Time - frequency representations for function spaces
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Some contributions to spectral analysis of ψDO's

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Matrix type operators

Matrix type operators with the off-diagonal decay of polynomial or sub-exponential types are revisited with weaker assumptions concerning row or column estimates, still giving the continuity results for the frame type operators. Such results are extended from Banach to Fréchet spaces. Moreover, the localization of Fréchet frames is used for the frame expansions of tempered distributions and a class of Beurling ultradistributions.

Matrix type operators

Localized frames were introduced independently by Gröchenig and Balan, Casazza, Heil, and Landau. The localization conditions are related to off-diagonal decay (of polynomial or exponential type) of the matrix determined by the inner products of the frame elements and the elements of a given Riesz basis. A localized frame in this sense leads to the same type of localization of the canonical dual frame as well as to the convergence of the frame expansions in all associated Banach spaces.

Notation and preliminaries

Throughout the exposition, $(H, \langle \cdot, \cdot \rangle)$ denotes a separable Hilbert space and G (resp. E) denotes the sequence $(g_n)_{n=1}^{\infty}$ (resp. $(e_n)_{n=1}^{\infty}$) with elements from H. Recall that G is called:

- *frame for H* if there exist positive constants A and B (called *frame bounds*) so that $A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, g_n \rangle|^2 \leq B\|f\|^2$ for every $f \in H$;
- *Riesz basis for H* if its elements are the images of the elements of an orthonormal basis under a bounded bijective operator on H.

Notation and preliminarities

Recall, if G is a frame for H, then there exists a frame $(f_n)_{n=1}^{\infty}$ for H so that

$$f = \sum_{n=1}^{\infty} \langle f, f_n \rangle g_n = \sum_{n=1}^{\infty} \langle f, g_n \rangle f_n, f \in H.$$

Such $(f_n)_{n=1}^{\infty}$ is called a *dual frame of* $(g_n)_{n=1}^{\infty}$. Furthermore, the *analysis operator* U_G, given by $U_G f = (\langle f, g_n \rangle)_{n=1}^{\infty}$, is bounded from H into ℓ^2; the *synthesis operator* T_G, given by $T_G f = \sum_{n=1}^{\infty} c_n g_n$, is bounded from ℓ^2 into H; the *frame operator* $S_G := T_G U_G$ is a bounded bijection of H onto H with unconditional convergence of the series $S_G f = \sum_{n=1}^{\infty} \langle f, g_n \rangle g_n$. The sequence $(S_G^{-1} g_n)_{n=1}^{\infty}$ is a dual frame of $(g_n)_{n=1}^{\infty}$, called the *canonical dual of* $(g_n)_{n=1}^{\infty}$, and it will be denoted by $(\widetilde{g_n})_{n=1}$ or $\widetilde{G}$. When G is a Riesz basis of H (and thus a frame for H), then only $\widetilde{G}$ is a dual frame of H, it is the unique biorthogonal sequence to G and it is also a Riesz basis for H. A frame G which is not a Riesz basis has other dual frames in addition to the canonical dual and in that case we use notation G^d or $(g_n^d)_{n=1}^{\infty}$ for a dual frame of G.

Notation and preliminarities

Next, $(X, \|\cdot\|)$ denotes a Banach space and $(\Theta, |||\cdot|||)$ denotes a Banach sequence space; Θ is called a *BK-space* if the coordinate functionals are continuous. If the canonical vectors form a Schauder basis for Θ, then Θ is called a *CB-space*. A *CB*-space is clearly a *BK*-space.

Fréchet frames

Problem

Definition

Let $\{X_k, \|\cdot\|_k\}_{k\in\mathbb{N}_0}$ be a sequence of Banach spaces and let $\{\Theta_k, |||\cdot|||_k\}_{k\in\mathbb{N}_0}$ be a sequence of BK-spaces. A sequence $(\phi_n)_{n=1}^{\infty}$ with elements from X_F^* is called a General Fréchet frame (in short, General F-frame) for X_F with respect to Θ_F if there exist sequences $\{\widetilde{s}_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{N}_0$, $\{s_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{N}_0$, which increase to ∞ with the property $s_k \leq \widetilde{s}_k$, $k \in \mathbb{N}_0$, and there exist constants $0 < A_k \leq B_k < \infty$, $k \in \mathbb{N}_0$, satisfying

$$(\phi_n(f))_{n=1}^{\infty} \in \Theta_F, \ f \in X_F, \tag{1}$$

$$A_k\|f\|_{s_k} \leq |||\{\phi_n(f)\}_{n=1}^{\infty}|||_k \leq B_k\|f\|_{\widetilde{s}_k}, \ f \in X_F, k \in \mathbb{N}_0, \tag{2}$$

and there exists a continuous operator $V : \Theta_F \to X_F$ so that $V(\phi_n(f))_{n=1}^{\infty} = f$ for every $f \in X_F$.

Localization of frames

We consider polynomially and exponentially localized frames and furthermore, sub-exponential localization. Let G be a Riesz basis for the Hilbert space H. A frame E for H is called:

- *polynomially localized with respect to G with decay* $\gamma > 0$ (in short, γ*-localized wrt* $(g_n)_{n=1}^{\infty}$) if there is a constant $C_\gamma > 0$ so that

$$\max\{|\langle e_m, g_n\rangle|, |\langle e_m, \widetilde{g}_n\rangle|\} \leq C_\gamma(1+|m-n|)^{-\gamma},\ m, n \in \mathbb{N};$$

- *exponentially localized with respect to G* if for some $\gamma > 0$ there is a constant $C_\gamma > 0$ so that

$$\max\{|\langle e_m, g_n\rangle|, |\langle e_m, \widetilde{g}_n\rangle|\} \leq C_\gamma \mathrm{e}^{-\gamma|m-n|},\ m, n \in \mathbb{N}.$$

- β*-sub-exponentially localized with respect to G* (for $\beta \in (0,1)$) if for some $\gamma > 0$ there is $C_\gamma > 0$ so that

$$\max\{|\langle e_m, g_n\rangle|, |\langle e_m, \widetilde{g}_n\rangle|\} \leq C_s \mathrm{e}^{-\gamma|m-n|^\beta},\ m, n \in \mathbb{N}.$$

Sequence and function spaces

Recall that a positive continuous function μ on $\mathbb{R}$ is called: a *k-moderate weight* if $k \geq 0$ and there exists a constant $C > 0$ so that $\mu(t+x) \leq C(1+|t|)^k \mu(x),\ \ t, x \in \mathbb{R}$; a *$\beta$-sub-exponential* (resp. *exponential*) weight, if $\beta \in (0,1)$ (resp. $\beta = 1$) and there exist constants $C > 0, \gamma > 0$, so that $\mu(t+x) \leq Ce^{\gamma |t|^\beta} \mu(x),\ \ t, x \in \mathbb{R}$. If β is clear from the context, we will write just *sub-exponential weight*. Let μ be a k-moderate, sub-exponential, or exponential weight so that $\mu(n) \geq 1$ for every $n \in \mathbb{N}$, and $p \in [1, \infty)$. Then the Banach space

$$\ell^p_\mu := \{(a_n)_{n=1}^\infty \ : |||(a_n)_n|||_{p,\mu} := (\sum_{n=1}^\infty |a_n|^p \mu(n)^p)^{1/p} < \infty\}$$

is a *CB*-space. We will need the following, easy to prove, statements.

Sequence and function spaces

Let $k \in \mathbb{N}_0$ and $\mu_k(x) = (1+|x|)^k$ (resp. $\mu_k(x) = e^{k|x|^\beta}$, $\beta \in (0,1]$), $x \in \mathbb{R}$. Then, with $\Theta_k := \ell^p_{\mu_k}$, $k \in \mathbb{N}_0$, the projective limit $\cap_k \Theta_k$ is the space $\mathbf{s}$ of rapidly (resp. $\mathfrak{s}^\beta$ of sub-exponentially when $\beta < 1$ and exponentially when $\beta = 1$) decreasing sequences determined by $\{(a_n)_{n=1}^\infty \in \mathbb{C}^\mathbb{N} \ : \ (\sum_{n=1}^\infty |a_n \mu_k(n))|^p)^{1/p} < \infty, \ \forall k \in \mathbb{N}_0\}$, which is the same set for any $p \in [1,\infty)$. The space $\mathbf{s}$ (resp. $\mathfrak{s}^\beta$) can also be derived as the projective limit of the Banach spaces $\mathbf{s}_k$ (resp. $\mathfrak{s}^\beta_k$) defined as $\{(a_n)_{n=1}^\infty \in \mathbb{C}^\mathbb{N} \ : \ |||(a_n)_{n=1}^\infty|||_{\sup,k} := \sup_{n\in\mathbb{N}} |a_n| n^k < \infty\}$, (resp. $|||\cdot|||^\beta_{\sup,k} := \sup_{n\in\mathbb{N}} |a_n| e^{kn^\beta} < \infty\}$), $k \in \mathbb{N}_0$; note that here instead of $k \in \{0,1,2,3,\dots\}$ one can also use any strictly increasing sequence of non-negative numbers $k \in \{0, q_1, q_2, q_3, \dots\}$.

Sequence and function spaces

Recall that the well known Schwartz space $\mathcal{S}$ is the intersection of Banach spaces

$$\mathcal{S}_k(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : ||f||_k = \sum_{m=0}^{k} ||(1+|\cdot|^2)^{k/2} f^{(m)}||_{L^2(\mathbb{R})}\}, k \in \mathbb{N}.$$

The dual $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions.
The space of sub-exponentially decreasing functions of order $1/\alpha$, $\alpha > 1/2$, is $\Sigma^\alpha := X_F = \cap_{k\in\mathbb{N}_0} \Sigma^{k,\alpha}$ where $\Sigma^{k,\alpha}$ are Banach spaces of L^2- functions with finite norms

$$||f||_k^\alpha = \sup_{n\in\mathbb{N}_0} ||\frac{k^n e^{k|x|^{1/\alpha}} |f^{(n)}(x)|}{n!^\alpha}||_{L^2(\mathbb{R})}, k \in \mathbb{N}.$$

Its dual $(\Sigma^\alpha(\mathbb{R}))'$ is the space of Beurling tempered ultradistributions.

