Orthogonal Fourier Analysis on domains and tiling problems

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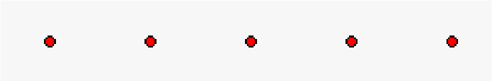
- The Hilbert space $L^2([0,1])$ has

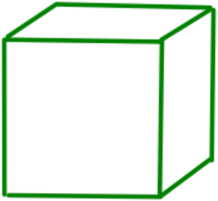
$$e_n(x) = e^{2\pi in\cdot x}, \quad n \in \mathbb{Z},$$

  as an orthogonal basis.
- Inner product is $\langle f, g\rangle = \int_{[0,1]} f(x)\overline{g(x)}\,dx$
- The $e_n(x)$ are orthogonal, normalized and complete.
- Unique expansion: $f(x) = \sum_{n\in\mathbb{Z}} \langle f, e_n\rangle e_n(x)$
- Here $\langle f, e_n\rangle = \int_{[0,1]} f(x)e^{-2\pi inx}\,dx = \widehat{f}(n)$ are the Fourier coefficients of f.

We call the frequencies $\mathbb{Z}$ a *spectrum* of $[0, 1]$.



Fourier Analysis at its simplest

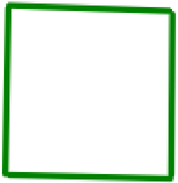


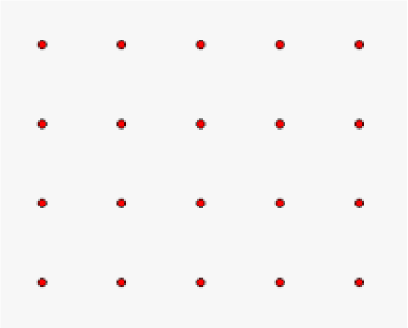
- d-dimensional Fourier series:

$$e_n(x) = e^{2\pi i n \cdot x}, \quad n = (n_1, \ldots, n_d) \in \mathbb{Z}^d,$$

  is an orthogonal basis of $L^2([0,1]^d)$.
- Here $n \cdot x = n_1 x_1 + n_2 x_2 + \cdots + n_d x_d$.

Spectra of domains





- We call $\mathbb{Z}^d$ a spectrum of $[0,1]^d$.
- **Observation**: In 1d and higher dim the set of frequencies has *density equal to the volume of space.*

Which domains are spectral?

Question

On which domains can we do Fourier Analysis?

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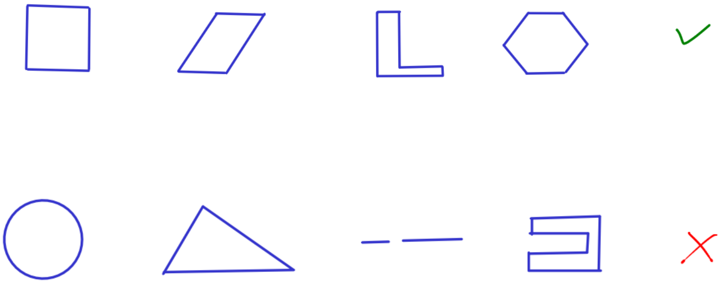
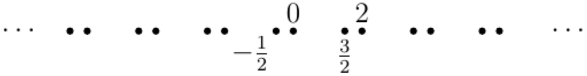
Examples



A more interesting example

has as spectrum the set



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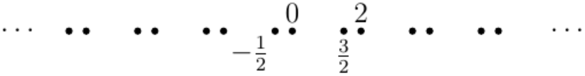
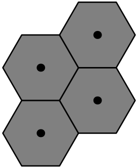


The spectrum of a domain is *not unique*.

The Fuglede Conjecture (1974)

"Ω is spectral $\iff$ it can tile space by translations"



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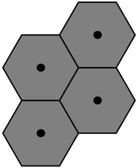


Def: Ω tiles when translated at the locations T if

$$\sum_{t \in T} \mathbf{1}_\Omega(x - t) = 1, \quad \text{for a.e. } x.$$

Its T translates cover $\mathbb{R}^d$ exactly (except for measure 0).

The Fuglede Conjecture (1974)

Was led to this by:

For which $\Omega \subseteq \mathbb{R}^d$ can the commuting operators

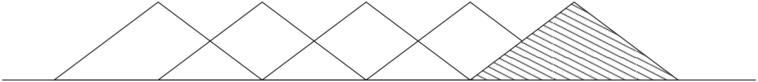
$$-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_d}$$

on $C_c^\infty(\Omega)$, extend to a set of commuting, self-adjoint operators

$$H_1, \ldots, H_d$$

on $L^2(\Omega)$?

When does a function tile by translations?



Let $f \in L^1(\mathbb{R}^d)$, $T \subseteq \mathbb{R}^d$.

Def: We say f tiles by translations with T at level ℓ if

$$\sum_{t \in T} f(x - t) = \ell$$

for almost every $x \in \mathbb{R}^d$ (absolute convergence).

Spectrality in Fourier space

Fourier Transform in $\mathbb{R}^d$:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x)\, dx$$

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- $\langle e_\lambda, e_\mu \rangle = \langle e^{2\pi i \lambda \cdot x}, e^{2\pi i \mu \cdot x} \rangle_{L^2(\Omega)} = \widehat{\mathbf{1}_\Omega}(\lambda - \mu)$
- So $\lambda \perp \mu$ (i.e. $e^{2\pi i \lambda \cdot x} \perp e^{2\pi i \mu \cdot x}$) $\Longleftrightarrow$

$$\lambda - \mu \in \mathrm{Z}\left(\widehat{\mathbf{1}_\Omega}\right) = \left\{\xi \in \mathbb{R}^d :\ \widehat{f}(\xi) = 0\right\}.$$

The zero set $\mathrm{Z}\left(\widehat{\mathbf{1}_\Omega}\right)$ is the crucial geometric object!

Take $\Lambda \subseteq \mathbb{R}^d$ a set of frequencies.
If orthogonal

- Bessel's inequality $\sum_{\lambda\in\Lambda} \left|\langle f, \frac{e^{2\pi i\lambda\cdot x}}{|\Omega|^{1/2}}\rangle\right|^2 \leq \|f\|_2^2$.

