dots (1-\sigma_m)^{\delta_m-1}\, \sigma_1^{\gamma_1} \dots \sigma_m^{\gamma_m}}{\Gamma(\delta_1) \dots \Gamma(\delta_m)} f(t\sigma_1^{1/\beta_1} \dots \sigma_m^{1/\beta_m})\, d\sigma_1 \dots d\sigma_m.$$

But their operational rules could be easier derived by using the special functions theory (G- and H-functions), with simple denotations and expressions.

3. Generalized fractional calculus (GFC)

Definition 3.1. Let $m \geq 1$ be integer, $\delta_1 \geq 0, ..., \delta_m \geq 0$, $\gamma_1, ..., \gamma_m$ and $\beta_1 > 0, ..., \beta_m > 0$ be arbitrary real numbers. By a *generalized (multiple, m-tuple) Erdélyi-Kober (E-K) operator of integration of multi-order* $\delta = (\delta_1, ..., \delta_m)$ we mean an integral operator of the form

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(t) = \int_0^1 G_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\gamma_k)_1^m \end{matrix} \right] f(t\sigma) d\sigma, \text{ if } \forall\, \beta_k := \beta > 0, \tag{14}$$

or in the more general case,

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t) = \int_0^1 H_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right] f(t\sigma) d\sigma, \tag{15}$$

where $m = 1, 2, 3, ...$; and the vector indices (multi-indices)

$$\delta = (\delta_1, \delta_2, ..., \delta_m), \ \gamma = (\gamma_1, \gamma_2, ..., \gamma_m), \ \beta = (\beta_1, \beta_2, ..., \beta_m)$$

play the role resp. of the *fractional multi-order*, multi-weight and additional multi-index. Operator (14) is the simpler case of (15) when the H-function reduces to a G-function.

Then, each operator of the form

$$\mathcal{I}f(t) = t^{\beta\delta_0}\, I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t), \quad \text{with arbitrary} \;\; \delta_0 \geq 0,$$

is called a generalized operator of fractional integration of R-L type, or briefly: a *generalized (R-L) fractional integral*.

Basic operational rules in GFC

Functional spaces: for example, the space C_α of power-weighted continuous functions of the form

$$C_\alpha^{(k)} := \left\{ f(t) = t^p \widetilde{f}(t),\; p > \alpha,\; \widetilde{f} \in C^{(k)}[0,\infty) \right\}, \quad C_\alpha^{(0)} := C_\alpha, \tag{16}$$

with real α; also – of the Lebesgue integrable or analytic functions with power weights, $L_{\alpha,p}(0,\infty)$; and resp. $H_\alpha(\Omega)$, Ω being a starlike domain in $\mathbb{C}$ containing the zero point. In general, we need that *the following parameters' conditions* are satisfied:

$$\gamma_k \geq -\frac{\alpha}{\beta_k} - 1, \quad \delta_k \geq 0, \quad \beta_k > 0, \quad k = 1, ..., m. \tag{17}$$

The first *basic result* for the generalized fractional integrals (15) and (14) *suggests their alternative name* as "multiple (m-tuple)" E-K fractional integrals.

Proposition. (Composition/Decomposition theorem) *Under the conditions* (17), *the classical E-K fractional integrals* (5)*:* $I_{\beta_k}^{\gamma_k,\delta_k}$, $k = 1, ..., m$, *commute in the spaces* C_α, $L_{\alpha,p}$, H_α, *and their product* (13) *can be represented as an m-tuple E-K operator* (15), *i.e. by means of a single integral involving H-function (resp. a G-function in the simpler case).*

Conversely, under the same conditions, each multiple E-K operator of form (15) *can be represented as a product* (13).

To prove the next operational rules we use the single integral representation (14)-(15), the tools of G- and H-functions and of the Mellin transform, instead of dealing with boring and lengthy repeated integrations (and differentiations).

Lemma. *The multiple E-K fractional integral* (15) *preserves the power functions in* C_α, *with* $\alpha \geq \max_k [-\beta_k(\gamma_k+1)]$ (*this means* (17) *holds*), *up to a constant multiplier:*

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \{t^p\} = c_p t^p,\ p > \alpha, \ \ \textit{where}\ \ c_p = \prod_{k=1}^{m} \frac{\Gamma(\gamma_k + \frac{p}{\beta_k} + 1)}{\Gamma(\gamma_k + \delta_k + \frac{p}{\beta_k} + 1)}\,,$$

and it is an invertible mapping $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} : C_\alpha \mapsto C_\alpha^{(\eta_1+\cdots+\eta_m)} \subset C_\alpha$.

Analogously, under the same conditions, (15) maps the class $H_\alpha(\Omega)$ into itself, preserving the power functions (up to constant multipliers like above) and the image of a power series has the same radius of convergence.

It is also shown that (15) has a Mellin type convolutional representation, based on its Mellin image. Another well expected result, say for $L_{\alpha,p}(0,\infty)$ is the following.