Sequence and function spaces

Remark

The case $\alpha = 1/2$ *leads to the trivial space* $\Sigma^{1/2} = \{0\}$. *There is another way in considering the test space which corresponds to that limiting Beurling case* $\alpha = 1/2$ *and can be considered also for* $\alpha < 1/2$. *We will not treat these cases in the first talk.*

In the sequel, $(h_n)_{n=1}^{\infty}$ is the Hermite orthonormal basis of $L^2(\mathbb{R})$ re-indexed from 1 to ∞ instead of from 0 to ∞. Recall that $h_n \in \Sigma^{\alpha}$, $\alpha > 1/2$, $n \in \mathbb{N}$.

Sequence and function spaces

Moreover, we know
- If $f \in \mathcal{S}$, then $(\langle f, h_n\rangle)_{n=1}^{\infty} \in \mathbf{s}$; conversely, if $(a_n)_{n=1}^{\infty} \in \mathbf{s}$, then $\sum_{n=1}^{\infty} a_n h_n$ converges in $\mathcal{S}$ to some f with $(\langle f, h_n\rangle)_{n=1}^{\infty} = (a_n)_{n=1}^{\infty}$.
- If $F \in \mathcal{S}'$, then $(b_n)_{n=1}^{\infty} := (F(h_n))_{n=1}^{\infty} \in \mathbf{s}'$ and $F(f) = \sum_{n=1}^{\infty} \langle f, h_n\rangle b_n$, $f \in \mathcal{S}$; conversely, if $(b_n)_{n=1}^{\infty} \in \mathbf{s}'$, then the mapping $F : f \to \sum_{n=1}^{\infty} \langle f, h_n\rangle b_n$ is well defined on $\mathcal{S}$, it determines F as an element of $\mathcal{S}'$ and $(F(h_n))_{n=1}^{\infty} = (b_n)_{n=1}^{\infty}$.

The above two statements also hold when $\mathcal{S}$, $\mathcal{S}'$, $\mathbf{s}$, and $\mathbf{s}'$ are replaced by Σ^{α}, $(\Sigma^{\alpha})'$, $\mathfrak{s}^{1/(2\alpha)}$, and $(\mathfrak{s}^{1/(2\alpha)})'$ with $\alpha > 1/2$, respectively.

Sequence and function spaces

We can consider $\mathcal{S}$ and Σ^α as the projective limit of Hilbert spaces H^k, $k \in \mathbb{N}_0$, with elements $f = \sum_n a_n h_n$, in the first case with norms

$$\|f\|_{H^k} := \||(a_n n^k)_n\||_{\ell^2} < \infty\}, k \in \mathbb{N}_0,$$

and in the second case with norms

$$\|f\|_{H^k} := \||(a_n e^{kn^{1/(2\alpha)}})_n\||_{\ell^2} < \infty\}, k \in \mathbb{N}_0.$$

Thus, $(h_n)_n$ is an F-frame for $\mathcal{S}(\mathbb{R})$ with respect to **s** as well as an F-frame for Σ^α with respect to $\mathfrak{s}^{1/2\alpha}$, $\alpha > 1/2$, (F- boundedness is trivial).

Matrix type operators

Grochenig and others uses the following condition: for some $\gamma > 0$ there is $C_\gamma > 0$ such that

$$|A_{m,n}| \le \frac{C_\gamma}{(1+|m-n|)^\gamma} \quad (\text{resp. } |A_{m,n}| \le C_\gamma e^{-\gamma|m-n|}),\ \forall n,m \in \mathbb{N}. \tag{3}$$

We consider matrices with more general off-diagonal type of decay (see $(***)$ below which is weaker condition compare to the polynomial type condition in (3)). Moreover, we consider matrices which have column decrease but allow row increase allowing sub-exponential type conditions as well.
In the sequel, for a given matrix $(A_{mn})_{m,n\in\mathbb{N}}$, the letter $\mathcal{A}$ will denote the mapping $(c_n)_{n=1}^\infty \to (a_m)_{m=1}^\infty$ determined by $a_m = \sum_{n=1}^\infty A_{m,n}c_n$ (assuming convergence), $m \in \mathbb{N}$; conversely, for a given mapping $\mathcal{A}$ determined on a sequence space containing the canonical vectors δ_n, $n \in \mathbb{N}$, the correspondent matrix $(A_{mn})_{m,n\in\mathbb{N}}$ is given by $A_{m,n} = \langle \mathcal{A}\delta_n, \delta_m\rangle$. We will sometimes use $\mathcal{A}$ with the meaning of $(A_{mn})_{m,n\in\mathbb{N}}$ and vice-verse.

Polynomial type conditions

Let us begin with some comparison of polynomial type of off-diagonal decay:

Lemma

Let $\gamma > 0$*. Consider the following conditions:*

$$(*) \quad |A_{m,n}| \leq C \begin{cases} \frac{(1+m)^{\gamma}}{(1+n)^{2\gamma}}, & n \geq m, \\ \frac{(1+n)^{\gamma}}{(1+m)^{2\gamma}}, & n \leq m, \ \textit{for some } C > 0. \end{cases}$$

$$(**) \quad |A_{m,n}| \leq C(1 + |n-m|)^{-\gamma}, \ \textit{for some } C > 0.$$

$$(***) \quad |A_{m,n}| \leq C \begin{cases} \frac{m^{\gamma}}{n^{\gamma}}, & n \geq m, \\ \frac{n^{\gamma}}{m^{\gamma}}, & n \leq m, \ \textit{for some } C > 0. \end{cases}$$

Then, the implications $(*) \Rightarrow (**) \Rightarrow (***)$ *hold. The converse implications are not valid.*

Polynomial type conditions

Proposition

Assume that the matrix $(A_{mn})_{m,n\in\mathbb{N}}$ *satisfies the condition*

$$|A_{m,n}| \leq C_0\, n^{\gamma_0}, n > m,$$

$$|A_{m,n}| C_1\, n^{\gamma_1} m^{-\gamma_1}, n \leq m.$$

for some $\gamma_0 \geq 0, \gamma_1 > 0, C_0 > 0, C_1 > 0$. *Then* $\mathcal{A}$ *is a continuous operator from* $\mathbf{s}_{\gamma_1+\gamma_0+1+\varepsilon}$ *into* $\mathbf{s}_{\gamma_1}$ *for any* $\varepsilon \in (0,1)$.

Note that $\mathbf{s}_\gamma$ is the Banach space of sequences $(a_n)_n$ with the norm

$$\sqrt{\sum_{n\in\mathbb{N}_0} |a_n|^2(1+|n|^2)^\gamma}$$

Polynomial type conditions

A direct consequence of Proposition 0.1 is:

Proposition

Assume that the matrix $(A_{mn})_{m,n\in\mathbb{N}}$ *satisfies: there exist* $\gamma_0 \geq 0$ *and* $C_0 > 0$*, and for every* $\gamma > 0$ *there is* $C_\gamma > 0$ *so that*

$$|A_{m,n}| \leq C_0 n^{\gamma_0}, n > m,$$

$$|A_{m,n}| \leq C_\gamma n^\gamma m^{-\gamma}, n \leq m.$$

Then, $\mathcal{A}$ *is a continues operator from* **s** *into* **s**.

Polynomial type conditions

In order to determine $\mathcal{A}$ as a mapping from a space $\mathbf{s}_{\gamma_1}$ into the same space, we have to change the decay condition.

Proposition

Let $(A_{mn})_{m,n\in\mathbb{N}}$ *satisfy:*

$$(\exists\varepsilon > 0, \gamma_1 \in \mathbb{N})(\exists C_0 > 0, C_1 > 0) \text{ such that}$$

$$|A_{m,n}| \leq \begin{cases} C_0 n^{-1-\varepsilon}, & n > m, \\ C_1 n^{\gamma_1} m^{-\gamma_1-1-\varepsilon}, & n \leq m. \end{cases} \tag{4}$$

Then $\mathcal{A}$ *is a continuous operator from* $\mathbf{s}_{\gamma_1}$ *into* $\mathbf{s}_{\gamma_1}$.

Polynomial type conditions

Remark

For the same conclusion as above, one has another condition non-comparible to (4):

$$|A_{m,n}| \leq C(1 + |n - m|)^{-\gamma_1 - 1 - \varepsilon} \tag{5}$$

Sub-exponential and exponential type conditions

Up to the end of the paper β will be a fixed number of the interval $(0,1]$; $\beta = 1$ is related to the exponential growth order while $\beta \in (0,1)$ corresponds to the pure sub-exponential growth order.

Proposition

Assume that the matrix $(A_{mn})_{m,n\in\mathbb{N}}$ satisfies the condition: there exist positive constants C_0, C_1 and $\gamma_0 \geq 0$, $\gamma_1 > 0$, so that

$$|A_{m,n}| \leq \begin{cases} C_0 \mathrm{e}^{\gamma_0 n^\beta}, & n > m, \\ C_1 \mathrm{e}^{-\gamma_1 (m^\beta - n^\beta)}, & n \leq m. \end{cases} \tag{6}$$

Then $\mathcal{A}$ is a continues operator from $\mathfrak{s}^\beta_{\gamma_1+\gamma_0+\varepsilon}$ into $\mathfrak{s}^\beta_{\gamma_1}$ for any $\varepsilon \in (0,1)$.

Now, $\mathfrak{s}^\beta_\gamma$ is the Banach space of sequences $(a_n)_n$ with the norm

$$\sqrt{\sum_{n\in\mathbb{N}_0} |a_n|^2 e^{2\gamma n^\beta}}$$

Sub-exponential and exponential type conditions

Remark

Since $e^{-\gamma(m-n)^\beta} \leq e^{-\gamma(m^\beta - n^\beta)}$ for $n \leq m$ ($\beta \in (0,1], \gamma \in (0,\infty)$), in (6) we consider $e^{-\gamma(m^\beta - n^\beta)}$ instead of $e^{-\gamma(m-n)^\beta}$.