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- Plugging in $f(x) = e^{2\pi i t\cdot x}$ we get

$$\forall t \in \mathbb{R}^d: \sum_{\lambda\in\Lambda} \left|\widehat{\mathbf{1}_\Omega}\right|^2 (t-\lambda) \le |\Omega|^2 \ (\textit{packing condition}).$$

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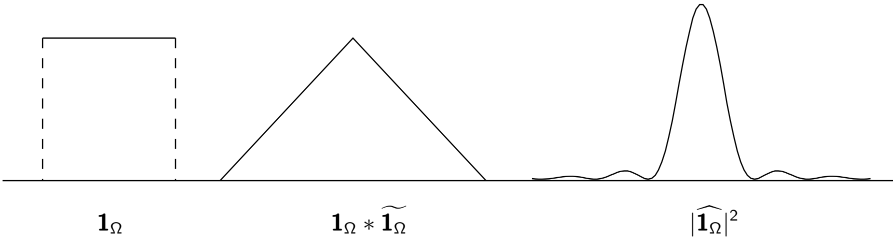
- By completeness of all exponentials in $L^2(\Omega)$

$$\Lambda \text{ orthogonal \& complete} \iff \forall t \in \mathbb{R}^d : \sum_{\lambda\in\Lambda} \left|\widehat{\mathbf{1}_\Omega}\right|^2 (t-\lambda) = |\Omega|^2.$$

(*tiling condition*)

Fuglede in Fourier space

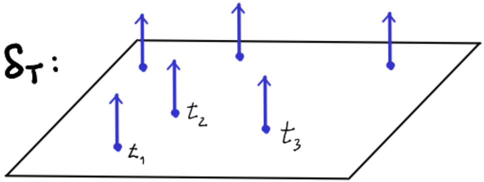


- Fuglede's Conjecture in geometric language:

$$\Omega \text{ tiles at level } 1 \iff \left|\widehat{\mathbf{1}_\Omega}\right|^2 \text{ tiles at level } |\Omega|^2.$$

Tiling in Fourier space

Define the measure $\delta_T = \sum_{t \in T} \delta_t$ (unit point masses at $t \in T$).



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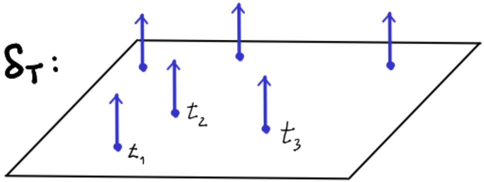

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- Express tiling via convolution:

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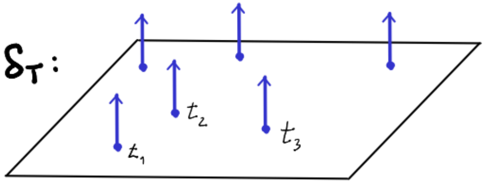

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- Convolution *loves and needs* the Fourier Transform

$$\widehat{f * \mu}(\xi) = \widehat{f}(\xi) \cdot \widehat{\mu}(\xi)$$

Tiling in Fourier space

- $f * \delta_T = \text{const.}$

  $\iff \widehat{f} \cdot \widehat{\delta_T} = \text{const.}\delta_0$ (taking Fourier Transform).

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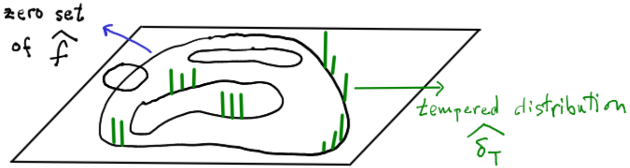
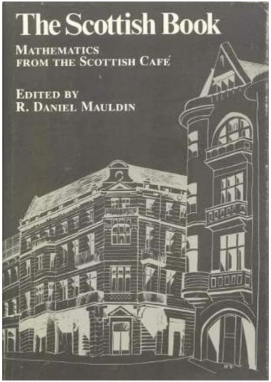
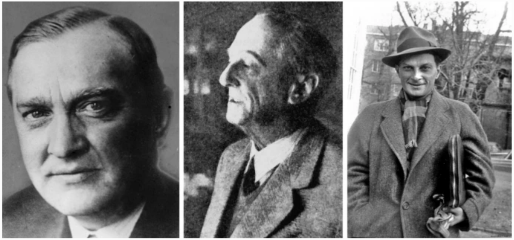
  $\Longleftrightarrow \widehat{f} \cdot \widehat{\delta_T} = \text{const.}\delta_0$ (taking Fourier Transform).
- *Almost* equivalent to:

$$\operatorname{supp} \widehat{\delta_T} \subseteq \{0\} \cup \left\{\widehat{f} = 0\right\}$$



The Scottish Café





Example: From the Scottish Book

(in the sense of H. Steinhaus) for every couple $t_1, t_2 (t_1 \neq t_2)$?

181

H. STEINHAUS

FIND A CONTINUOUS function (or perhaps an analytic one) $f(x)$, positive and such that one has

$$\sum_{n=-\infty}^{\infty} f(x+n) \equiv 1$$

(identically in x in the interval $-\infty < x < +\infty$); examine whether $(1/\sqrt{\pi})e^{-x^2}$ is such a function; or else prove the impossibility; or else prove uniqueness.

Addendum. The function $(1/\sqrt{\pi})e^{-x^2}$ does not have the property — this follows from the sign of the second derivative for $x = 0$ of the expression

$$\sum_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-(x+n)^2}.$$

H. STEINHAUS

- *H. Steinhaus*:
  Is there *analytic* $f > 0$ s.t.

  $$f + \mathbb{Z} = \mathbb{R}? \text{ (shorthand for: } f \text{ tiles } \mathbb{R} \text{ with } \mathbb{Z})$$

  Is $f(x) = Ce^{-x^2}$ such a function?

Example: From the Scottish Book

- $f + \mathbb{Z} = \mathbb{R} \iff \widehat{f} = 0$ at $\mathbb{Z} \setminus \{0\}$ and $\int f = 1$.

