Orthogonal Fourier Analysis on domains and tiling problems

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The Steinhaus tiling problem and its variants

The classical Steinhaus question

- Steinhaus (1950s): Are there $A, B \subseteq \mathbb{R}^2$ such that

$$|\tau A \cap B| = 1, \quad \text{for every rigid motion } \tau?$$

Are there two subsets of the plane which, no matter how moved, always intersect at exactly one point?

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- Sierpiński, 1958:

  

  Yes.

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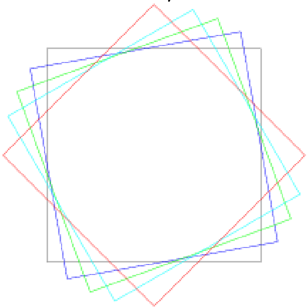
- In tiling language:

$$\rho A \oplus B = \mathbb{R}^2, \quad \text{for all rotations } \rho.$$

Every rotation of A tiles (partitions) the plane when translated at the locations B.

Fixing $B = \mathbb{Z}^2$: the lattice Steinhaus question

- Can we have $\rho A \oplus \mathbb{Z}^2 = \mathbb{R}^2$ for all rotations ρ?



Can a domain behave simultaneously like all those squares?

- Equivalent: A is a fundamental domain of all $\rho\mathbb{Z}^2$.
  Or, A tiles the plane by translations at any $\rho\mathbb{Z}^2$.

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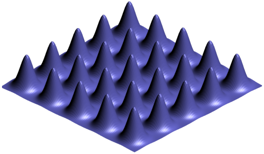
- In higher dimension:
  K. & Wolff (1999), K. & Papadimitrakis (2002):
  $\Longrightarrow$ No measurable Steinhaus sets exist for $\mathbb{Z}^d$, $d \geq 3$.

Lattice Steinhaus in Fourier space

- For f to tile with $\mathbb{Z}^2$ its periodization



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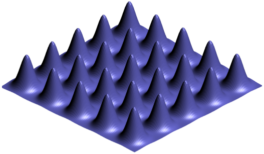

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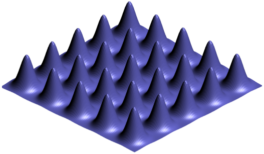

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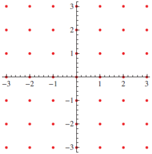
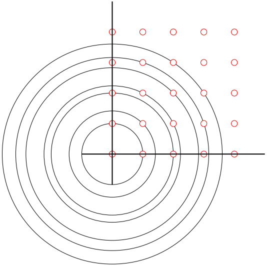
- 

For constant F all $\widehat{F}(n)$ must vanish $(n \neq 0)$.

Equivalently $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}^2 \setminus \{0\}$.

The zeros of the Fourier Transform

- Applying to $f = 1_{\rho A}$ for all rotations ρ we get



that $\widehat{1_A}$ must vanish on all circles through lattice points.

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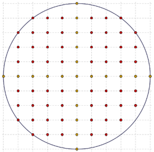
- 

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In dimension $d = 2$ this gives

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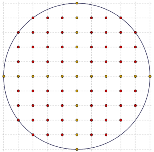
- 
From the estimates on Gauss' Circle Problem:
In dimension $d = 2$ this gives
$$\int_A |x|^{\frac{46}{27}+\epsilon} \, dx = \infty.$$
- In dimension $d \geq 3$: better control of circle gap.
We get 1_A is continuous (contradiction)

$d \geq 4$: No lattice Steinhaus: number theory

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- **Important:** Tiling level must be an integer!

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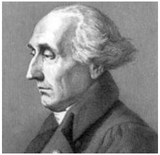
- 

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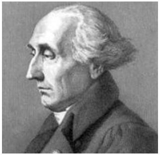
-  Every $n = \square + \square + \square + \square$ (Lagrange)

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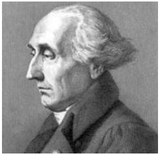
-  Every $n = \square + \square + \square + \square$ (Lagrange)

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- $E + \Lambda$ is a tiling at level

$$|E| \cdot \mathrm{dens}\,\Lambda = \mathrm{vol}\,\Lambda^* = \sqrt{\det B} = \sqrt{5}/4.$$

Not an integer!

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- If E is Steinhaus then $E + \Lambda$ is a tiling.
- But then $\mathrm{dens}\,\Lambda = \mathrm{vol}\,\Lambda^*$ must be an integer.
- And $\mathrm{vol}\,\Lambda^* = \sqrt{2 \cdot 11 \cdot 6}$ is *not* an integer.