As a consequence of Proposition 0.4, we have:

Corollary

Assume that the matrix $(A_{mn})_{m,n\in\mathbb{N}}$ satisfies the condition: there exist constants $C_0 > 0$ and $\gamma_0 \geq 0$, and for every $\gamma > 0$, there is a positive constant C_γ so that

$$|A_{m,n}| \leq \begin{cases} C_0 e^{\gamma_0 n^\beta}, & n > m, \\ C_\gamma e^{\gamma(n^\beta - m^\beta)}, & n \leq m. \end{cases}$$

Then $\mathcal{A}$ is a continuous operator from $\mathfrak{s}^\beta$ into $\mathfrak{s}^\beta$.

Sub-exponential and exponential type conditions

Proposition

Let $(A_{mn})_{m,n\in\mathbb{N}}$ *satisfy the condition: there exist positive constants* $\varepsilon, \gamma_1, C_0, C_1$*, so that*

$$|A_{m,n}| \leq \begin{cases} C_0 e^{-\varepsilon n^\beta}, & n > m, \\ C_1 e^{\gamma_1 n^\beta} e^{-(\gamma_1+\varepsilon)m^\beta}, & n \leq m. \end{cases}$$

Then $\mathcal{A}$ *is a continues operator from* $\mathfrak{s}^\beta_{\gamma_1}$ *into* $\mathfrak{s}^\beta_{\gamma_1}$.

Sub-exponential and exponential type conditions

Remark

One can simply show that the assumption $|A_{m,n}| \leq Ce^{-\gamma|m-n|^{\beta}}$, $m, n \in \mathbb{N}$, leads to similar continuity results. We will consider this condition later in relation to the the invertiblity of such matrices and the Jaffard theorem.

We discus. continuity of the frame-related operators under relaxed “decay” conditionsbut I will stop here with this introductory part

Matrix type operators

We study the decay rate of the entries in the inverse of an invertible matrix type operators $A = (A_{s,t})_{s,t}$ where the entries $A_{s,t}$ decrease exponentially as the distance $d(s,t) \to \infty$. We call this decay exponential off-diagonal decay. First we explain the general framework of the investigation.
Recall that the index set $\Lambda \subseteq \mathbb{R}^d$ is called a lattice if $\Lambda = G\mathbb{Z}^d$, where G is a non-singular matrix. Given a complex, separable Hilbert space H we denote by $\mathcal{L}(H)$ the Banach algebra of all bounded linear operators on H. If $A \in \mathcal{L}(H)$ then we can represent A as a matrix (which we still denote by A) with respect to any complete orthonormal set. We do not loose on generality if we restrict our investigations to the case $H = \ell^2(\Lambda)$ with the ordinary operator norm $\|A\|$.

Matrix type operators

A convolution Banach algebra of sequences of polynomial decay $\mathcal{A}$ is analysed by Grochenig as a tool for the analysis of the corresponding extension of the Sjöstrand class $M^{\infty,1}(\mathbb{R}^{2d})$. The new symbol class $\widetilde{M}^{\infty,\mathcal{A}}$ is also characterised by the mean of the Toeplitz matrices $\mathcal{C}_A$ with the diagonal elements determined by the elements of A.
Recall

$$M_w^{p,q} = \{f \in \mathcal{S}' : \int_{\mathbb{R}^d_\xi} (\int_{\mathbb{R}^d_x} (|V_\phi f(x,\xi)| w(x,\xi))^p dx)^{q/p} d\xi)^{1/q}\}$$

where

$$V_\phi f(x,\xi) = \int_{\mathbb{R}^d_t} f(t)\overline{\phi(t-x)} e^{-2\pi i<t,\xi>} dt.$$

Matrix type operators

Consider the class of matrix type operator A satisfying the estimate

$$|A_{s,t}| \leq C\omega,\ \ s,t \in \Lambda,$$

with $\omega = \omega_p = (1 + |s-t|)^{-k/2}$, for some $k > 0$, (i.e. the class of matrices with polynomial off-diagonal decay) or $\omega = \omega_{se} = e^{k|s-t|^{\beta}}$ for some $\beta \in (0,1)$, (i.e. the class of matrices with sub-exponential off-diagonal decay). One of the most important properties that these classes share is that they are spectral invariant. This is explained by Jaffard and later by Grochenig and his collaborators; here we just recall that spectral invariance (or Wiener type property) means that if A is invertible in ℓ^2, then its inverse has the same off-diagonal decay order (polynomial or sub-exponential).

Matrix type operators

Recently, a series of papers Grochenig, Cordero, Rodino, Nicola were related to the matrix type characterisation of various classes of pseudo-differential operators and Fourier integral operators through the matrix representation related to the Gabor wave packets and the almost diagonalisation of ΨDOs and FIOs as was suggested for Sjöstrand or Gröchenig–Rzeszotnik class of ΨDOs.

Matrix type operators

In our main result we consider different exponential off-diagonal decay rate assumptions and obtain new sufficient conditions characterising matrix type operators. Finally, we consider invertible matrices of infinite order which satisfy estimates of the form

$$|A_{s,t}| \leq C_p e^{-pd(s,t)}, \; s,t \in \Lambda, \quad \text{with} \quad C_p \leq K e^{p\varphi(p)}, \; p \in \mathbb{N},$$

where $\varphi(p)$ is a strictly increasing function which satisfies certain conditions. In the main result, we provide estimates on the off-diagonal decay rate of the entries of A^{-1}.

Band limited matrices and sub-exponential off-diagonal decay matrices

From the chronological point of view, important results were obtained by Demko, where a special class of $m-$banded matrices was considered. Recall that a matrix $A=(A_{s,t})_{\Lambda\times\Lambda}$ is $m-$banded if $A_{s,t}=0$ for $s,t\in\Lambda$, $|s-t|>m$. It is proved by Demko that for an $m-$banded invertible matrix operator $A\in\mathcal{L}(\ell^2(\Lambda))$ with $(A^{-1}_{s,t})_{\Lambda\times\Lambda}=A^{-1}\in\mathcal{L}(\ell^2(\Lambda))$, the following estimate holds true

$$|A^{-1}_{s,t}|\leq Ce^{\frac{1}{m}(\ln(1-\frac{2}{\sqrt{\kappa}+1})|s-t|)},\quad s,t\in\Lambda, \tag{7}$$

where C is a certain positive constant depending on m,A, where $[a,b]$, $a>0$, is the smallest interval containing the spectrum of AA^*, while $\kappa=b/a$.

Band limited matrices and sub-exponential off-diagonal decay matrices

The following result of Jaffard is more general since the band limitation is not assumed. Denote by $\mathcal{E}_\gamma$, $\gamma > 0$, the space of matrices $A = (A_{s,t})_{\Lambda\times\Lambda}$ whose entries satisfy:

$$(\exists C_{A,\gamma} \geq 1)\ |A_{s,t}| \leq C_{A,\gamma} e^{-\gamma d(s,t)}, \quad s,t \in \Lambda. \tag{8}$$

Theorem

Let $A : \ell^2(\Lambda) \to \ell^2(\Lambda)$ be an invertible matrix on $\ell^2(\Lambda)$. If $A \in \mathcal{E}_\gamma$, then $A^{-1} \in \mathcal{E}_{\gamma_1}$ for some $\gamma_1 \in (0,\gamma)$.

Band limited matrices and sub-exponential off-diagonal decay matrices

The next simple example shows that for any $k \in \mathbb{Z}_+$ one can construct a matrix A that belongs to $\mathcal{E}_\gamma$ for all $\gamma > 0$, but the inverse A^{-1} is in $\mathcal{E}_\gamma$ only for $\gamma \leq 1/k$. Actually this example is mentioned by Demko in another general form (more precisely, a matrix AA^* is considered) as one for which (7) gives useless estimate (as it was written by Demko). Because of simplicity, the explicit estimate of the growth rate and for the sake of further comments concerning the super-exponential growth, we proceed with it.

Band limited matrices and sub-exponential off-diagonal decay matrices

EXAMPLE Let $k \in \mathbb{Z}_+$. Consider the matrix $A = I - \Gamma$, where I is the identity matrix and the elements of $\Gamma = (\Gamma_{s,t})_{\mathbb{Z}\times\mathbb{Z}}$ are given by $\Gamma_{s,t} = e^{-1/k}$ if $s+1=t,\ s,t \in \mathbb{Z}$ and $\Gamma_{s,t} = 0$ otherwise, i.e.

$$\Gamma = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 0 & e^{-1/k} & 0 & 0 & \dots \\ \dots & 0 & 0 & e^{-1/k} & 0 & \dots \\ \dots & 0 & 0 & 0 & e^{-1/k} & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and}$$

$$A = I - \Gamma = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 1 & -e^{-1/k} & 0 & 0 & \dots \\ \dots & 0 & 1 & -e^{-1/k} & 0 & \dots \\ \dots & 0 & 0 & 1 & -e^{-1/k} & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Band limited matrices and sub-exponential off-diagonal decay matrices

(in other words, Γ is $e^{-1/k}$ times the backward shift on ℓ^2). Clearly, for every $\gamma > 0$ there exists $C_\gamma > 0$ such that $|A_{s,t}| \leq C_\gamma e^{-\gamma|s-t|}$, $s,t \in \mathbb{Z}$; i.e. A belongs to $\bigcap_{\gamma>0} \mathcal{E}_\gamma$. As $\|\Gamma\| = e^{-1/k} < 1$, A^{-1} exists and $A^{-1} = \sum_{n=0}^{\infty} \Gamma^n$. One easily verifies that the entries in $\Gamma^n = (\Gamma^n_{s,t})_{\mathbb{Z}\times\mathbb{Z}}$ are given by $\Gamma^n_{s,t} = e^{-n/k}$ if $s+n=t$, $n \in \mathbb{Z}_+$, $s,t \in \mathbb{Z}$, and $\Gamma^n_{s,t} = 0$ otherwise. Consequently, the entries in the inverse $A^{-1} = (B_{s,t})_{\mathbb{Z}\times\mathbb{Z}}$ are as follows: $B_{s,t} = e^{-(1/k)|s-t|}$ for $s \leq t$, $s,t \in \mathbb{Z}$ and $B_{s,t} = 0$ for $s > t$, $s,t \in \mathbb{Z}$; i.e.

$$A^{-1} = B = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 1 & e^{-1/k} & e^{-2/k} & e^{-3/k} & \dots \\ \dots & 0 & 1 & e^{-1/k} & e^{-2/k} & \dots \\ \dots & 0 & 0 & 1 & e^{-1/k} & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Hence, one obtains

$$|B_{s,t}| = e^{-(1/k)|s-t|}, \quad \text{for} \quad s \le t,\ s,t \in \mathbb{Z},$$

so $A^{-1} \in \mathcal{E}_\gamma$ only for $\gamma \le 1/k$.
The most general result for the sub-exponential growth is: Assume that $\rho : [0,\infty) \to [0,\infty)$ is a strictly increasing concave and normalised ($\rho(0) = 0$) function that satisfies

$$\lim_{\xi\to\infty} \frac{\rho(\xi)}{\xi} = 0. \tag{9}$$

A function u is an admissible weight if $u(x) = e^{\rho(|x|)}$, $x \in \mathbb{Z}^d$. Let $s > d$ and let the weight v be given by $v(x) = u(x)(1+|x|)^s$, $x \in \mathbb{Z}^d$.