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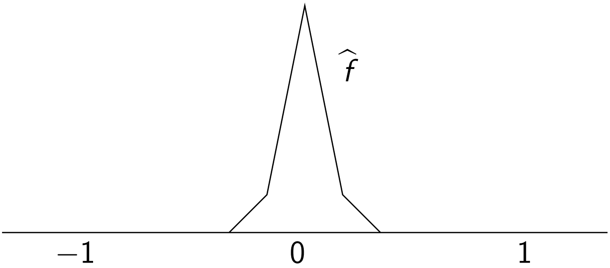
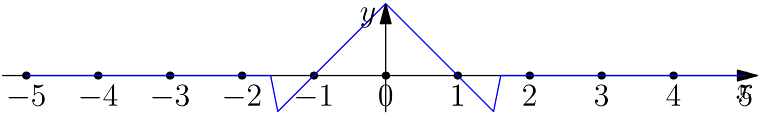
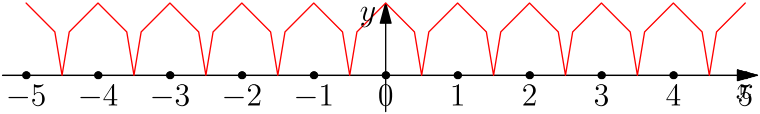


- **Solution:**
  For $f > 0$ we sum in $\widehat{f}$ two triangles with incommensurable bases.

Periodization of a function





- $\mathbb{Z}$-Periodization of $f : \mathbb{R} \to \mathbb{C}$ is $F : \mathbb{T} \to \mathbb{C}$

$$F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$$

and

$$\widehat{F}(n) = \widehat{f}(n).$$

Periodization of a function

- $f + \mathbb{Z} = \mathbb{R}$ is a tiling $\iff$

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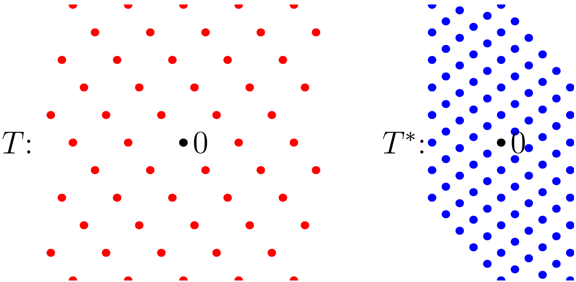
- Equivalently

$$\iff \forall n \in \mathbb{Z} \setminus \{0\} : \ \widehat{f}(n) = 0.$$

Tiling by a lattice

- Lattice case: $T = A\mathbb{Z}^d$, $A \in GL(n, \mathbb{R})$.
- Dual lattice: $T^* = A^{-\top}\mathbb{Z}^d$.



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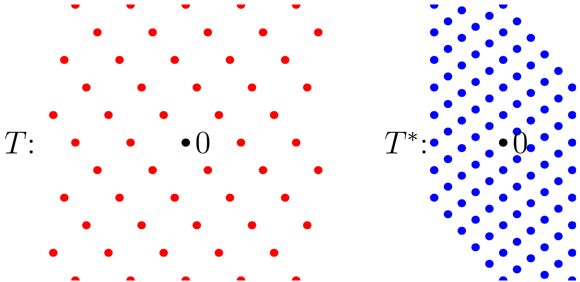


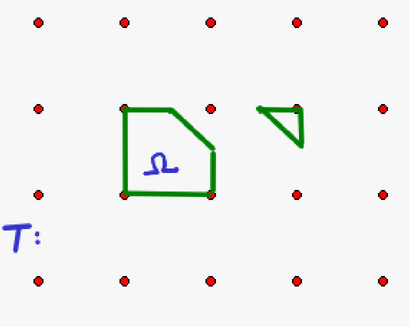
- *Poisson Summation Formula*:

$\widehat{\delta_T} = \frac{1}{|\det A|}\delta_{T^*}$ usually first seen as: $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$

implies

$$f * \delta_T = \text{const.} \iff \widehat{f} \equiv 0 \text{ on } T^* \setminus 0.$$

Lattice Fuglede is true (Fuglede 1974)



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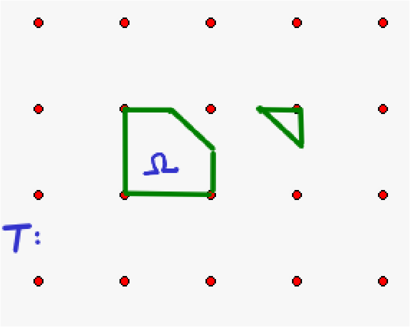

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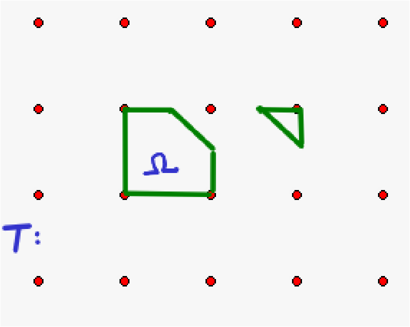

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- FT of $\left|\widehat{\mathbf{1}_\Omega}\right|^2$ is $\mathbf{1}_\Omega * \mathbf{1}_{-\Omega}$ whose support is $\overline{\Omega - \Omega}$.

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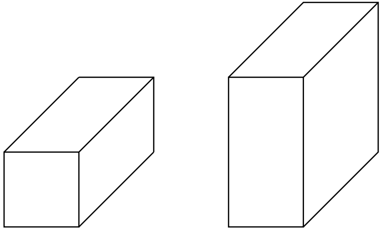
(by volume-density matching & periodicity).

Example: filling a box with 2 kinds of bricks)

Two types of bricks:

$A = a_1 \times a_2 \times a_3$ and

$B = b_1 \times b_2 \times b_3$.



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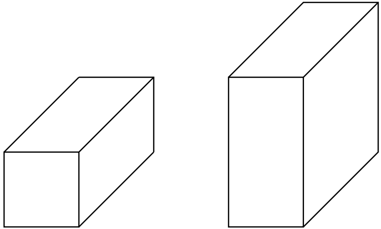

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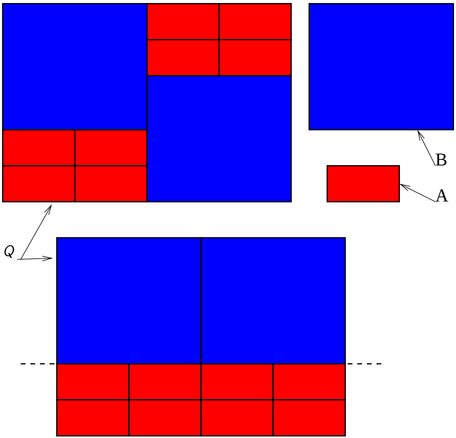
Theorem (Bower and Michael, 2004)

$\iff$ *can cut the box Q into 2 boxes, filling one with brick A, the other with brick B.*

True in all dimensions.