Lemma. *Under conditions* (17) *the generalized fractional integral* $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t)$ *exists almost everywhere on* $(0, \infty)$ *and it is a bounded linear operator from the Banach space* $L_{\alpha,p}$ *into itself. More exactly,*

$$\left\| I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f \right\|_{\alpha,p} \leq h_{\alpha,p} \; \|f\|_{\alpha,p}, \qquad \textit{i.e.} \quad \left\| I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f \right\| \leq h_{\alpha,p}$$

with $h_{\alpha,p} = \prod_{k=1}^{m} \Gamma(\gamma_k - \frac{\alpha}{p\beta_k} + 1) \; / \; \Gamma(\gamma_k + \delta_k - \frac{\alpha}{p\beta_k} + 1) < \infty.$

Proposition. *Suppose conditions* (17) *hold. Then, in* C_α, $L_{\alpha,p}$, H_α, *the following basic operational rules hold, confirming that the operators of our GFC satisfy the axioms of FC:*

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \{\lambda f(ct) + \eta g(ct)\} = \lambda \left\{ I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f \right\} (ct) + \eta \left\{ I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} g \right\} (ct)$$

(*bilinearity of* (15));

$$I^{(\gamma_1,\dots,\gamma_s,\gamma_{s+1},\dots,\gamma_m),(0,\dots,0,\delta_{s+1},\dots,\delta_m)}_{(\beta_1,\dots,\beta_m),m} f(t) = I^{(\gamma_{s+1},\dots,\gamma_m)(\delta_{s+1},\dots,\delta_m)}_{(\beta_{s+1},\dots,\beta_m),m-s} f(t)$$

(*i.e., if* $\delta_1 = \delta_2 = \cdots = \delta_s = 0$, *then the multiplicity reduces to* $(m-s)$);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} z^\lambda f(t) = t^\lambda I^{(\gamma_k+\frac{\lambda}{\beta_k}),(\delta_k)}_{(\beta_k),m} f(t), \quad \lambda \in \mathbb{R}$$

(*generalized commutability with power functions*);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n} f(t) = I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t)$$

(*commutability of operators of form* (15));

$$\text{the left-hand side of above} = I^{((\gamma_k)_1^m,(\tau_j)_1^n)((\delta_k)_1^m,(\alpha_j)_1^n)}_{((\beta_k)_1^m,(\varepsilon_j)_1^n),m+n} f(t)$$

(*compositions of m-tuple and n-tuple integrals* (15) *give* $(m+n)$-*tuple integrals of same form*);

$$I^{(\gamma_k+\delta_k),(\sigma_k)}_{(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) = I^{(\gamma_k),(\sigma_k+\delta_k)}_{(\beta_k),m} f(t), \quad \text{if } \delta_k > 0,\ \sigma_k > 0,\ k = 1, \dots, m$$

(*law of indices, product rule or semigroup property*);

$$\left\{ I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \right\}^{-1} f(t) = I_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)} f(t)$$

(*formal inversion formula*).

The above inversion formula follows from the previous index law for $\sigma_k = -\delta_k < 0,\ k = 1, ..., m$ and the definition for zero multi-order of integration, since:

$$I_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t) = I_{\beta_k,m}^{(\gamma_k),(0,...,0)} f(t) = f(t).$$

But the symbols (15) are not yet defined for negative multi-orders of integration $-\delta_k < 0$, $k = 1, ..., m$. The problem is to propose an appropriate meaning for them and so to avoid the appearance of divergent integrals. The situation is the same as in the classical case when the R-L and E-K operators of fractional order $\delta > 0$ are inverted by using an additional differentiation of suitable integer order $\eta = [\delta] + 1$, thus defining the R-L or Caputo fractional derivatives. In GFC, the problem is resolved by means of auxiliary differential operator D_η, polynomial of $t\dfrac{d}{dt}$, using a set of integers $\eta = (\eta_1, ..., \eta_m)$ related to the multi-order $\delta = (\delta_1, ..., \delta_m)$.

Generalized fractional derivatives (GFD)

Definition. Assume the same parameters, and same conditions (17) hold. Taking the integers

$$\eta_k = \begin{cases} \delta_k & \text{if } \delta_k \text{ is integer,} \\ [\delta_k]+1, & \text{if } \delta_k \text{ is noninteger,} \end{cases} \qquad k = 1, \ldots, m, \qquad (18)$$

we introduce the auxiliary differential operator

$$D_\eta = \left[\prod_{r=1}^{m} \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta_r} t \frac{d}{dt} + \gamma_r + j \right) \right]. \qquad (19)$$

Then, we define the *multiple (m-tuple) Erdélyi-Kober fractional derivative* of multi-order $\delta = (\delta_1 \geq 0, \ldots, \delta_m \geq 0)$ and of *R-L type* by means of the differ-integral operator:

$$D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) = D_\eta \; I^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}_{(\beta_k),m} f(t) \qquad (20)$$

$$= D_\eta \int_0^1 H^{m,0}_{m,m} \left[\sigma \left| \begin{array}{c} (\gamma_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{array} \right. \right] f(t\sigma) \, d\sigma.$$

For this definition, we rely on a differential property we derived for the H-function.