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Then Grochenig and Lenert states the following: if there exists $C > 0$ such that

$$|A_{s,t}| \leq Cv(s-t)^{-1}, \quad s,t \in \mathbb{Z}^d,$$

then there is $C_1 > 0$ such that

$$|A^{-1}_{s,t}| \leq C_1 v(s-t)^{-1}, \quad s,t \in \mathbb{Z}^d.$$

This assertion implies the next one which shows that stronger sub-exponential off-diagonal decay for the matrix A implies the stronger sub-exponential off-diagonal decay for the inverse matrix A^{-1}

Band limited matrices and sub-exponential off-diagonal decay matrices

Proposition

Let $A = (A_{s,t})_{\mathbb{Z}^d \times \mathbb{Z}^d} \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$ be invertible with inverse $A^{-1} = (A^{-1}_{s,t})_{\mathbb{Z}^d \times \mathbb{Z}^d}$. Assume that

$$(\forall k > 0)(\exists C_k > 0) \quad |A_{s,t}| \leq C_k e^{-k\rho(|s-t|)}, \; s,t \in \mathbb{Z}^d, \tag{10}$$

where ρ satisfies the above assumptions. Moreover, assume that for every $\varepsilon > 0$ there exists $\tilde{C}_\varepsilon > 0$ such that

$$\tilde{C}_\varepsilon \frac{e^{\varepsilon\rho(\xi)}}{(1+\xi)^{d+1}} \geq 1, \quad \textit{for all } \xi \geq 0. \tag{11}$$

Then

$$(\forall k > 0)(\exists C'_k > 0) \quad |A^{-1}_{s,t}| \leq C'_k e^{-k\rho(|s-t|)}, \; s,t \in \mathbb{Z}^d. \tag{12}$$

Remark

Clearly, the function $\rho(\xi) = \xi^\beta$, $\beta \in (0,1)$, $\xi \geq 0$, satisfies the conditions in Proposition 0.6.

Jaffard result and additional estimates

We analyse, in details, the decay of the inverse matrices of infinite order if the original matrices have exponential off-diagonal decay, that is, if $\rho(\xi)/\xi$ does not converge to zero. For example this is the case if $\rho(\xi) = c\xi$, $\xi \geq 0$. Actually, we follow the proof of Jaffard theorem in and give precise estimates. We consider the special case $\rho(\xi) = c\xi$, $\xi \geq 0$, and $v(x) = e^{\rho(|x|)}$, $x \in \mathbb{Z}^d$.
Let Λ be a discrete index set endowed with a distance d which satisfies the following assumption:

$$m_\varepsilon := \sup_{s\in\Lambda} \sum_{t\in\Lambda} e^{-\varepsilon d(s,t)} < \infty, \quad \forall \varepsilon > 0. \tag{13}$$

Note that $m_\varepsilon \geq 1$, $\forall \varepsilon > 0$, the function $\varepsilon \mapsto m_\varepsilon$, $(0, \infty) \to [1, \infty)$, is decreasing and $m_\varepsilon \to \infty$, as $\varepsilon \to 0^+$.

Jaffard result and additional estimates

Theorem

Let $A = (A_{s,t})_{\Lambda\times\Lambda} : \ell^2(\Lambda) \to \ell^2(\Lambda)$ *be an invertible matrix on* $\ell^2(\Lambda)$ *with inverse* $(A^{-1}_{s,t})_{\Lambda\times\Lambda} = A^{-1} \in \mathcal{L}(\ell^2(\Lambda))$. *Assume that* $A \in \mathcal{E}_\gamma$, *that is, there exists* $C_\gamma \geq 1$ *so that*

$$|A_{s,t}| \leq C_\gamma e^{-\gamma d(s,t)} \quad \textit{for all} \quad s,t \in \Lambda.$$

Then there exist constants $\gamma_1 \in (0,\gamma)$ *and* $C_{A,\gamma_1} > 0$ *such that*

$$|A^{-1}_{s,t}| \leq C_{A,\gamma_1} e^{-\gamma_1 d(s,t)} \quad \textit{for all} \quad s,t \in \Lambda. \tag{14}$$

Furthermore, (14) *holds true with*

$$\gamma_1 = \min\left\{\delta, \frac{(\gamma'-\delta)\ln(1/r)}{\ln\left(\tilde{C}C_\gamma^2 r^{-1} m^2_{(\gamma-\gamma')/2}\right)}\right\}, \quad C_{A,\gamma_1} = \frac{2C_\gamma m_{\gamma-\gamma_1}}{(1-r)\|A\|^2} \tag{15}$$

where $\gamma', \delta \in (0,\gamma)$, $0 < \delta < \gamma' < \gamma$ *are arbitrary,* $r = \|\operatorname{Id} - \|A\|^{-2}AA^*\|$ *and* $\tilde{C} = 1 + \|A\|^{-2}$.

Jaffard result and additional estimates

Remark

As we mentioned before, if $\gamma' \to \gamma$ *then* $m_{\gamma-\gamma'} \to \infty$. *Therefore, the constant* γ_1 *can be arbitrary close to* 0 *when* γ' *approaches* γ *and* δ *approaches* 0.

SECOND LECTURE
Spectral analysis of symbols in the limiting case $s = 1/2$

We analyse symbols of ΨDOs belonging to $(\mathcal{S}^{1/2}\hat{\otimes}_\pi\widetilde{\Sigma}^{1/2})'$ through the analysis of a new class of matrix type operators. The result is almost inverse closedness for those ΨDOs for which the corresponding matrix type operator A satisfies $|A_{s,t}| \leq C_{p,k}e^{-kpd(s,t)}$ where contstants $C_{p,k}$satisfy an appropriate bound as $p \to \infty$, $\forall k > 0$.

Notation and definitions

We use notation $X = (x, \xi)$, $Y = (y, \eta) \in \mathbb{R}^{2d}$. Assume that a locally bounded function v is positive, even and sub-multiplicative. i.e. $v(X+Y) \leq v(X)v(X), X, Y \in \mathbb{R}^{2d}$. Then we say that a positive function m, called weight function, is v-temperate if for some $C > 0$,

$$m(X+Y) \leq Cm(X)v(Y), \; X, Y \in \mathbb{R}^{2d}. \tag{16}$$

In the sequel we take $s \geq 1/2$ and consider weights $m_r^s(X) = e^{r\langle X\rangle^{1/s}}$, $r \in \mathbb{R}$, where $\langle X\rangle = (1 + |X|^2)^{1/2}$. Above weights are also defined over $\mathbb{R}^d$. Notation will always distinguish such weights (and corresponding spaces).
The Fourier and inverse Fourier transforms of $\varphi \in \mathcal{S}(\mathbb{R}^d)$ are given by $\mathcal{F}(\varphi) = \widehat{\varphi} := \int_{\mathbb{R}^d} \varphi(x)e^{-2\pi i\langle x,\cdot\rangle}\, dx$, $\mathcal{F}^{-1}(\varphi) = \mathcal{F}(\check{\varphi})$, where $\check{\varphi}(t) = \varphi(-t)$.

Notatiton and definitions

The space of (sub-, super- or)exponentially decreasing smooth functions of growth order $1/s$, $s \geq 1/2$, $\mathcal{S}^{h,s}$ is a Banach space with the norm

$$||f||_h^s = \sup{}_{n \in \mathbb{N}_0^d} || \frac{h^n e^{h|x|^{1/s}} |f^{(n)}(x)|}{n!^{\alpha}} ||_{L^2(\mathbb{R}^d)}, h > 0.$$

Then

$$\mathcal{S}^s = \bigcup_{h \to 0} \mathcal{S}^{s,h}, \text{ for } s \geq 1/2 \text{ and for } s > 1/2, \ \Sigma^s = \bigcap_{h \to \infty} \mathcal{S}^{s,h}.$$

with the corresponding inductive, respectively, projective topology.

Notatiton and definitions

Σ^s is a dense subspace of $\mathcal{S}^s$, for $s > 1/2$ and $\mathcal{S}^s$, $s \geq 1/2$ is dense in Σ^t, $t > s$. These spaces are very well known in the literature. Their duals S'^s, respectively, Σ'^s are the spaces of Gelfand-Shilov ultradistributions, or Roumieu tempered ultradistributions, respectively, the space of Beurling tempered ultradistributions, cf. [SP], [T], [T]....
In the sequel, $(h_n)_{n\in\mathbb{N}_0^d}$ is the Hermite orthonormal basis of $L^2(\mathbb{R}^d)$. We also recall, $\widetilde{\Sigma}^{1/2}(\mathbb{R}^d) = \widetilde{\Sigma}^{1/2}$, [SP] and later discussed and extende much more by Toft and his collaborators [T],...,[T] as

$$\widetilde{\Sigma}^{1/2}(\mathbb{R}^d) = \{f \in L^2 \cap C^\infty : ||f||_{h,\mathfrak{R}} = \sup_{x\in\mathbb{R}^d, k\in\mathbb{N}_0} \frac{|\mathfrak{R}^k f(x)|}{h^k k!} < \infty, h > 0\},$$

where $\mathfrak{R} = \prod_{i=1}^d (\partial^2_{x_i x_i} - x_i^2)$. Recall that $\mathfrak{R}^k h_m = \prod_{i=1}^d (2m_i + 1)^k$.