Example:
A: 4×2,
B: 8×7,
Q: 16×11.



We cut the box horizontally

Fourier Transform

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$$C = \left(-\frac{c_1}{2}, \frac{c_1}{2}\right) \times \left(-\frac{c_2}{2}, \frac{c_2}{2}\right)$$

$$\widehat{\mathbf{1}_C}(\xi,\eta) = \frac{\sin(\pi c_1 \xi)}{\xi} \cdot \frac{\sin(\pi c_2 \eta)}{\eta}.$$

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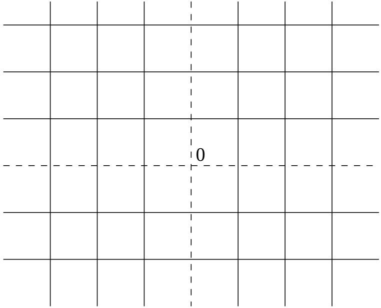


- Where does $\widehat{\mathbf{1}_C}(\xi,\eta)$ vanish?

Fourier Transform

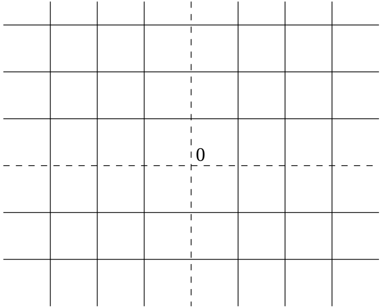
- Fourier transform of f:

$$\widehat{f}(\xi,\eta) = \int_{\mathbb{R}^2} f(x,y)e^{-2\pi i(\xi x+\eta y)}\,dxdy.$$

- Box

$$C = \left(-\frac{c_1}{2},\frac{c_1}{2}\right)\times\left(-\frac{c_2}{2},\frac{c_2}{2}\right)$$

$$\widehat{\mathbf{1}_C}(\xi,\eta) = \frac{\sin(\pi c_1\xi)}{\xi}\cdot\frac{\sin(\pi c_2\eta)}{\eta}.$$



- Where does $\widehat{\mathbf{1}_C}(\xi,\eta)$ vanish?

- When ($0 \neq \xi$ is a multiple of $\frac{1}{c_1}$) or ($0 \neq \eta$ is a multiple of $\frac{1}{c_2}$).

Filling the box in Fourier space

- Brick A at locations T, brick B at locations S:

  Filling box Q: $\forall x \in \mathbb{R}^2 : \quad \mathbf{1}_Q(x) = \sum_{t \in T} \mathbf{1}_A(x-t) + \sum_{s \in S} \mathbf{1}_B(x-s).$

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- We have $Q = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ hence

$$\widehat{\mathbf{1}_Q}(\xi, \eta) = 0 \Longleftrightarrow [\xi \in \mathbb{Z} \setminus \{0\} \text{ or } \eta \in \mathbb{Z} \setminus \{0\}]$$

Consequences of the common zeros

- $\widehat{\mathbf{1}_Q}$ vanishes at

$$(1/a_1, 1/b_2) \in \left(\left\{ \frac{1}{a_1} \right\} \times \mathbb{R} \right) \cap \left(\mathbb{R} \times \left\{ \frac{1}{b_2} \right\} \right).$$

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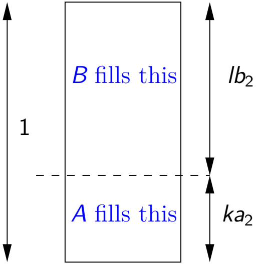
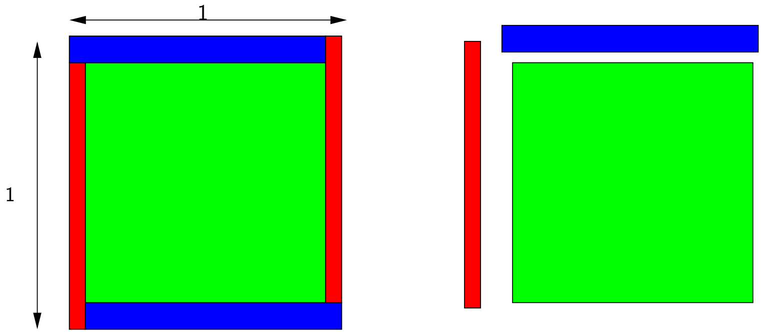
- Let $\frac{1}{a_1}, \frac{1}{b_1} \in \mathbb{Z}$.
  Crossing the box along the y-axis:

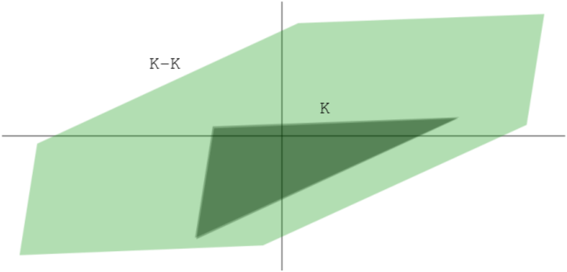
$$1 = ka_2 + lb_2,$$

  for some $k, l \in \mathbb{Z}$.