Changing Faces: The Mistaken Portrait of Legendre

Peter Duren

Adrien-Marie Legendre (1752–1833) made great contributions to analysis, number theory, celestial mechanics, and practical science. His name is attached to the Legendre differential equation, Legendre polynomials, the Legendre transformation, the Legendre symbol in number theory, the Legendre conditions in calculus of variations, the Legendre relation for elliptic integrals, the Legendre duplication formula for the gamma function, and the list goes on. He wrote important books on advanced calculus, number theory, and elliptic integrals. His textbook adaptation of Euclid's ge-

Messrs Legendre et Fourier

Lattice Steinhaus for finitely many lattices

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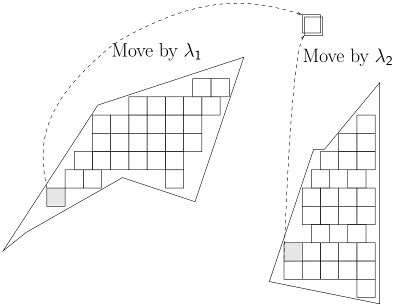
  Generically yes!

  If the sum $\Lambda_1^* + \cdots + \Lambda_n^*$ is direct then Kronecker-type density theorems allow us to rearrange a fundamental domain of one lattice to accomodate the others.



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  Fails in general.
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- No good condition is known!

An application in Gabor analysis

- **Question:** If K, L are two lattices in $\mathbb{R}^d$ with
$$\mathrm{vol}\, K \cdot \mathrm{vol}\, L = 1,$$
can we find $g \in L^2(\mathbb{R}^d)$, such that the (K, L) time-frequency translates
$$g(x-k)e^{2\pi i \ell \cdot x}, \quad (k \in K, \ell \in L)$$
form an orthogonal basis of $L^2(\mathbb{R}^d)$?

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  E is a **common tile** for K, L^*.
- Then L forms an orthogonal basis for $L^2(E)$.
- Space partitioned in K-copies of E and on each copy L is an orthogonal basis.

- **Reminder of the general question**: Is there $A \subseteq \mathbb{R}^2$ such that

$$\rho A \oplus B \text{ is a tiling of } \mathbb{R}^2 \text{ for all rotations } \rho?$$

$B = \mathbb{Z} \times \{0\}$ OR B A FINITE SET

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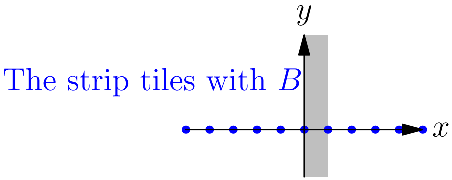

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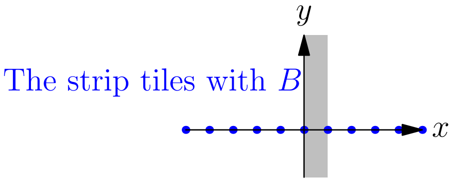

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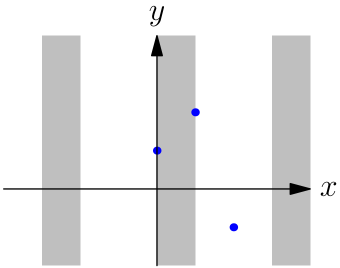

The shaded set tiles with B

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We show here

- A Komjáth set cannot be Lebesgue measurable.
- For any finite $B \subseteq \mathbb{R}^2$ there is no Lebesgue measurable Steinhaus set A.

Finite B: a Fourier condition

Write $\delta_B = \sum_{b\in B} \delta_b$.

$\implies \widehat{\delta_B}(x) = \sum_{b\in B} e^{-2\pi i b\cdot x}$ is a trig. polynomial.

- Suppose $1_A * \delta_B(x) = \sum_{b\in B} 1_A(x-b) = 1$ a.e.(tiling).

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- We conclude

$$\operatorname{supp} \widehat{1_A} \subseteq \{0\} \cup \left\{ \widehat{\delta_B} = 0 \right\}.$$

(Note $\widehat{1_A}$ is a *tempered distribution*.)

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$$\mathrm{supp}\,\widehat{1_A} \subseteq \{0\}\cup\left\{\widehat{\delta_B}=0\right\}.$$

  (Note $\widehat{1_A}$ is a *tempered distribution*.)
- Valid for all rotations ρ:

$$\bigcup_{\rho} \rho\left(\mathrm{supp}\,\widehat{1_A}\right) \subseteq \{0\}\cup\left\{\widehat{\delta_B}=0\right\}.$$

  $\implies$ The zeros of $\widehat{\delta_B}$ contain a *circle*.

Zeros of trigonometric polynomials

Theorem

If $\psi(x) = \sum_{b \in B} c_b e^{2\pi i b \cdot x}$ *is a trigonometric polynomial on* $\mathbb{R}^d$ *which vanishes on a sphere then* $\psi(x) \equiv 0$.

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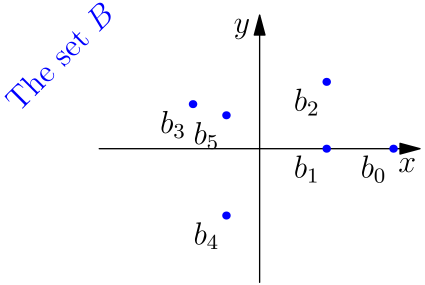

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  May assume zeros at unit circle centered at origin.