Notatiton and definitions

We endow this space with the projective topology with respect to the family of normes $||\cdot||_{h,\mathfrak{R}}$, $h>0$, and denote by $\widetilde{\Sigma}'^{1/2}(\mathbb{R}^d)$ its dual space. We know well that $\widetilde{\Sigma}^{1/2}$ is a dense subset of $\mathcal{S}^{1/2}$. Also, that $f\in\widetilde{\Sigma}^{1/2}(\mathbb{R}^d)$ is equivalent to

$$f=\sum_{n\in\mathbb{N}^d} a_n h_n \ \text{ and } \ \forall h>0\ \exists C_h>0 \sum_{n\in\mathbb{N}_0^d} |a_n|^2 e^{h|n|} < C_h.$$

Recall that $h_n\in\widetilde{\Sigma}^{1/2}\subset\mathcal{S}^{1/2}$, $s\geq 1/2$, $n\in\mathbb{R}^d$.

Spaces of sequences

Following [GR], and the definition of $\mathcal{A}$, in this paper we denote by $\mathcal{A}_r^{2s}$ the sequence space over $\mathbb{N}_0^d$ with respect to the weight m_r^{2s} (instead of the polynomial weights [GR]) consisting of sequences $(a_\lambda)_{n\in\mathbb{N}_0^d}$ with the finite norm $\sum_{n\in\mathbb{N}^d}|a_n|^2 e^{2r|\lambda|^{1/(2s)}} < \infty$. It is a Banach algebra under the convolution

$$(a_n)_n * (b_n)_n = (c_n)_n, \text{ where } c_n = \sum_m a_{n-m}b_m, n \in \mathbb{N}_0^d.$$

Let

$$\mathcal{A}^{2s} = \bigcup_{r\to 0} \mathcal{A}_r^{2s}, \ \mathcal{A}_0^{2s} = \bigcap_{r\to\infty} \mathcal{A}_r^{2s}, \ s \geq 1/2. \tag{17}$$

As we rote for $\widetilde{\Sigma}^{1/2}$, recalling old results, by the use of Hermite expansions, we have isomorphisms of $\mathcal{S}^s$ and $\mathcal{A}^{2s}$, $s \geq 1/2$, of Σ^s and $\mathcal{A}_0^{2s}$, $s > 1/2$ and of $\widetilde{\Sigma}^{1/2}$ and $\mathcal{A}_0 = \mathcal{A}_0^1$. We have the corresponding isomorphisms of dual spaces.

Time-Frequence analysis

Let $X=(x,\xi)\in\mathbb{R}^{2d}$ and $\pi(X)f(t)=M_\xi T_x f=e^{2\pi i t\cdot\xi}f(t-x)$, $f\in\mathcal{S}'_s(\mathbb{R}^d)$. With this notation, and $g\in\mathcal{S}_s(\mathbb{R}^d)$, the short-time Fourier transform of f is

$$V_g f(x,\xi)=\langle f,\pi(X)g\rangle=\int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\xi}\,dt, \tag{18}$$

where we have used the formal integral representation of the dual pairing given in the middle of (18). The same formula (with integration over $\mathbb{R}^{2d}$), with $(X,Y)=(x,\xi,y,\eta)\in\mathbb{R}^{4d}$ instead of X and with

$$\Phi(X)=W(g,g)(X) \tag{19}$$

instead of g, is denoted by $\mathcal{V}_\Phi f(X,Y)$. Recall that W is the Wigner transformation defined over $\mathbb{R}^{2d}$ by

$$W(f,h)(X)=\int_{\mathbb{R}^{2d}} f((x,\xi)-(t,\tau)/2)\overline{h((x,\xi)+(t,\tau)/2)}e^{2\pi(t,\tau)\cdot(x,\xi)}\,dtd\tau,$$

where f and h are $L^2(\mathbb{R}^{2d})$ functions. (Definition of $W(f,h)$ over $\mathbb{R}^d$ is clear.)

Time-Frequence analysis

Assume that m and v satisfy (16). Recall that the modulation space $M_m^{p,q}(\mathbb{R}^d)$ consists of ultradistributions f with the property

$$||f||_{M_m^{p,q}} = \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p |m(x,\xi) dx^{q/p} d\xi\right)^{1/q} < \infty,$$

where $1 \leq p,q \leq \infty$, $g \in M_v^{1,1}(\mathbb{R}^d)$ (the shorten notation is M_v^1)
If

$$g = e^{-\pi x \cdot x} \text{ then } W(g,g)(x,\xi) = \Phi(x,\xi), x,\xi \in \mathbb{R}^d = e^{-\frac{\pi}{2}(x,\xi)\cdot(x,\xi)}. \tag{20}$$

Time-Frequence analysis

Let $\sigma \in \mathcal{S}_s'(\mathbb{R}^{2d})$. Then by [GR],

$$|\langle \sigma^w \pi(X)g, \pi(Y)g \rangle| = |\mathcal{V}_\Phi \sigma(\frac{X+Y}{2}, j(Y-X))| \tag{21}$$

which gives

$$|\mathcal{V}_\Phi \sigma(U,V)| = |\langle \sigma^w \pi(U - \frac{j^{-1}(V)}{2})g, \pi(U + \frac{j^{-1}V}{2})g \rangle|, U, V \in \mathbb{R}^{2d}, \tag{22}$$

$(j(X) = j(x,\xi) = (\xi, -x).)$

Frames

Let $(\mathcal{H}, <\cdot,\cdot>)$ be a separable Hilbert space and $G = (g_\lambda)_{\lambda\in\Lambda} \subset \mathcal{H}^\Lambda$, where $\Lambda \subset \mathbb{R}^{2d}$ is a denumerable directed set of indices.
Denote by Λ a discrete subgroup of $\mathbb{R}^{2d}$ realised as $A\mathbb{Z}^{2d}$, where A is a $2d$-dimensional, regular matrix with determinant $|A| < 1$; Λ is called a set of lattice points, lattices.
Recall that G is called: a frame for $\mathcal{H}$ if there exist positive constants c_1 and c_2 such that

$$c_1 \sum_{\lambda\in\Lambda} |\langle f, g_\lambda\rangle|^2 \leq ||f||^2_{\mathcal{H}} \leq \; c_2 \sum_{\lambda\in\Lambda} |\langle f, g_\lambda\rangle|^2. \tag{23}$$

If $c_1 = c_2$, it is called tight frame.

Frames

if G is a frame for $\mathcal{H}$, then there exists a frame $E = (e_\lambda)_{\lambda \in \Lambda} \in \mathcal{H}^\Lambda$ so that

$$f = \sum_{\lambda \in \Lambda} < f, e_\lambda > g_\lambda = \sum_{\lambda \in \Lambda} < f, g_\lambda > e_\lambda, f \in \mathcal{H}.$$

Such E is called a dual frame of G. If G is a tight frame, its dual frame is G; it is called self dual.

Frames

By a window function $g \in M^1_v$ is determined a Gabor frame

$$G = \mathcal{G}(g, \lambda) = \{\pi(\lambda)g; \lambda \in \Lambda\}.$$

Then by appropriate window $\gamma \in M^1_v$, every $f \in \mathcal{H}$ has the frame expansion

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda) g.$$

Let $H_0(x) = e^{-a|x|^2}, x \in \mathbb{R}^d, a > 0$. It is known that $\mathcal{G}(H_0, \lambda)$ forms a frame in $\mathcal{H}$. Its dual frame is $\mathcal{G}(\gamma)$ which satisfies

$$|\gamma(x)| + |\widehat{\gamma}(x)| \leq Ce^{-c|x|^2}, x \in \mathbb{R}^d.$$

Moreover, it is known that for given $s > 1$, there exists a tight Gabor frame with $|V_g(g)(X)| \leq Ce^{a\langle X \rangle^{1/s}}$ for some positive C and a.

Frames

Recall that the analysis operator U_G, given by $U_G f = (< f, g_\lambda >)_{\lambda\in\Lambda}$, is bounded from $\mathcal{H}$ into ℓ^2; the synthesis operator T_G, given by $T_G(c_\lambda)_\lambda = \sum_{\lambda\in\Lambda} c_\lambda g_\lambda$, is bounded from ℓ^2 into $\mathcal{H}$; the frame operator $S_G := T_G U_G$ is a bounded bijection of $\mathcal{H}$ onto $\mathcal{H}$ with unconditional convergence of the series $S_G f = \sum_{\lambda\in\Lambda} < f, g_\lambda > g_\lambda$.