Theorem not true for 3 bricks





THEOREM

If $K \subseteq \mathbb{R}^d$ is a convex body then

$$|K - K| \geq 2^d |K|,$$

with equality ***if and only if*** *K is symmetric: $K = -K$.*

Convex translational tiles are symmetric

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- Brunn-Minkowski: $|L| > |K|$ if K not symmetric
- Contradiction: in the packing $L + \Lambda$ we must have

$$|L| \cdot \operatorname{dens} \Lambda \leq 1,$$

but $|K| \cdot \operatorname{dens} \Lambda = 1$ from the tiling $K + \Lambda$

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- Fourier condition for tiling:

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \{0\} \cup \left\{\widehat{f} = 0\right\}.$$

which becomes

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \{0\} \cup D^c$$

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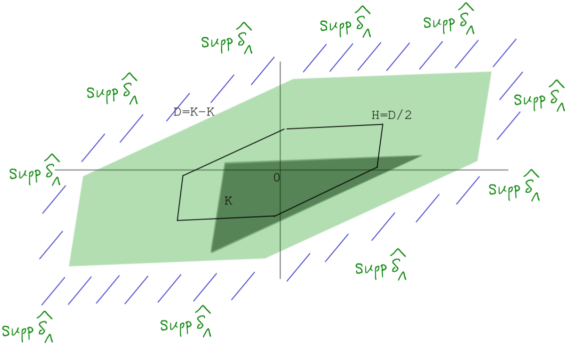

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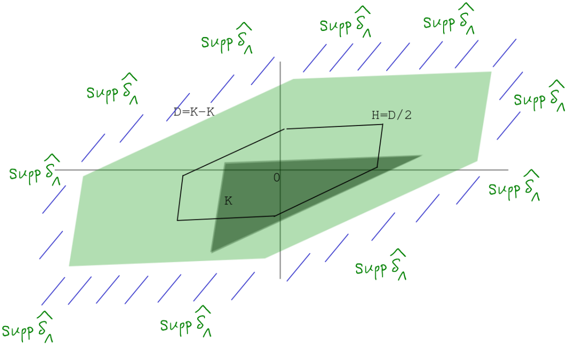

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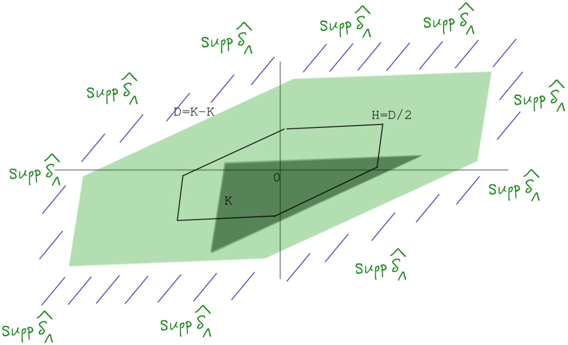

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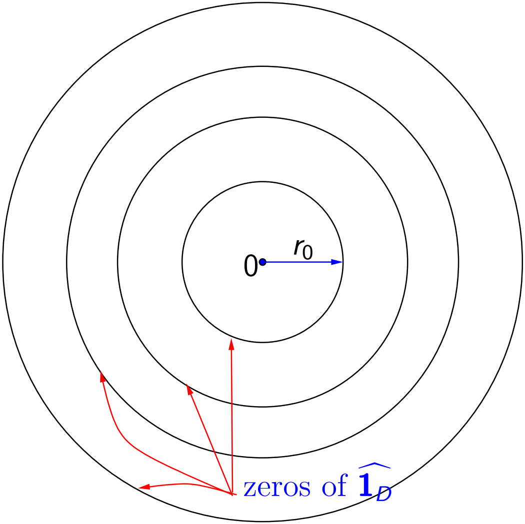
- But $g \geq 0$ and $g(0) = |H|^2 > |H|$, contradiction.

The disk in dimension 2 is not spectral

- $D = \left\{ x \in \mathbb{R}^2 : \ |x| \le \frac{1}{\sqrt{\pi}} \right\}$ is the unit-area disk in the plane.



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  (from the Bessel function $\widehat{\mathbf{1}_D}(\xi) = J_1((2\sqrt{\pi}|\xi|))$
- Placing a disk of radius $r_0/2$ around each point of Λ is a packing.

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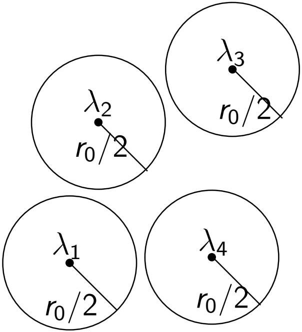

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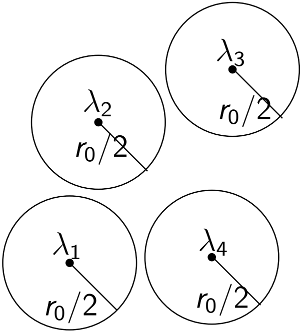

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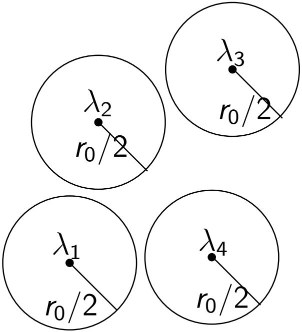

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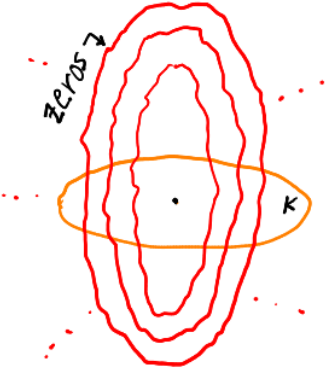
- **Contradiction**: $\mathrm{dens}\,\Lambda = 1$ hence this packing has density
$$1 \cdot \pi\frac{r_0^2}{4} = 0.917751703\cdots ,$$

Smooth convex body Fourier zeros



$K \subseteq \mathbb{R}^d$ is a smooth convex body, K^o its polar (also smooth).
If $\widehat{\mathbf{1}_K}(\xi) = 0$ then, as $|\xi| \to \infty$

$$\|\xi\|_{K^o} = \left(\frac{\pi}{2} + \frac{d\pi}{4}\right) + k\pi + o(1), \qquad k \in \mathbb{Z}.$$

- A result in *Geometric Ramsey Theory*:
  Bourgain, 1986,
  Furstenberg, Katznelson and Weiss, 1990,
  K., 2004:

  If $E \subseteq \mathbb{R}^d$ has positive upper *Lebesgue* density then

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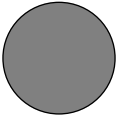
- Or, any *separated* Λ of positive upper *counting* density defines all large-enough distances up to any $\epsilon > 0$.
- If Λ spectrum of K then this contradicts the asymptotics

$$\|\lambda_1 - \lambda_2\|_{K^o} = \left(\frac{\pi}{2} + \frac{d\pi}{4}\right) + k\pi + o(1).$$



- Iosevich, Katz & Pedersen (2001):
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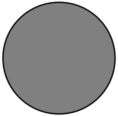
The disk: orthogonal families are finite



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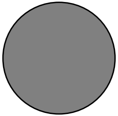

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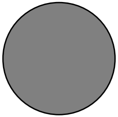
Still unknown

Is there an upper bound for the size of an orthogonal set?

open problem

Disk: size and growth of orthogonal families



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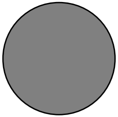
DISK: SIZE AND GROWTH OF ORTHOGONAL FAMILIES



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IOSEVICH & K. (2011)

If Δ is the smallest distance between two elements of Λ then

$$|\Lambda| = O(\Delta),$$

and also $\left|\Lambda \cap [-R, R]^2\right| = O(R^{2/3})$.