In the case of equal β_k's, we can use a simpler representation involving the Meijer G-function, corresponding to generalized fractional integral (14):

$$D_{\beta,m}^{(\gamma_k),(\delta_k)} f(t) = D_\eta \; I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t) \tag{21}$$

$$= \left[\prod_{r=1}^{m} \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta} t \frac{d}{dt} + \gamma_r + j \right) \right] I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t).$$

More generally, all differ-integral operators of the form

$$\mathcal{D} f(t) = D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} t^{-\beta\delta_0} f(t) = t^{-\beta\delta_0} D_{(\beta_k),m}^{(\gamma_k-\delta_0),(\delta_k)} f(t) \;\; \text{with} \;\; \delta_0 \geq 0,$$

are called *generalized (multiple, multi-order) fractional derivatives*, of R-L type.

Recently, in a joint paper with Luchko (CEJP, 2013), we have introduced also the *Caputo type generalized fractional derivative*, as the integro-differential operator

$$
{}^{*}D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) := I^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}_{(\beta_k),m} \, D_\eta f(t)
$$

$$
= \int_0^1 H^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{matrix}\right] \left[\prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta_r} t\frac{d}{dt}+\gamma_r+j\right) f(t\sigma)\right] d\sigma, \tag{22}
$$

with the same parameters as in the previous definition, but the order of the auxiliary differential operator D_η *is interchanged* with the multiple E-K fractional integration.

In the space $C_\alpha^{(\eta_1+\ldots+\eta_m)}$, both R-L type and Caputo-type generalized fractional derivatives (20), (22) exist. Also, both they are left-inverse operators to the generalized (multiple E-K) fractional integral (15) in C_α, namely:

Proposition. *For* $f \in C_\alpha, \ \alpha \geq \max_k[-\beta_k(\gamma_k+1)]$,

$$
D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \, I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) = {}_{*}D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \, I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \, f(t) = f(t). \tag{23}
$$

However, like in the classical Calculus, the fractional derivative is NOT in general a right-inverse operator to the fractional integral unless some kind of initial conditions are all equal to zero. The difference between the identity operator I and the composition of an operator and its left-inverse operator is usually called a *projector of the operator*, or its *operator of the initial conditions*.

The two kind of projectors, related to the R-L or C-type generalized fractional derivatives have the following forms:

$$Ff(t) := f(t) - I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \, D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \, f(t) = \sum_{k=1}^{m} \sum_{j=1}^{\eta_k} A_{k,j} \, t^{-\beta_k(\gamma_k+j)} \,, \tag{24}$$

and

$$_*Ff(t) := f(t) - I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \, {}_*D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \, f(t) = \sum_{k=1}^{m} \sum_{j=1}^{\eta_k} C_{k,j} \, t^{-\beta_k(\gamma_k+j)}, \tag{25}$$

where the coefficients $A_{k,j}$ and $C_{k,j}$ are depending on the initial conditions (at $t = 0+$) for the fractional differ-integrals of $f(t)$, or resp. only integer-order derivatives of $f(t)$ in the second case, as follows:

$$A_{k,j} = \prod_{k=1}^{m} \frac{\Gamma(\eta_k + 1 - j)}{\Gamma(\delta_k + 1 - j)} \times \tag{26}$$

$$\lim_{t\to 0} \left[t^{\beta_k(\gamma_k+j)} \prod_{i=j+1}^{\eta_k - 1} \left(\frac{1}{\beta_k} t \frac{d}{dt} + \gamma_k + i \right) I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t) \right],$$

$$C_{k,j} = \lim_{t\to 0} \left[t^{\beta_k(\gamma_k+j)} \prod_{i=j+1}^{\eta_k - 1} \left(\frac{1}{\beta_k} t \frac{d}{dt} + \gamma_k + i \right) f(t) \right]. \tag{27}$$

Remark. In the case of *integer multi-order of differentiation*, i.e. if $\forall\, \delta_k = \eta_k$, *the generalized R-L and Caputo type fractional derivatives coincide and both are the differential operators* D_η of integer order $\eta = \eta_1 + ... + \eta_m$, namely:

$$_*D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{(\beta_k),m}^{(\gamma_k),(\eta_k)} = D_\eta, \tag{28}$$

$$\text{since} \quad I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} = I_{(\beta_k),m}^{(\gamma_k+\delta_k),(0,0,\dots,0)} = I.$$

Especially, if $\delta_k = \eta_k = 1$, $\beta_k = \beta > 0$, $k = 1, ..., m$, then these generalized fractional derivatives are both reduced to the *hyper-Bessel differential operators*, mentioned next in this lecture.

More generally, the Caputo type and the Riemann-Liouville type generalized fractional derivatives ${}_*D^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ and $D^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ coincide for functions $f \in C_\alpha^{(\eta_1+\ldots+\eta_m)}$ if and only if the equalities for the coefficients of the projector operators (24) and (25): $C_{k,j} = A_{k,j}$ are fulfilled for all $k = 1, ..., m$; $j = 1, ..., \eta_k$.