Zeros of trigonometric polynomials

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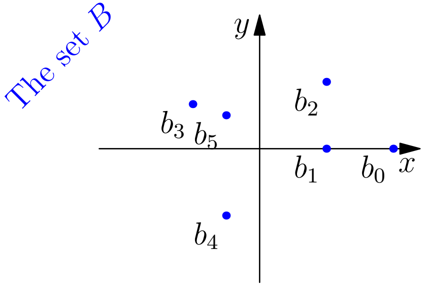

- Enough to prove for $d = 2$.
  May assume zeros at unit circle centered at origin.
- May also assume $(b_0, 0) \in B$ is unique with maximal modulus.

Zeros of trigonometric polynomials, cont'd

- Write $b = b_x + ib_y$, for $b \in B$, and $z = x - iy$, with $|z| = 1$. Then $(b_x, b_y) \cdot (x, y) = \Re(bz)$ and

$$\psi(x, y) = \sum_{b \in B} c_b e^{2\pi i \Re(bz)} \overset{|z|=1}{=} \sum_{b \in B} c_b e^{\pi i (bz + \frac{\overline{b}}{z})} =: g(z)$$

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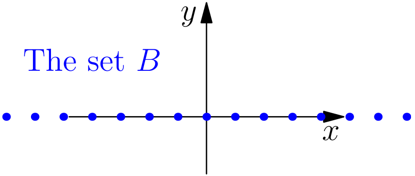
- For real $t \to +\infty$ we have

$$0 = g(-it) = c_{b_0} e^{\pi b_0 t + O(1/t)} + \sum_{b \in B \setminus \{(b_0, 0)\}} c_b e^{\pi bt + O(1/t)}$$

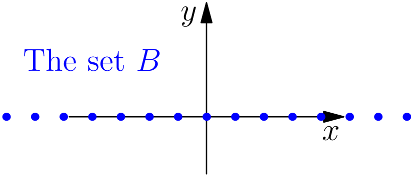
  Contradiction:
  unique exponential with highest exponent.



- Suppose $B = \{(n,0) : n \in \mathbb{Z}\} \subseteq \mathbb{R}^2$ and measurable A so that

$$\sum_{n\in\mathbb{Z}} 1_{\rho A}(x-n,y) = 1, \quad \text{for all rotations } \rho \text{ and a.e. } (x,y).$$

Komjáth sets



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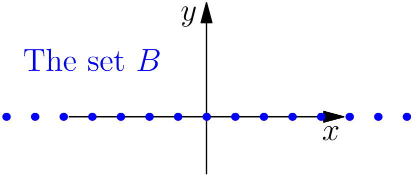

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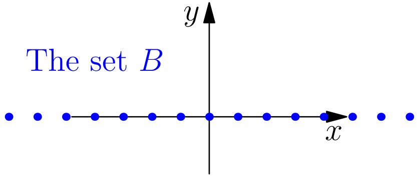

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- Hence A intersects almost all lines of the plane at measure 1.

Meeting the lines thus is too much

Theorem

There is no measurable $A \subseteq \mathbb{R}^2$ which intersects almost all lines of the plane in measure (length) at least C_1 and at most C_2, where $0 < C_1, C_2 < \infty$.

- We only need $C_1 = C_2 = 1$ for showing there are no measurable Komjáth sets.

Line integrals bounded above and below

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Line integrals bounded above and below

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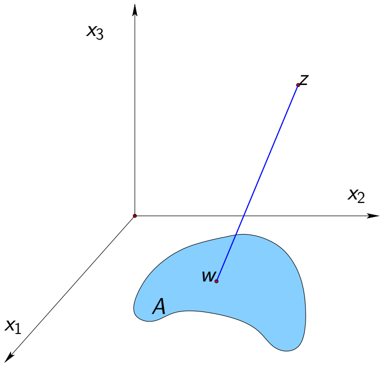
- Define $f : \mathbb{R}^3 \to \mathbb{R}^+$ by (convergence is clear)

$$f(z) = \int_{\mathbb{R}^2} 1_A(w) \frac{1}{|z - w|} \, dw.$$



- **Claim:** $C_1\pi \leq f(z) \leq C_2\pi$ for almost all $z \in \mathbb{R}^2$

$$
\begin{aligned}
f(z) &= \int_{\mathbb{R}^2} 1_A(w) \frac{dw}{|z-w|} \\
&= \int_{\mathbb{R}^2} 1_A(z+w) \frac{dw}{|w|} \quad \text{(change of variable)} \\
&= \int_{[0,\pi]} \int_{\mathbb{R}} 1_A(z + r(\cos\theta, \sin\theta))\, dr\, d\theta \quad \text{(polar coordinates)} \\
&= \int_{[0,\pi]} |A \cap (z + L_\theta)|\, d\theta \quad \text{(where } L_\theta \text{ is the line with angle } \theta\text{)} \\
&\in [C_1\pi, C_2\pi].
\end{aligned}
$$

Line integrals bounded above and below, continued

- f is continuous on $\mathbb{R}^3$.
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$$H = \{(x_1, x_2, x_3) : x_3 > 0\}.$$

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- Contradiction: Clearly $\lim_{t\to+\infty} f(x, y, t) = 0$.