Improve this upper bound. open problem

Orthogonality at the Fourier side

- Fourier Transform: $\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i \xi \cdot x} f(x)\, dx$.
- $\langle e^{2\pi i \lambda \cdot x}, e^{2\pi i \mu \cdot x} \rangle_{L^2(D)} = \widehat{\mathbf{1}_D}(\mu - \lambda)$

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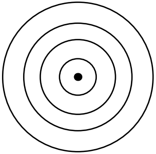

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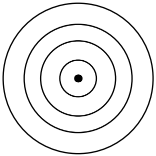

  Zero radii: roots $0 < r_1 < r_2 < \cdots$ of Bessel function $J_1(2\pi r)$.
- Known asymptotic estimates:

$$r_n = \frac{1}{2} n + \frac{1}{8} + O\left(\frac{1}{n}\right)$$

$$r_m - r_n = \frac{1}{2}(m - n) + O\left(\frac{K}{n^2}\right), \quad (m > n).$$

Three orthogonal exponentials

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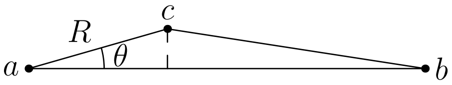

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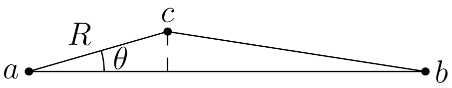
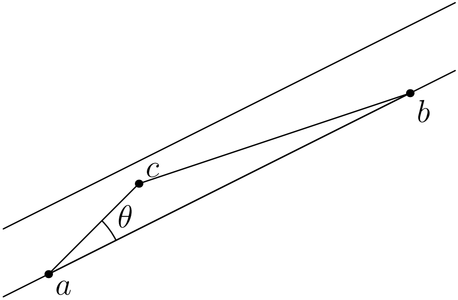


- • $a, b, c \in \mathbb{R}^2$ are orthogonal exponentials.
  • $|a-b|, |b-c|, |c-a| \geq R$, root asymptotics
  $\Longrightarrow$ all angles but one are $\geq \frac{C}{\sqrt{R}}$.

Orthogonal exponentials in a strip



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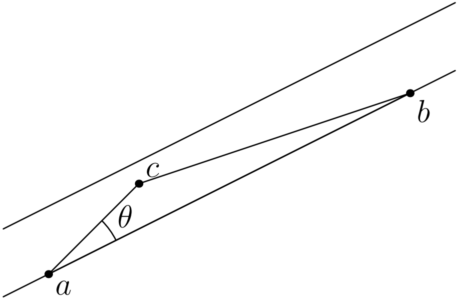
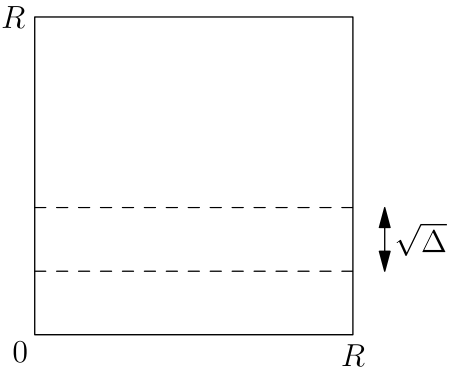


- Any strip of width $\sqrt{L}$ cannot contain more than two orthogonal points of distance $\gtrsim L$.
- If $\Delta = \min_{\lambda \neq \mu \in \Lambda} |\lambda - \mu|$ then in a strip of width $\Delta^{1/2}$ there are at most 2 points of Λ.

Previous bound on Λ (strip covering)



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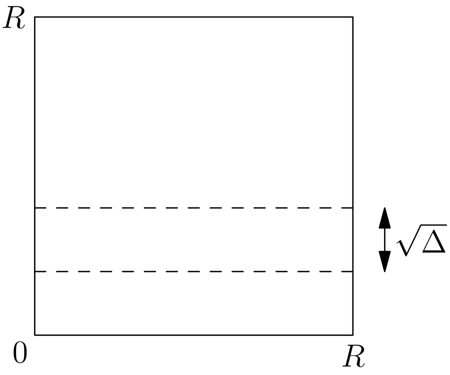
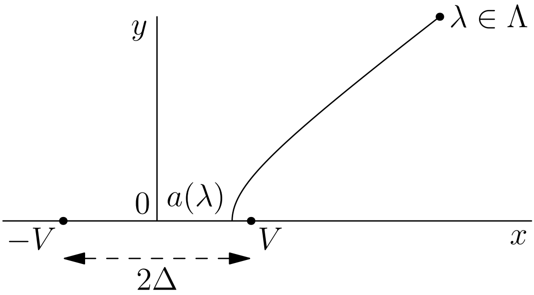


- Cover $[0, R]^2$ by $O\left(\frac{R}{\Delta^{1/2}}\right)$ strips of width $\Delta^{1/2}$.
- Each of them has at most two points of Λ.
- Total is $O\left(\frac{R}{\Delta^{1/2}}\right) = O(R)$ as $\Delta \gtrsim 1$ (Iosevich & Jaming).