Note that the generalized fractional derivatives (20) and (22) reduce for $m = 1$ to the "*classical E-K derivatives*" of R-L type and resp. of Caputo type, introduced in Kiryakova (1994), Yakubovich-Luchko (1994), Luchko-Trujillo (FCAA, 2007), as left inverse differentiations to the E-K integral (5):

$$D_\beta^{\gamma,\delta} f(t) := D_n\, I_\beta^{\gamma+\delta,n-\delta} f(t) = \prod_{j=1}^{n} \left(\frac{1}{\beta} t \frac{d}{dt} + \gamma + j\right)\, I_\beta^{\gamma+\delta,n-\delta}\, f(t), \tag{29}$$

and

$${}_*D_\beta^{\gamma,\delta} f(t) := I_\beta^{\gamma+\delta,n-\delta}\, D_n f(t) = I_\beta^{\gamma+\delta,n-\delta} \prod_{j=1}^{n} \left(\frac{1}{\beta} t \frac{d}{dt} + \gamma + j\right) f(t). \tag{30}$$

4. The hint by Dimovski's hyper-Bessel operators

Since 1966, Dimovski introduced and started to develop a Mikusinski-type approach to an operational calculus for the *general differential operator of Bessel type* of arbitrary (integer) order $m > 1$, called later (see Kiryakova, 1994)) as the *hyper-Bessel differential operator*:

$$B = t^{\alpha_0} \frac{d}{dt} t^{\alpha_1} \frac{d}{dt} \cdots t^{\alpha_{m-1}} \frac{d}{dt} t^{\alpha_m} , \quad 0 < t < \infty , \qquad (31)$$

with some real parameters $\alpha_k, k = 1, ..., m$ and $\beta = m - (\alpha_0 + \alpha_1 + \cdots + \alpha_m) > 0$.

Further, Dimovski duplicated his "algebraic" approach, by using a modification of the Obrechkoff integral transform (1958) as a Laplace-type transform basis of the operational calculus for the differential operator (31). This is a very far reaching generalization of the Laplace and Mejjer transforms, with special cases studied by many other authors. The details on these developments can be found in [Ch.3] of Kiryakova book (1994).

The hyper-Bessel differential operator (31) has also alternative representations in the following equivalent forms, either as:

$$B = t^{-\beta} \prod_{k=1}^{m} \left(t\frac{d}{dt} + \beta\gamma_k \right) = t^{-\beta} Q_m(t\frac{d}{dt}), \quad 0 < t < \infty, \quad (32)$$

which is symmetric with respect to the zeros $\mu_k = -\beta\gamma_k$, $k = 1, \ldots, m$ of the m-th degree polynomial $Q_m(\mu)$, where $\gamma_k = \frac{1}{\beta}(\alpha_k + \cdots + \alpha_m - m + k)$, $k = 1, ..., m$; or as

$$B = t^{-\beta} \left[t^m \frac{d^m}{dt^m} + a_1 t^{m-1} \frac{d^{m-1}}{dt^{m-1}} + \cdots + a_m \right],$$

with coefficients

$$a_{m-k} = \sum_{j=0}^{m} \left[\frac{(-1)^j}{j!(k-j)!} \prod_{i=1}^{m} (\beta\gamma_i + k - j) \right], \quad k = 0, 1, ..., m-1,$$

The above representation gives a better impression on the nature of the hyper-Bessel differential operators, appearing often in problems of mathematical physics, and extending the Bessel differential operator B_ν of 2nd order $(m = 2)$ with $\gamma_{1,2} = \pm\nu/2$, $\beta = 2$.

The notion *hyper-Bessel integral operator L* is used for the linear right inverse operator of B, defined by means of the Cauchy problem

$$\begin{cases} By(t) = f(t), \quad \lim_{t\to+0} B_i\, y(t) = b_i = 0, \quad i = 1, \dots, m, \\ \text{where} \;\; b_i = B_i\, y(t) = t^{\alpha_i} \dfrac{d}{dt} t^{\alpha_{i+1}} \dots \dfrac{d}{dt} t^{\alpha_m}\, y(t) \\ \qquad = t^{\beta\gamma_i} \displaystyle\prod_{j=i+1}^{m} \left(t\frac{d}{dt} + \beta\gamma_j \right) y(t) \end{cases} \tag{33}$$

denote the so-called *Bessel-type initial conditions*. This integral operator has the following explicit form (as repeated integrals), as was used initially in the works of Dimovski:

$$y(t) = Lf(t) = \frac{t^\beta}{\beta^m} \int_0^1 \cdots \int_0^1 \left[\prod_{k=1}^{m} t_k^{\gamma_k} \right] f\left[t(t_1 \dots t_m)^{1/\beta} \right] dt_1 \dots dt_m, \tag{34}$$

and considered in the space C_α as $L : C_\alpha \mapsto C_{\alpha+\beta} \subset C_\alpha$ with $\alpha := \max_{1\le k\le m} [-\beta(\gamma_k + 1)]$.

But using the tools of the special functions (Meijer G-functions), we were able to represent the hyper-Bessel integral operator (32) in a more concise (single integral) form, as

$$Lf(t) = \frac{t^\beta}{\beta^m} \int_0^1 G^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+1)_1^m \\ (\gamma_k)_1^m \end{matrix}\right] f(t\sigma^{1/\beta})d\sigma. \tag{35}$$

In 1968, Dimovski considered also *fractional powers of the integral operator* L (34), namely the operators $L^\lambda,\ \lambda > 0$. To express them, he used the *notion of convolution* (the basic one in his "Convolutional Calculus", 1990). Namely, he represented the fractional powers of L as the convolutional products

$$L^\lambda f = \{l_\lambda\}*f, \quad \text{with} \quad l_\lambda = \left\{ \frac{t^{\beta(\lambda-\delta-1)}}{\prod_{k=1}^m \Gamma(\lambda-\delta+\gamma_k)} \right\}, \ \ \delta \geq \max_k \gamma_k, \tag{36}$$

where the convolution $(*)$ has a very complicated form. And he proved that under this definition, the FC semigroup property is satisfied: $L^\lambda L^\mu = L^{\lambda+\mu}$, $\lambda > 0, \mu > 0$, $L^n = L \cdot L \cdots L$ for $n \in \mathbb{N}$.