Almost diagonalization in graded and Fréchet algebras

Repeat that $s \geq 1/2$. Let C be an open ball (or any bounded open set). It is said that $F(Y)$, $Y \in \mathbb{R}^{2d}$ belongs to an amalgam type space $W(\mathcal{A}_r^s)$ if

$$(sup_{Y \in \lambda + C} |F(Y)|)_{\lambda \in \Lambda} = (F_\lambda)_\lambda \in \mathcal{A}_r^s. \tag{24}$$

The space of symbols is defined by the norm

$$||\sigma||_{\widetilde{M}^{\infty,\mathcal{A}_r^{2s}}} = ||(\sup_{Y \in \lambda + C} \mathcal{G}(\sigma) \circ j(Y))_\lambda||_{\mathcal{A}_r^{2s}} = ||(\sup_{Y \in \lambda + C} (\sup_{X \in \mathbb{R}^{2d}} |\mathcal{V}_\Phi \sigma(X, \cdot)|) \circ j(Y))_\lambda||_{\mathcal{A}_r^{2s}}, \tag{25}$$

where $j(y, \eta) = (\eta, -y))$, $Y = (y, \eta) \in \mathbb{R}^{2d}$. Space $\widetilde{M}^{\infty,\mathcal{A}_r^s}$ is defined by

$$\widetilde{M}^{\infty,\mathcal{A}_r^{2s}} = \{\sigma \in \mathcal{S}'(\mathbb{R}^{2d}) : ||(||\mathcal{V}_\Phi(\sigma)(X, j(Y))||_{L^\infty(\mathbb{R}^{2d} \times L^\infty(\lambda + C)})_\lambda||_{\mathcal{A}_r^{2s}}\}.$$

Almost diagonalization in graded and Fréchet algebras

Recall that $W(\mathcal{A})$ is defined replacing $\mathcal{A}_r^{2s}$ by $\mathcal{A}$ while the generalized Sjöstrand class $\widetilde{M}^{\infty,\mathcal{A}}$ is defined by $\mathcal{A}$ in ([GR]). As in the case of $\mathcal{A}$, the definition does not depend on $\Psi \in L^2(\mathbb{R}^{2d})$ if

$$F(Y) = \int_{\mathbb{R}^{2d}} |V_\Phi \Psi(X, j(Y))| dX \in W(\mathcal{A}_r^s), \tag{26}$$

see (19) for Φ. In the sequel we continue to use windows g and Φ so that (26) holds.

Graded and Fréchet algebras $\mathcal{A}^{2s}$ and $\mathcal{A}_0^{2s}$

We assume $s > 1/2$.
Define, $F(Y)$, $Y \in \mathbb{R}^{2d}$ belongs to $W(\mathcal{A}^{2s})$ if $(\sup_{Y\in\lambda+C}|F(Y)|)_{\lambda\in\Lambda} = (F_\lambda)_\lambda \in \mathcal{A}^{2s}$, that is, $(F_\lambda)_\lambda \in \mathcal{A}_r^{2s}$, for some $r > 0$. We note that $\mathcal{A}^{2s}$, defined in (17), is closed under the convolution. We define

$$W(\mathcal{A}^{2s}) = \varinjlim_{r\to 0} W(\mathcal{A}_r^{2s}).$$

Now by the use of the grand symbol $(\mathcal{G}(\sigma)(Y) = \sup_{X\in\mathbb{R}^{2d}} |\mathcal{V}_\Phi(\sigma)(X,Y)|)$, we define the space $\widetilde{M}^{\infty,\mathcal{A}^{2s}}$ as

$$\widetilde{M}^{\infty,\mathcal{A}^{2s}} = \varinjlim_{r\to 0} \widetilde{M}^{\infty,\mathcal{A}_r^{2s}}$$

Graded and Fréchet algebras $\mathcal{A}^{2s}$ and $\mathcal{A}_0^{2s}$

It is the space of symbols with the property $(\sup_{Y\in\lambda+C}\mathcal{G}(\sigma)\circ j(Y))_\lambda \in \mathcal{A}^{2s}$ that is $\mathcal{G}(\sigma)\circ j(Y) \in W(\mathcal{A}^{2s})$. We supply $\widetilde{M}^{\infty,\mathcal{A}^{2s}}$ with the inductive limit topology.
Frécht convolution algebra $\mathcal{A}_0^{2s}$ is also defined in (17). Supplied by the projective topology, it is an $FS-$ space.

Graded and Fréchet algebras $\mathcal{A}^{2s}$ and $\mathcal{A}_0^{2s}$

We define

$$W(\mathcal{A}_0^{2s}) = \varprojlim_{r\to\infty} W(\mathcal{A}_r^{2s}).$$

So, $F(Y)$, $Y \in \mathbb{R}^{2d}$ belongs to $W_{\mathcal{A}_0^{2s}}$ if $(F_\lambda)_\lambda \in \mathcal{A}_r^{2s}$, for every $r > 0$.

We define the space $\widetilde{M}_{1\otimes v}^{\infty,\mathcal{A}_0^{2s}}$ as

$$\widetilde{M}^{\infty,\mathcal{A}_0^{2s}} = \varprojlim_{r\to\infty} \widetilde{M}^{\infty,\mathcal{A}_r^{2s}}$$

It is the space of symbols with the property $\mathcal{G}(\sigma) \circ j(Y) \in W(\mathcal{A}_r^{2s})$, for every $r > 0$
$M^{\infty,\mathcal{A}_0^{2s}}$ has the projective limit topology.
Moreover, we assume that $V_g g \in W(\mathcal{A}_r^{2s})$ for every $r > 0$. Then we have also independence of the definition with respect to g of a modulation space $\widetilde{M}^{\infty,\mathcal{A}_0^{2s}}(\mathbb{R}^{2d})$.

Graded and Fréchet algebras $\mathcal{A}^{2s}$ and $\mathcal{A}_0^{2s}$

Theorem

The following conditions are equivalent

1. $\sigma \in \widetilde{M}^{\infty,\mathcal{A}^{2s}}$*, resp.,* $\sigma \in \widetilde{M}^{\infty,\mathcal{A}_0^{2s}}$

2. There exists $H \in W(A^{2s})$*, resp.,* $H \in W(A_0^{2s})$*, such that*

$$|\langle \sigma^w \pi(X)g, \pi(Y)g\rangle| \leq H(Y-X), X, Y \in \mathbb{R}^{2d} \tag{27}$$

3. There exists a sequence $h \in \mathcal{A}^{2s}$*, resp.,* $h \in \mathcal{A}_0^{2s}$*, such that*

$$|\langle \sigma^w \pi(\mu)g, \pi(\nu)g\rangle| \leq h(\nu-\mu), \mu, \nu \in \Lambda$$

Since $g_\lambda = \pi(\lambda)g$ is a frame, we obtain that in the frame expansion of σ the dominant role have coefficients near the matrix diagonal. This is why this theorem gives almost diagonalisation.

Matrix representation

Let A be a matrix on the set of lattice points Λ, $A = (A_{m,n})_{m,n\in\Lambda^2}$ and let $\mathbf{a}$ be a sequence defined by

$$\mathbf{a}_m = \sup_{n\in\Lambda} |a_{n,n-m}|, m \in \Lambda.$$

Then, it is said that $A \in C_{\mathcal{A}_r^{2s}} = C_r^{2s}, r \in \mathbb{R}$ if $(\mathbf{a}(m))_{m\in\Lambda} \in \mathcal{A}_r^{2s}, r \in \mathbb{R}$. We supply C_r^{2s} by the topology transferred from $\mathcal{A}_r^{2s}$. Moreover, define

$$C^{2s} = \varinjlim_{r\to 0} C_r^{2s}, \quad C_0^{2s} = \varprojlim_{r\to\infty} C_r^{2s}$$

and supply C^{2s} by the inductive limit topology and C_0^{2s} by the projective limit topology. Again, C^{2s} is a *DFS* space, while C_0^{2s} is an *FS*-space. With the notation from the previous subsection, put

$$M(\sigma)_{m,n} = \langle \sigma^w \pi(m)g, \pi(n)g\rangle, m, n \in \Lambda.$$

Matrix representation

Assume that $\mathcal{G}(g,\Lambda)$ makes a frame with the dual frame $\mathcal{G}(\gamma,\Lambda)$. By Theorem 1, 3, $\sigma \in \widetilde{M}^{\infty,\mathcal{A}^{2s}}$, respectively, $\sigma \in \widetilde{M}^{\infty,\mathcal{A}_0^{2s}}$, if for some $r>0$ and $C>0$, respectively, for every $r>0$ and corresponding $C_r>0$

$$|M(\sigma)_{m,n}| \leq Ce^{-r|m-n|^{1/(2s)}}, \text{ respectively, } |M(\sigma)_{m,n}| \leq C_r e^{-r|m-n|^{1/(2s)}}.$$

For an $L^2(\mathbb{R}^d) \ni f = \sum_{m\in\Lambda}\langle f,\pi(m)\gamma\rangle\pi(m)g$ (note $m=(m_1,m_2)\in\Lambda$), one has

$$V_g f(m) = \langle f,\pi(m)g\rangle,$$

and with $n\in\Lambda$,

$$\langle\sigma^w f,\pi(n)g\rangle = \sum_{m\in\Lambda}\langle f,\pi(m)g\rangle\langle\sigma^w\pi(m)g,\pi(n)g\rangle, \text{ i.e. } V_g(\sigma^w f) = M(\sigma)V_g f.$$

In the next theorem we discuss invertibility.

Matrix representation

Theorem

a) $\sigma(x,\xi) \in \widetilde{M}^{\infty,\mathcal{A}^{2s}}$, *resp.,* $\sigma(x,\xi) \in \widetilde{M}^{\infty,\mathcal{A}_0^{2s}}$ *if and only if* $M(\sigma) \in C_r^{2s}$ *for some* $r > 0$*, respectivly, for every* $r > 0$*.*
Moreover, there exist $r > 0, C > 0, c > 0$*, resp., for every* $r > 0$ *and corresponding* $C_r > 0, c_r > 0$ *there holds*

$$c||\sigma||_{\widetilde{M}^{\infty,\mathcal{A}_r^s}} \leq ||M(\sigma)||_{C_r^{2s}} \leq C||\sigma||_{\widetilde{M}^{\infty,\mathcal{A}_r^{2s}}}, \tag{28}$$

respectively,

$$c_r||\sigma||_{\widetilde{M}^{\infty,\mathcal{A}_r^{2s}}} \leq ||M(\sigma)||_{C_r^{2s}} \leq C_r||\sigma||_{\widetilde{M}^{\infty,\mathcal{A}_r^{2s}}}. \tag{29}$$

Matrix representation

Theorem

b) The following conditions are equivalent:

(1) $\sigma(x,D) : M^p_{1\otimes w_r^{2s}} \to M^p_{1\otimes w_r^{2s}}$ *is continuous for some* $r>0$*, respectively, for every* $r>0$

(2) *There exist* $r>0$ *and* $C>0$*, respectively, for every* $r>0$ *and corresponding* $C_r>0$,

$$\langle \sigma(x,D) g_{l,n}, g_{l',n'}\rangle e^{r|(l,n)-(l',n')|^{1/(2s)}} \leq C, (l,n),(l',n') \in \Lambda,$$

respectively,

$$\langle \sigma(x,D) g_{l,n}, g_{l',n'}\rangle e^{r|(l,n)-(l',n')|^{1/(2s)}} \leq C_r, (l,n),(l',n') \in \Lambda.$$

c) If $\sigma \in \widetilde{M}^{\infty,\mathcal{A}^{2s}}$*, respectively,* $\sigma \in \widetilde{M}^{\infty,\mathcal{A}_0^{2s}}$ *is invertible on* $L^2(\mathcal{R})$ *then its inverse* $\tau^w = (\sigma^w)^{-1}$ *has a symbol in* $\widetilde{M}^{\infty,\mathcal{A}^{2s}}$*, respectively, in* $\widetilde{M}^{\infty,\mathcal{A}_0^{2s}}$.