Location of λ with respect to 2 fixed points



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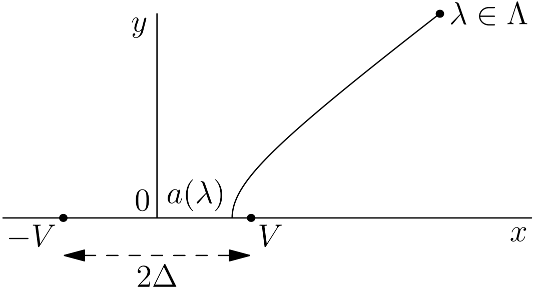
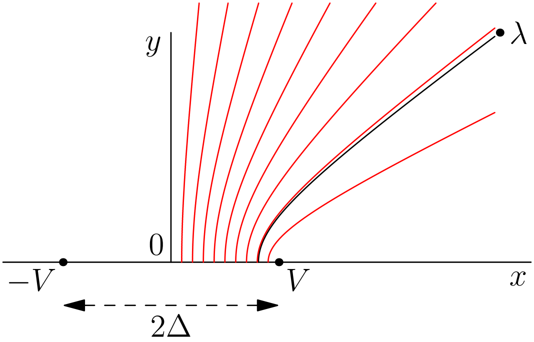


- $V = (\Delta, 0)$ and $-V = (-\Delta, 0)$ are in Λ.
- 2Δ is the smallest distance in Λ.
- Consider the hyperbola with foci at $\pm V$, through λ.
- By the root asymptotics

$$2a(\lambda) = |\lambda + V| - |\lambda - V| = \frac{k}{2} + O(\Delta|\lambda|^{-2}).$$

The hyperbolas with foci at $\pm V$



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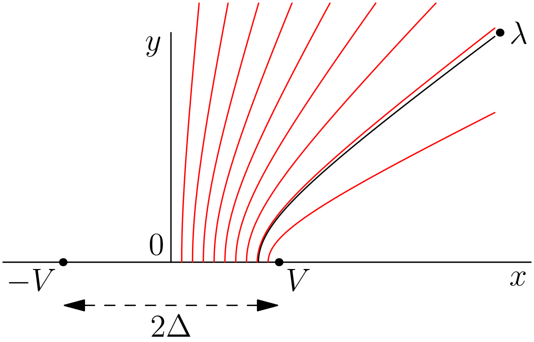

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- Have $O(\Delta)$ of them.
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$|\Lambda| = O(\Delta)$. Covering by hyperbolas.

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- In disk of radius $\Delta^{3/2}$ apply previous

$$O(R\Delta^{-1/2})$$

bound to get $O(\Delta)$ points.

$\left|\Lambda \cap [-R, R]^2\right| = O(R^{2/3})$.

- Bound 1, from strip covering:

$$\left|\Lambda \cap [-R, R]^2\right| = O\left(\frac{R}{\Delta^{1/2}}\right).$$

- Bound 2, from covering by hyperbolas:

$$|\Lambda| = O(\Delta).$$

- Minimum of two bounds is

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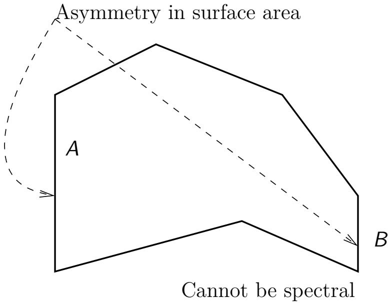
- Conjecture true for convex bodies in $\mathbb{R}^2$ (Iosevich, Katz and Tao, 2003).

$\Longrightarrow$ only parellelograms and symmetric hexagons are spectral among planar convex sets.

For each normal direction of a spectral polytope the same area measure looks forward and backward.
(K. and Papadimitrakis, 2002)

Same is obviously true for polytopes that are tiles.



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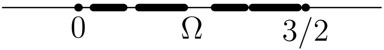

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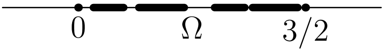

$\implies$ conjecture true for Ω (K. and Łaba, 2001).

- Conjecture true for unions of 2 intervals in $\mathbb{R}$ (Łaba, 2001).
- "Tiling $\implies$ Spectral" for 3 intervals (Bose, Kumar, Krishnan and Madan, 2010)
- "Spectral $\implies$ Tiling" for 3 intervals not known. open problem

Disaster: Failure for “spectral $\implies$ tile”

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- In the group $\mathbb{Z}_2^n$ orthogonal exponentials (characters) on

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- Conjecture still open in both directions for $d = 1, 2$.

open problem

Varying the group: the easy case of $\mathbb{Z}_p$

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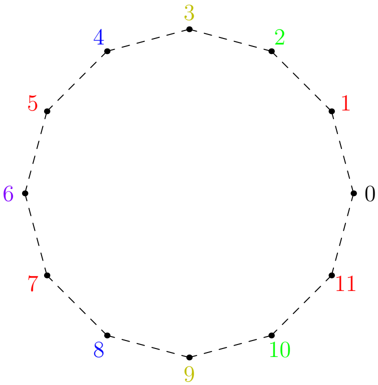
  where $\zeta_\nu = e^{2\pi i \nu / p}$ is a p-th root of unity ($\nu \neq 0$).
- Minimal polynomial of ζ_ν is $\Phi_p(x) = 1 + x + x^2 + \cdots + x^{p-1}$, so

$$\Phi_p(x) \mid \sum_{j \in E} x^j.$$

- So $E = \mathbb{Z}_p$.

The name of the game: algebraic conjugates



Roots	$\{1, 5, 7, 11\}$	$\{2, 10\}$	$\{3, 9\}$	$\{4, 8\}$	$\{6\}$	$\{0\}$
Polynomial	Φ_{12}	Φ_6	Φ_4	Φ_3	Φ_2	Φ_1

Integer polynomials vanish on whole algebraic conjugacy classes.