However, from our point of view based on the use of the G-functions, we were able to find a representation similar to its form in (35),

$$L^{\lambda} f(t) = \frac{t^{\beta}}{\beta^{m}} \int_{0}^{1} G_{m,m}^{m,0} \left[\sigma \left| \begin{array}{c} (\gamma_k + \lambda)_1^m \\ (\gamma_k)_1^m \end{array} \right. \right] f(t\sigma^{1/\beta}) d\sigma. \qquad (37)$$

This representation of the fractional powers of the hyper-Bessel integral operators was first published in a joint paper Dimovski-Kiryakova (1983), and in subsequent Kiryakova's works. This coincided also with the results by McBride (1982) found in completely different way.

Then our step was to think about: what was to be if we replace the parameters in the upper row of the above kernel G-function

$(\gamma_1 + \lambda, \gamma_2 + \lambda, ..., \gamma_m + \lambda)$ by $(\gamma_1 + \delta_1, \gamma_2 + \delta_2, ..., \gamma_m + \delta_m)$

with arbitrary and different $\delta_1 > 0, \delta_2 > 0, ..., \delta_m > 0$? *This led us to the idea of the operators of fractional integration of multi-order (vector order)* $(\delta_1, \delta_2, ..., \delta_m)$, whose theory (GFC) was developed in details in Kiryakova (1994).

Then, the *hyper-Bessel integral and differential operators appear as one of the typical examples of the generalized fractional integrals and derivatives* of arbitrary multiplicity $m \geq 1$ but of integer multi-orders $\delta = (\delta_1, \delta_2, ..., \delta_m) = (1, 1, ..., 1)$, $\forall\, \beta_k = \beta > 0$, $k = 1, ..., m$. Namely, up to a constant multiplier,

$$Lf(t) = ct^{\beta} I_{\beta,m}^{(\gamma_k),(1,\ldots,1)} f(t), \quad Bf(t) = (1/c) D_{\beta,m}^{(\gamma_k),(1,\ldots,1)} t^{-\beta} f(t),$$
$$B^{\lambda} f(t) = (1/c)^{\lambda} D_{\beta,m}^{(\gamma_k),(\lambda,\ldots,\lambda)} t^{-\beta\lambda} f(t). \tag{38}$$

In this way, the hyper-Bessel operators of integer order m gave us the hint for the appropriate definitions of the operators of the GFC. Let us mention that in this special case the R-L and Caputo type generalized "fractional" derivatives both coincide with the hyper-Bessel differential operators B, (31), (32).

Let us compare the situation for the Cauchy problems involving the R-L derivative and the Caputo derivative with the two kinds of initial value problems for hyper-Bessel differential equations, once with the so-called Bessel-type initial conditions (R-L type) as in (33), and on the other side – with the classical-type initial conditions (Caputo-type).

Namely, we consider the general form of the *hyper-Bessel ordinary differential equations*

$$By(t) = \lambda y(t) + f(t), \quad 0 < t < \infty, \tag{39}$$

with conventional (Caputo-type) initial conditions

$$\lim_{t \to +0} y^{(k-1)}(t) = c_k, \quad k = 1, 2, ..., m, \tag{40}$$

or by the equivalent set of Bessel-type initial conditions (R-L type) as in (33),

$$\lim_{t \to 0} B_k y(t) = b_k, \quad k = 1, 2, ..., m. \tag{41}$$

In all cases, including $\lambda \neq 0$ and $f(t) \neq 0$, in our works we provided the explicit forms of the solutions of (39), and the fundamental system of solutions of $By(t) = \lambda y(t)$ that consists of the so-called hyper-Bessel functions (to be discusses in the next lecture on SF of FC).

The sets of the above initial conditions (40), (41) are shown to be *equivalent*, in a sense that $\{c_1, ..., c_m\}$ can be pre-calculated in terms of $\{b_1, ..., b_m\}$, and vice versa, and lead to same solutions.

In particular, let $\beta = m > 1$, one of the γ-parameters of the hyper-Bessel operator (32) be zero, and:

$$\gamma_1 < \gamma_2 < ... < \gamma_{m-1} < \gamma_m = 0 < \gamma_1 + 1.$$

Then, the simpler Cauchy problems with $\lambda = -1,\ f(t) = 0$:

$$By(t) = -y(t), \quad y(0) = 1,\ y'(0) = ... = y^{(m-1)}(0) = 0 \qquad (42)$$

and

$$By(t) = -y(t),$$

$$\lim_{t\to+0} B_k y(t) = b_k = 0, k = 1, 2, ..., m-1;\ \lim_{t\to+0} B_m y(t) = b_m = 1 \qquad (43)$$

with initial conditions being equivalent, have the same (unique) solution as the *normalized hyper-Bessel function*:

$$y(t) = {}_0F_{m-1}\left((1+\gamma_j)_1^{m-1}; -(\frac{t}{m})^m\right) := j^{(m-1)}_{\gamma_1,...,\gamma_m-1}(t).$$