Estimates depending on the growth rate of constants

Our main results are devoted to the case $s = 1/2$. Results are based on papers PS and PPZ. We consider matrices $A = (A_{s,t})_{\mathbb{Z}\times\mathbb{Z}}$. The transfer to the multidimensional case ($(s,t) \in \mathbb{Z}^d \times \mathbb{Z}^d$) is than, simple but cumbersome and so omitted.
Following PPZ, let φ be a strictly increasing, unbounded function $\varphi : [0,\infty) \to [0,\infty)$, such that $\varphi(\xi p) \leq \xi^{a-1}\varphi(p),\ p,\xi \in [1,\infty)$ for some $a > 1$. We denote the space of such functions by $\mathfrak{m}$. For example $\varphi(p) = p^\alpha$, $\alpha > 0$, and $\varphi(p) = \ln(p + e - 1)$, $\varphi : [1,\infty) \to [1,\infty)$, satisfy these conditions, as well as the products of such functions.

Estimates depending on the growth rate of constants

Band limited matrices, those withe elements $A_{s,t} = 0$ for $|t-s| > k$ for some $k \in \mathbb{N}$, satisfy

$$(\forall p \geq 0)(\exists C \geq 1)\ |A_{s,t}| \leq Ce^{-pd(s,t)},\ \text{for all } s,t \in \mathbb{Z}. \tag{30}$$

Example 2.2 in PPZ shows that for every $\varepsilon > 0$ one can construct a band limited matrix operator A so that A^{-1} satisfies

$$|A^{-1}_{s,t}| \leq e^{-\varepsilon d(s,t)}, s,t \in \mathbb{Z}.$$

This shows that the super-exponential decrease rate ($|A_{s,t}| \leq C^{-pd^r(s,t)}$, with $r > 1$) does not imply the super-exponential decrease rate of A^{-1}, if A^{-1} exists.
Our goal is to measure jointly the diagonal-of exponential decrease of elements of an invertible matrix $A \in L(\ell^2)$, continuously acting on ℓ^2 and its inverse A^{-1}.
For the later use, put $m_2 = \sup_{s\in\Lambda} \sum_{t\in\Lambda} e^{-2d(s,t)} < \infty$.

Estimates depending on the growth rate of constants

Fix φ, as above. We consider the space of matrix type operators $A = (A_{s,t})$, $s, t \in \mathbb{Z}$, which satisfy the following condition:

$$(\exists K > 0)(\forall p > 0)(\exists C_p > 0) \;\; |A_{s,t}| \leq K e^{p\varphi(p)} e^{-pd(s,t)}, \;\; s, t \in \mathbb{Z}. \tag{31}$$

Estimates depending on the growth rate of constants

Theorem

Assume the matrix type operator $A = (A_{s,t})_{\mathbb{Z}\times\mathbb{Z}}$ *satisfies (31) Assume that* $\varphi \in \mathfrak{m}$. *Then for every* $D > 0$ *there exists* $C_D > 0$ *such that*

$$|A^{-1}_{s,t}| \leq C_D e^{-Dd(s,t)}, \quad \text{for all } s,t \in \mathbb{Z}. \tag{32}$$

Characterization of ΨDO's in $\mathcal{S}\widetilde{\Sigma}'^{1/2}(\mathbb{R}^n)$

Every $\phi \in \widetilde{\Sigma}^1(\mathbb{R}^d)$, has the expansion $\phi = \sum_{n\in\mathbb{N}_0^d} \langle \phi, h_n\rangle h_n$. Let $\sigma \in \widetilde{\Sigma}'^{1/2}(\mathbb{R}^{2d})$ be a symbol of the corresponding Weyl operator. We have

$$\langle \sigma^w \phi, h_m\rangle = \sum_{n\in\mathbb{N}_0^d} \langle \phi, h_m\rangle \langle \sigma^w h_n, h_m\rangle, \ \ m \in \mathbb{N}_0^d. \tag{33}$$

Recall $U_G\phi = (a_m)_m$ beelongs to the corresponding sequence space, where $\phi = \sum_{m\in\mathbb{N}_0^d} a_m h_m$. It follows that

$$U_G(\sigma^w \phi) = (\sum_{m\in\mathbb{N}_0^d} A_{n,m} a_m)_n = A((a_m)_m) = (b_n)_n.$$

the elements $A_{n,m}$ of the matrix A are $A_{n,m} = \langle \sigma^w h_n, h_m\rangle, \ \ n, m \in \mathbb{N}_0^d$.

Characterization of ΨDO's in $\mathcal{S}\widetilde{\Sigma}'^{1/2}(\mathbb{R}^n)$

Theorem

Let A satisfy (31) be invertible so that

$$A_{n,m} = \langle \sigma^w h_n, h_m \rangle, n, m \in \mathbb{N}_0^d, \text{ for some } \sigma \in \widetilde{\Sigma}'^{1/2}(\mathbb{R}^{2d}).$$

Then A^{-1} *determines an element* $f \in \widetilde{\Sigma}'^{1/2}(\mathbb{R}^{2d})$ *so that the corresponding Weyl operator* f^w *satisfies*

$$A^{-1}_{n,m} = \langle f^w h_n, h_m \rangle, n, m \in \mathbb{N}_0^d.$$

Stoeva and PS analysed the action of the matrix type operator on a corresponding space sequence spaces so that in the notation of this paper we have $A((a_m)_n) = ((b_n)_n)$, where

$$b_n = \sum_{m \in \mathbb{Z}} b_{n,m} a_m, s \in \mathbb{N}_0^d.$$

Characterization of ΨDO's in $\mathcal{S}\widetilde{\Sigma}'^{1/2}(\mathbb{R}^n)$

Let us note that for $\phi, \psi \in \widetilde{\Sigma}^{1/2}(\mathbb{R}^d)$ and $A_{n,m} = \langle \sigma^w h_n(x), h_m(x) \rangle$, $n, m \in \mathbb{N}_0^d$ one has

$$\langle \sigma^w \phi(x), \psi(x) \rangle = \langle \sigma^w \sum_{n \in \mathbb{N}_0^d} \langle \phi(x), h_n \rangle h_n, \sum_{m \in \mathbb{N}_0^d} \langle \psi(x), h_m \rangle h_m \rangle = \langle K(x,y), \phi(x)\psi(y) \rangle,$$

where $K(x,y) = \int e^{2\pi i (x-y)\xi} \sigma(\frac{x+y}{2}, \xi) d\xi$ is the element of $\widetilde{\Sigma}'^{1/2}(\mathbb{R}^{2d})$. Therefore,

$$\langle \sigma^w \phi(x), \psi(x) \rangle = \sum_{n \in \mathbb{N}_0^d} \sum_{m \in \mathbb{N}_0^d} A_{n,m} \langle \psi, h_m \rangle \langle \phi, h_n \rangle. \tag{34}$$

This implies

$$\mathcal{F}_2 \sigma(\frac{x+y}{2}, x-y) = K(x,y) = \sum_{n \in \mathbb{N}_0^d} \sum_{m \in \mathbb{N}_0^d} A_{n,m} h_n(x) h_m(y) \Rightarrow$$

$$\langle \mathcal{F}_2 \sigma^w(\frac{x+y}{2}, x-y) = \sum_{n \in \mathbb{N}_0^d} \sum_{m \in \mathbb{N}_0^d} A_{n,m} h_n(x) h_m(y) \Rightarrow$$

$$\mathcal{F}_2 \sigma(p,q) = \sum_{n \in \mathbb{N}^d} \sum_{m \in \mathbb{N}_0^d} A_{n,m} h_n(p + q/2) h_m(p - q/2) \Rightarrow$$

$$\sigma(x,\xi) = \sum_{n \in \mathbb{N}_0^d} \sum_{m \in \mathbb{N}_0^d} A_{n,m} W(h_n, h_m)(x,\xi) \tag{35}$$

Characterization of ΨDO's in $\mathcal{S}\widetilde{\Sigma}'^{1/2}(\mathbb{R}^n)$

In the opposite direction, (35) implies that the action of σ^w defined by (34) is determined by the matrix operator $A = (A_{s,t})_{s,t}$ In the opposite direction, (35) implies that the action of σ^w defined by (34) is determined by the matrix operator $A = (A_{n,m})_{n,m}$ So, we define f as follows

$$f(p,r) = \sum_{n\in\mathbb{N}_0^d} \sum_{m\in\mathbb{N}_0^d} A_{n,m}^{-1} W(h_n, h_m)(p,r)$$

and in the same way as above have that

$$\langle f^w \phi(x), \psi(x)\rangle = \langle \sum_{n\in\mathbb{N}_0^d} \sum_{m\in\mathbb{N}_0^d} A_{n,m}^{-1} h_n(x) h_m(y), \phi(x)\psi(y)\rangle,$$

where $A_{n,m}^{-1} = \langle f^w h_n, h_m\rangle$, $n, m \in \mathbb{N}_0^d$ This implies that with $A_{n,m}^{-1}$ is determined the corresponding ΨDO.