Varying the group: what's known

- Fuglede true in $\mathbb{Z}_{p^m}$ (Łaba, 2002)
- "tile $\implies$ spectral" OK in $\mathbb{Z}_{p^m q^n}$ (Łaba, 2002)
- Fuglede true in $\mathbb{Z}_p \times \mathbb{Z}_p$ (Iosevich, Mayeli, Pakianathan, 2015)
- Fuglede true in $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ (Shi, 2019)
- "tile $\implies$ spectral" OK in $\mathbb{Z}_p^3$ (K., 2015 and Aten et al, 2015)
- Fuglede Conj. FAILS in $\mathbb{Z}_p^4$ for prime $p \geq 3$
  (Ferguson and Sothanaphan, 2019)
- Fuglede true in $\mathbb{Z}_{p^n q}$ (Malikiosis and K., 2016)
- Fuglede true in $\mathbb{Z}_{pqr}$ (Shi, 2018) and $\mathbb{Z}_{p^2 qr}$ (Vizer, 2019)
- Fuglede true in $\mathbb{Z}_{p^n q^2}$ (Kiss, Malikiosis, Somlai and Vizer, 2018)
- p-adics: Fuglede true in $\mathbb{Q}_p$ (Fan, Fan, Liao and Shi, 2015)
- "spectral $\implies$ tile" in $\mathbb{Z}_{p^m q^n}$ (Malikiosis, 2020)
  but if ($p < q$ and $m \leq 9$ or $n \leq 6$) or $p^{m-2} < q^4$.
- "tile $\implies$ spectral" in $\mathbb{Z}_{p_1^n p_2 \cdots p_k}$ (Malikiosis, 2020)

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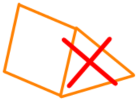

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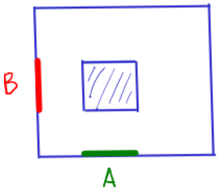
- Lev and Matolcsi, 2019: Conjecture true for all convex polytopes, in all dimensions.

The product question



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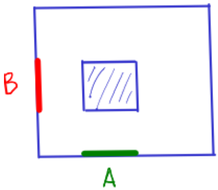

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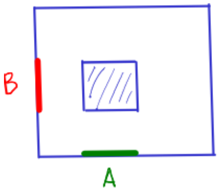

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  $A \times B$ has spectrum $\Lambda_1 \times \Lambda_2$.
- **Unknown:** If $A \times B$ is spectral must A and B also be? *open problem*
- Greenfeld and Lev, 2016: Yes, if $A \subseteq \mathbb{R}$ is an interval.
- K., 2016: Yes, if $A \subseteq \mathbb{R}$ is a union of 2 intervals.
- Greenfeld and Lev, 2018: Yes, if $A \subseteq \mathbb{R}^2$ is a convex polygon.

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- Still, all spectra for bounded $\Omega \subseteq \mathbb{R}$ are periodic
  Bose and Madan, 2010, K., 2011:
  for finite unions of intervals, and
  Iosevich and K., 2011: for general bounded sets.

The weak-tiling of Lev and Matolcsi (2019)

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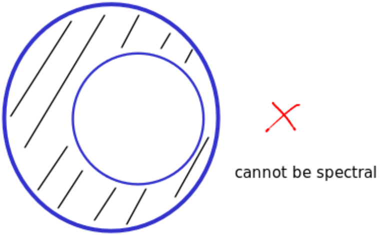
- Some immediate topological obstructions to spectrality:



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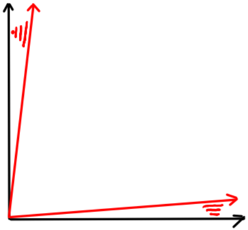
- But $\left|\widehat{\mathbf{1}_\Lambda}\right|^2(0) = |\Lambda|^2$ so Ω weak-tiles its complement with

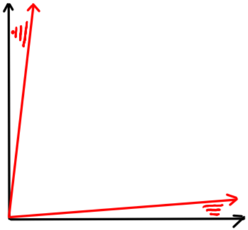
$$\mu = \left|\widehat{\mathbf{1}_\Lambda}\right|^2 - |\Lambda|^2 \delta_0.$$



What if we do not insist on orthogonality?
Riesz basis of exponentials:
Must have

$$\sum_\lambda |a_\lambda|^2 \approx \left\| \sum_\lambda a_\lambda e^{2\pi i \lambda x} \right\|_2^2, \quad \forall a_\lambda.$$

Relaxing orthogonality: Riesz bases



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- **Main Question:** Which domains $\Omega \subseteq \mathbb{R}^d$ admit a Riesz basis of exponentials?
- Major differences from spectrality. E.g., any RB can be perturbed.

- Finite unions of aligned rectangles in $\mathbb{R}^d$ have RBs (Kozma and Nitzan, 2015 and 2016).

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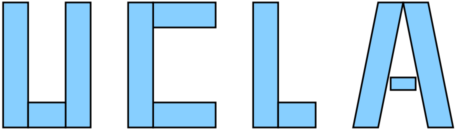

Riesz bases for some domains

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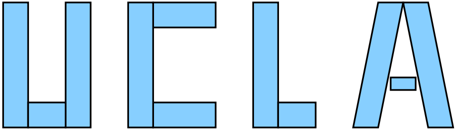

- All zonotopes in $\mathbb{R}^d$ have a RB (Debernardi and Lev, 2019, based on an approach of Walnut).
  Polytopes: centrally sym., with all faces also centrally sym.

Windowed waves (Gabor bases)

- Seeking orthogonal bases of *time-frequency translates*

$$g^{(a,b)}(x) = g(x-a)e^{2\pi i b\cdot x}, \quad (a,b) \in \Lambda \subseteq \mathbb{R}^d \times \mathbb{R}^d.$$

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- Short-time Fourier Transform:

$$V_g(f)(x,\nu) = \langle f, g^{(x,\nu)} \rangle = \int f(t)\overline{g(t-x)}e^{-2\pi i \nu\cdot x}\, dt$$

- Orthogonality for Λ:

$$\Lambda - \Lambda \subseteq \{0\} \cup \mathcal{Z}(V_g g)$$

- Orthogonality and completeness for Λ:

$$|V_g g|^2 + \Lambda \text{ is a tiling at level } \|g\|^4$$

Gabor bases

- A lot of work done for $\Lambda = K \times L$, with $K, L \subseteq \mathbb{R}^d$ lattices. Much less known for general $\Lambda \subseteq \mathbb{R}^{2d}$.

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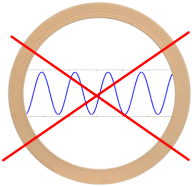

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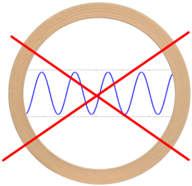
- Characterize window indicator functions. Must they tile? Be spectral? open problem



Thank you