And the *fundamental system of solutions* of this hyper-Bessel differential equation of multi-order $m = (1, 1, ..., 1)$ consists of the Meijer's G-functions, reduced to the *system of the hyper-Bessel functions*:

$$y_k(t) = G^{1,0}_{0,m}\left[\frac{t^\beta}{\beta^m}\,|-\gamma_k, \gamma_1, ..., -\gamma_{k-1}, -\gamma_{k+1}, ..., -\gamma_m\right] = ...$$

5. Examples and other related operators

The operators of this GFC (in the sense as discussed here) include as special cases the classical FC operators as well as various examples of generalized integration and differentiation operators of arbitrary integer or fractional multi-orders, and thus enjoy all their numerous applications in different areas.

1) Consider the trivial case $m=1$, and denote $\gamma_1 = \gamma, \delta_1 = \delta$, $\beta_1 = \beta$, then the kernel-functions of the generalized fractional integrals reduce to

$$H_{1,1}^{1,0}\left[\sigma \left| \begin{array}{c} (\gamma+\delta+1-1/\beta, 1/\beta) \\ (\gamma+1-1/\beta, 1/\beta) \end{array} \right.\right]$$
$$= \beta\sigma^{\beta-1} G_{1,1}^{1,0}\left[\sigma^\beta \left| \begin{array}{c} \gamma+\delta \\ \gamma \end{array} \right.\right] = \frac{\beta}{\Gamma(\delta)}(1-\sigma^\beta)^{\delta-1}\,\sigma^{\beta\gamma+\beta-1}.$$

Thus, (15) and (14) are nothing else but the (single) Erdélyi-Kober fractional integrals (5):

$$I_{\beta,1}^{\gamma,\delta} f(t) = \int_0^1 \frac{(1-\sigma^\beta)^{\delta-1}}{\Gamma(\delta)}\,\sigma^{\beta\gamma} f(t\sigma) d(\sigma^\beta),$$

and the corresponding generalized fractional derivatives are the E-K fractional derivatives of R-L and Caputo type.

Additionally, if $\gamma = 0$, $\beta = 1$, the R-L integral and R-L and Caputo derivatives appear. *For other choices of γ and β one obtains many other differential and integral operators introduced by various authors*.

Example. An operational calculus for solving Cauchy problems for fractional differential equations with Caputo type E-K derivatives has been developed by Hanna-Luchko (ITSF, 2014). Its effectiveness is illustrated by some examples of problems, say for the two-term equation

$$\begin{cases} t^{-\mu *}D_{\mu/\delta}^{\gamma,\delta} y(t) - \lambda y(t) = g(t), \quad t > 0, \ \ \mu > 0, \ \lambda \in \mathbb{C}, \\ \lim_{t\to 0} t^{(\mu/\delta)(1+\gamma+k)} \prod_{i=k+1}^{n-1} \left(1 + \gamma + i + \frac{\delta}{\mu} t \frac{d}{dt}\right) y(t) = p_k, \\ \qquad k = 0, 1, ..., n-1, \end{cases} \tag{44}$$

where the explicit solutions are found in terms of M-L functions.

2) The case $m = 2$ gives the so-called *hypergeometric fractional integrals* like (6) and the corresponding fractional derivatives.

For example, if $\beta_1 = \beta_2 = \beta$, the kernel function can be represented by the Gauss hypergeometric function:

$$H^{2,0}_{2,2}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_1+\delta_1+1-1/\beta),(\gamma_2+\delta_2+1-1/\beta) \\ (\gamma_1+1-1/\beta),(\gamma_2+1-1/\beta) \end{matrix}\right]$$

$$= \frac{\sigma^{\beta\gamma_2}(1-\sigma)^{\delta_1+\delta_2-1}}{\Gamma(\delta_1+\delta_2)}\, {}_2F_1(\gamma_2+\delta_2-\gamma_1,\delta_1;\delta_1+\delta_2;1-\sigma^{\beta}).$$

Thus, the GFC for $m = 2$ contains as special cases the operators considered by Love, Saxena, Kalla, Tricomi, etc., and especially *the Saigo operator* $I^{\alpha,\beta,\eta}$, the *Hohlov hypergeometric fractional integral* $F(a, b, c)$ and their corresponding derivatives.

In view of the composition / decomposition property, all such operators can be considered also as a composition of 2 commuting E-K operators, and thus also as weigthed compositions of R-L operators,

$$I^{(\gamma_1,\gamma_2),(\delta_1,\delta_2)}_{(\beta_1,\beta_2),2} = I^{\gamma_1,\delta_1}_{\beta_1} I^{\gamma_2,\delta_2}_{\beta_2} = I^{\gamma_2,\delta_2}_{\beta_2} I^{\gamma_1,\delta_1}_{\beta_1}.$$

Examples. Example of a Cauchy problem with hypergeometric differential operators of R-L type is the nonlinear problem considered recently by Furati (FCAA, 2013):

$$\begin{cases} D_r^{\alpha,\beta}\, y(t) = f(t, y(t)), \quad t > 0, \;\; 0 \le r < \alpha < 1, \; 0 < \beta < 1, \\ \lim\limits_{t \to 0+} I^{1-\beta} y(t) = c_0, \quad \lim\limits_{t \to 0+} I^{1-\alpha}\left[t^r D^\beta y(t)\right] = c_1, \end{cases} \tag{45}$$

where I^δ and D^δ denote the R-L fractional integral and R-L derivative or order $0 < \delta \le 1$, and the operator $D_r^{\alpha,\beta}$ is the so-called *weighted sequential derivative* (Podlubny, 1999) $D_r^{\alpha,\beta}\, f(t) := D^\alpha\, t^r\, D^\beta\, f(t)$, with a right inverse integral operator $I_r^{\alpha,\beta}\, f(t) = I^\alpha\, t^{-r}\, I^\beta\, f(t)$, which is a hypergeometric integral of the form (7).

Another example, but for a Caputo-type fractional derivative corresponding to the *Saigo hypergeometric integral* $I^{\alpha,\beta,\eta}$, is provided by Rao-Garg-Kalla (2010), following the same approach as by Luchko-Trujillo (FCAA, 2007), and this has been extended to the case of arbitrary multiplicity $m > 1$ in Kiryakova-Luchko (CEJP, 2013).

3) When $m = 3$, there is also an interesting example of the generalized fractional integrals, that nowadays became very popular in articles by other authors who are exercising to find their images of SF.

The kernel-function $G^{3,0}_{3,3}$ of (14) with some special parameters gives the so-called *Horn's (Appell's) F_3-function*. Operators with such kernel have been considered by Marichev (1974), and by Saigo and Maeda (1985, 1998), abbreviated recently as M-S-M operators. They have the form

$$\mathcal{F}f(t) = \int_0^t \frac{(t-\tau)^{c-1}}{\Gamma(c)}\, F_3(a, a', b, b', 1-\frac{t}{\tau}, 1-\frac{\tau}{t})\, f(\tau)d\tau$$
$$= t^c\, I_{1,3}^{(a,b,c-a'-b'),(b,c-a'-b,a')}\, f(t).$$

Many authors ignore the fact these are also 3-tuple commutable compositions of E-K operators $I_{\beta_j}^{\gamma_j,\delta_j}$, $j = 1, 2, 3$. The corresponding fractional derivatives of R-L and Caputo type are also considered and involved in problems for FO differential equations.

4) As mentioned, for arbitrary multiplicity $m = 1, 2, 3...$, the *hyper-Bessel integral and differential operators* (31), (35) and their

Gelfond-Leontiev operators: A more general example with arbitrary $m > 1$ is given by the *fractional indices analogues of the hyper-Bessel operators*, based on the notion of the *Gelfond-Leontiev (G-L) operators of generalized integrations and differentiations* (1951).

Let $z \in \mathbb{C}$ and $\mu_1, ..., \mu_m$ be arbitrary real and $\rho_1 > 0, ..., \rho_m > 0$. With these parameters, for a power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, convergent in a disk $\{|z| < R\} \subset \mathbb{C}$, we consider the following G-L operator of generalized integration (generated by the multi-index M-L functions, Kiryakova - 1999, 2000, 2010; next lecture):

$$\mathcal{I}_{(\mu_k),(\rho_k)} f(z) = \sum_{k=0}^{\infty} a_k \, \frac{\Gamma(\mu_1 + k/\rho_1)...\Gamma(\mu_m + k/\rho_1)}{\Gamma(\mu_1 + (k+1)/\rho_1)...\Gamma(\mu_m + (k+1)/\rho_1)} \, z^{k+1}, \tag{46}$$

and resp. the *G-L generalized differentiation*,

$$\mathcal{D}_{(\mu_k),(\rho_k)} f(z) = \sum_{k=1}^{\infty} a_k \, \frac{\Gamma(\mu_1 + k/\rho_1)...\Gamma(\mu_m + k/\rho_1)}{\Gamma(\mu_1 + (k-1)/\rho_1)...\Gamma(\mu_m + (k-1)/\rho_1)} \, z^{k-1}. \tag{47}$$

It happens that the analytical continuations of the above G-L operators in starlike complex domains Ω are cases of the generalized fractional integrals and derivatives of the form

$$\mathcal{I}_{(\mu_k),(\rho_k)} f(z) = z\ I_{(\rho_k),m}^{(\mu_k-1),(1/\rho_k)} f(z), \tag{48}$$

$$\mathcal{D}_{(\mu_k),(\rho_k)} f(z) = z^{-1}\ D_{(\rho_k),m}^{(\mu_k-1-1/\rho_k),(1/\rho_k)} f(z) - \left[\prod_{k=1}^{m} \frac{\Gamma(\mu_k)}{\Gamma(\mu_k - 1/\rho_k)}\right] \frac{f(0)}{z}. \tag{49}$$

Evidently, for parameters taken as

$$\mu_k = \gamma_k + 1, \forall \rho_k = 1,\ k = 1, ..., m,$$

these G-L operators coincide with the hyper-Bessel operators L and resp. B (for functions with $a_0 = 0$) with parameter $\beta = 1$.

The generalized differentiation operator $D^{(\mu_k-1-1/\rho_k),(1/\rho_k)}_{(\rho_k),m}$ can be considered as a "*fractional multi-order analogue*" *of the hyper-Bessel differential operator*, a kind of *weighted sequential derivative* of the form

$$\mathcal{D}f(z) := D^{(\mu_k-1-1/\rho_k),(1/\rho_k)}_{(\rho_k),m} f(z) = t^{-1} \prod_{k=1}^{m} \Big(z^{1+(1-\mu_k)\rho_k} \, D^{1/\rho_k}_{z^{\rho_k}} z^{(\mu_k-1)\rho_k} \Big) \, f(z)$$

and the G-L generalized integration is:

$$\mathcal{I}f(z) := I^{(\mu_k-1-1/\rho_k),(1/\rho_k)}_{(\rho_k),m} f(z) = z \, I^{(\mu_k-1),(1/\rho_k)}_{(\rho_k),m} f(z)$$

$$= z \int_0^1 H^{m,0}_{m,m} \left[\sigma \,\middle|\, \begin{matrix} (\mu_k, \frac{1}{\rho}_k)_1^m \\ (\mu_k - \frac{1}{\rho}_k, \frac{1}{\rho}_k)_1^m \end{matrix} \right] f(z\sigma) d\sigma.$$

For $\rho_k = 1,\ k = 1, \dots, m$, this evidently reduces to the hyper-Bessel integral operator $Lf(z)$ with $\beta = 1$ and $\gamma_k = \mu_k - 1,\ k = 1, \dots, m$, written in the form of generalized fractional integral:

$$Lf(z) = z \int_0^1 H^{m,0}_{m,m} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + 1, 1)_1^m \\ (\gamma_k, 1)_1^m \end{matrix} \right] f(z\sigma) d\sigma.$$

Example. Consider differential equations with G-L generalized fractional derivatives $\mathcal{D} = D_{(\rho_k),m}^{(\mu_k-1-1/\rho_k),(1/\rho_k)}$, of the form

$$\mathcal{D}y(z) - \lambda y(z) = f(z)$$

that have a fractional multiorder $(1/\rho_1, ..., 1/\rho_m)$ instead of $(1, ..., 1)$.

Their explicit solutions were provided by Kiryakova (1996 and on), Ali-Kiryakova-Kalla (2002) in terms of the *multi-index Mittag-Leffler functions* (see next lecture)

$$E_{(\frac{1}{\rho_k}),(\mu_k)}(\lambda) = \sum_{k=0}^{\infty} \varphi_k \lambda^k = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\mu_1 + k/\rho_1) \ldots \Gamma(\mu_m + k/\rho_m)} . \tag{50}$$

These are to replace the role of the hyper-Bessel functions for the initial value problems with the hyper-Bessel differential equations.

And in particular,

$$\mathcal{D}\, E_{(\frac{1}{\rho_k}),(\mu_k)}(z) = \lambda\, E_{(\frac{1}{\rho_k}),(\mu_k)}(z).$$

That is, the multi-index M-L functions appear as eigen functions of the Gelfond-Leontiev operators.

THANK YOU FOR ATTENTION !

Some References

- A.O. Gelfond, A.F. Leontiev, *Mat. Sbornik* **29**, 477-500 (1951), In Russian.
- I.N. Sneddon, *Mixed BVP in Potential Theory*, North Holland, Amsterdam (1966).
- S.L. Kalla, *Lecture Notes in Math.* **798**, 258-280 (1980), Springer, doi: 10.1007/BFb0097270
- V. Kiryakova, *C.R. Acad. Bulg. Sci.* **41**, 11-14 (1988).
- V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman & J. Wiley, Harlow–N. York (1994).
- I. Dimovski, V. Kiryakova, *Numer. Funct. Anal. and Optimization* **21** 121-144 (2000), doi:10.1080/01630560008816944
- F. Al-Musallam, V. Kiryakova, Vu Kim Tuan, *Rocky Mountain J. Math.* **32**, 409-428 (2002;, doi:10.1216/rmjm/1030539678

- I. Ali, V. Kiryakova, S. Kalla, *J. Math. Anal. and Appl.* **269**, 172-199 (2002); doi: 10.1016/S0022-247X(02)00012-4
- Y. Luchko, J.J. Trujillo, *Fract. Calc. Appl. Anal.* **10**, 249-268 (2007), http://www.math.bas.bg/∼fcaa.
- V. Kiryakova, *Computers and Mathematics with Appl.*, **59**, (2010), 1128-1141; doi:10.1016/j.camwa.2009.05.014
- V. Kiryakova, Yu. Luchko, *Central Europ. J. Physics* **11**, 1314-1336 (2013); doi: 10.2478/s11534-013-0217-1.
- Kh. Furati, *Fract. Calc. Appl. Anal.* **16**, 171-188 (2013); doi: 10.2478/s13540-013-0012-0.
- L.A-M. Hanna, Yu.F. Luchko, *Integral Transf. Spec. Functions* **25**, 359-373 (2014).
- V. Kiryakova, *Fract. Calc. Appl. Anal.* **17**, No 4 (2014), 977-1000; doi: 10.2478/s13540-014-0210-4
- V. Kiryakova, In: *Handbook of Fractional Calculus with Applications*. Volume **1**: *Basic Theory*, Ch. 4, 87-110 (2019)