STFT with the window in $\widetilde{\Sigma}^{1/2}$

We define $\mathcal{S}\widetilde{\Sigma}^{1/2}_{h_1,h_2}$, $h_1 > 0$, $h_2 > 0$, as the space of smooth functions Θ over $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$ such that the norm

$$||\Theta||_{h_1,h_2} = \sup_{k\in\mathbb{N}_0,\alpha,\beta\in\mathbb{N}_0^d} \frac{|\mathfrak{R}^k_x \xi^\alpha D^\beta_\xi \Theta(x,\xi)|}{h_2^k h_1^{|\alpha|+|\beta|} k!\alpha!^{1/2}\beta!^{1/2}},$$

is finite. We will show in the appendih that the family of norms with the $\xi^\alpha D^\beta_\xi \mathfrak{R}^k_x \Theta(x,\xi)$ is equivalent to the given one.
We define

$$\mathcal{S}\widetilde{\Sigma}^{1/2} = \bigcup_{h_1>0} \bigcap_{h_2>0} \mathcal{S}\widetilde{\Sigma}^{1/2}_{h_1,h_2}, h_1 > 0, h_2 > 0,$$

and endow it with the corresponding proj-ind topology.

Appendix

Clearly, we have

$$\mathcal{S}\widetilde{\Sigma}^{1/2} = \mathcal{S}^{1/2}\widehat{\otimes}_\pi \widetilde{\Sigma}^{1/2},$$

since both spaces are Montel spaces.

Theorem

Let $g \in \widetilde{\Sigma}^{1/2}$ and $\theta \in \mathcal{S}^{1/2}$. Then $V_g\theta(x,\xi) \in \mathcal{S}\widetilde{\Sigma}^{1/2}$. It is a continuous bilinear mapping $\widetilde{\Sigma}^{1/2} \times \mathcal{S}^{1/2} \to \mathcal{S}\widetilde{\Sigma}^{1/2}$. In particular, the same holds for the Wigner transform.

Appendix

Recall that in one dimensional case

$$\frac{d}{dt}h_n = \sqrt{n/2}h_{n-1} - \sqrt{(n+1)/2}h_{n+1}, ..,$$

$$\frac{d^3}{dt^3}h_n = \sqrt{(n-2)(n-1)n/2^3}h_{n-3} - (\sqrt{(n-1)^2n/2^3} + \sqrt{(n^3/2^3} + \sqrt{n(n+1)^2)/2^3}h_{n-1}$$

$$+(\sqrt{n^2(n+1)/2^3} + \sqrt{(n+1)^3/2^3} + \sqrt{(n+1)(n+2)^2/2^3}h_{n+1} + \sqrt{n+1)(n+2)(n+3)/2^3}h_{n+}$$

We know $\mathcal{F}(t^s h_n(t))(\xi) = (2\pi)^s \frac{d^s}{dx^s} h_n(\xi)$, we obtain

$$t_1^{s_1} \dots t_n^{s_n} \frac{\partial_t^j}{\partial_{x_1}^{j_1} \dots \partial_{x_d}^{j_d}} h_n \leq C^{n+j+s} j!^{1/2} s!^{1/2} \leq C^{n+j+s}(s+j)!^{1/2}.$$

Appendix

Together with the fact that $\mathfrak{R}^k h_n(t) = (2n_1)^k ...(2n_d)^k h_n$ we obtain that the families of norms

$$||\Theta||_{h_1,h_2} = \sup_{k\in\mathbb{N}_0,\alpha,\beta\in\mathbb{N}_0^d} \frac{|\mathfrak{R}_x^k \xi^\alpha D_\xi^\beta \Theta(x,\xi)|}{h_2^k h_1^{|\alpha|+|\beta|} k!\alpha!^{1/2}\beta!^{1/2}},$$

and

$$||\Theta||_{h_1,h_2}^1 = \sup_{k\in\mathbb{N}_0,\alpha,\beta\in\mathbb{N}_0^d} \frac{|\xi^\alpha D_\xi^\beta \mathfrak{R}_x^k \Theta(x,\xi)|}{h_2^k h_1^{|\alpha|+|\beta|} k!\alpha!^{1/2}\beta!^{1/2}}, h_1, h_2 > 0,$$

are equivalent.

III LECTURE Ellipticity and fredholmness within Hörmanders metrics

The main result is that the ellipticity and the Fredholm property of a ΨDO acting on Sobolev spaces in the Weyl-Hörmander calculus are equivalent when the Hörmander metric is geodesically temperate and its associated Planck function vanishes at infinity. The proof is essentially related to the following result that we prove for geodesically temperate Hörmander metrics: if $\lambda \mapsto a_\lambda \in S(1,g)$ is a $\mathcal{C}^N$, $0 \leq N \leq \infty$, map such that each a_λ^w is invertible on L^2 then the mapping $\lambda \mapsto b_\lambda \in S(1,g)$, where b_λ^w is the inverse of a_λ^w, is again of class $\mathcal{C}^N$. At the very end, we give an example to illustrate our main result.

Oscilatory integral

Set of symbols :$a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C|\xi|^{m-\rho\beta+\delta\alpha}, |\xi| > R.$$

Set of phase functions: $\Phi(x,\theta) \in C^\infty(X \times (\mathbb{R}^N \setminus 0))$, real positively homogeneous of order 1 with respect to θ:

$$\Phi(x, t\theta) = t\phi(x,\theta), x \in X, \theta \in \mathbb{R}^N \setminus 0, t > 0,$$

Φ does not have critical points: If $\theta \neq 0$, then $\Phi'_{x,\theta}(x,\theta) \neq 0, x \in X, \theta \in \mathbb{R}^N \setminus 0$.

Oscilatory integral

There exists

$$L = \sum_{j=1}^{N} a_j(x,\theta)\partial_{\theta_j} + \sum_{k=1}^{d} b_k(x,\theta)\partial_{x_k} + c(x,\theta)$$

such that

$$a_j \in S^0_{1,0}(X \times \mathbb{R}^N), c, b_k \in S^{-1}_{1,0}(X \times \mathbb{R}^N)$$

so that

$${}^tLu(x,\theta)e^{i\Phi} = e^{i\Phi},$$

where the transposed operator ${}^tL$ is defined by

$${}^tLu(x,\theta) = -\sum_{j=1}^{N} \partial_{\theta_j}(a_j u) - \sum_{k=1}^{d} \partial_{x_k}(b_k u) + cu,$$

Oscilatory integral

By the use of this operator we have

$$I_\Phi(au) = \int\int e^{i\Phi(x,\theta)} L^k(a(x,\theta)u(x))dxd\theta =$$

$$\lim_{\varepsilon\to 0} I_{\Phi,\varepsilon}(au) = \lim_{\varepsilon\to 0}\int\int \kappa(\varepsilon\theta)e^{i\Phi(x,\theta)}a(x,\theta)u(x)dxd\theta$$

where $\kappa \in C_0^\infty(\mathbb{R}^N)$ and equals one in a neighbourhood of zero.
The second line gives that this definition does not depend on k.

Symbol classes

Symbol classes.
Let φ and Φ be sub-linear and temperate. Let M be a temperate weight. Then $S(M;\varphi,\Phi)$ is the space of smooth functions a (symbols) on $\mathbb{R}^{2d}$ such that for every α,β

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq M(x,\xi)\Phi(x,\xi)^{-|\alpha|}\varphi(x,\xi)^{-|\beta|}$$

We fix φ, Φ and change M...
If $M =< \xi >^m, \varphi = 1, \Phi =< \xi >^\rho$, $0 \leq \rho \leq 1$, one obtains Hörmader's class $S^M_{\rho,0}$.
Family of seminorms is given by

$$||a||_{k,S(M,\varphi,\Phi)} =$$

$$\sup_{|\alpha+\beta|\leq k,(x,\xi)\in\mathbb{R}^{2d}} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)|M(x,\xi)^{-1}\Phi(x,\xi)^{|\alpha|}\varphi(x,\xi)^{|\beta|}$$

Symbol classes

Strong uncertanty principle (SUP)

$$\varphi(x,\xi)\Phi(x,\xi) \geq (1+|x|+|\xi|)^{\delta} \text{ for some } \delta > 0.$$

Put $h(x,\xi) = \varphi^{-1}(x,\xi)\Phi^{-1}(x,\xi)$. We write

$$a(x,\xi) \sim \sum_n a_n(x,\xi) \text{ if } a - \sum_{j=0}^{N-1} a_j \in S(h^N M; \varphi, \Phi), N \in \mathbb{N}.$$

If SUP holds and $a_n \in S(h^n M; \varphi, \Phi)$, then there exists $a \in S(M; \varphi, \Phi)$ such that $a(x,\xi) \sim \sum_n a_n(x,\xi)$. It is uniquely determined modulo Schwartz function.

ΨDO

Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$. Then psudodifferential operator is defined by

$$Op_\tau(a)u(x) = \int e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi)u(y)d'yd'\xi, u \in \mathcal{S}(\mathbb{R}^d)$$

and a_τ is its τ symbol. It maps $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.
Left quantization is obtain for $\tau = 0$; then we put $a(x, D)$ (Hörmander's calculus).
Let $a_1 \in S(M_1; \varphi, \Phi)$, $a_2 \in S(M_2; \varphi, \Phi)$. Then $b(x, D) = a_1(x, D)a_2(x, D)$ is given by

$$b(x, \xi) = e^{iD_y D_\eta} a_1(x, \eta)a_2(y, \xi)|_{\eta=\xi, y=x} \in S(M_1 M_2; \varphi, \Phi)$$

with the asymptotic expansion

$$b(x, \xi) \sim \sum_\alpha (\alpha!)^{-1} \partial_\xi^\alpha a_1(x, \xi) D_x^\alpha a_2(x, \xi)$$

and the corresponding mapping of symbols is continuous.

ΨDOEllipticity and Fredholmness

A symbol a is globally elliptc in $S(M,\varphi,\Phi)$ if $|a(x,\xi)| \geq CM(x,\xi), |x|+|\xi| \geq R > 0$.
Then here exists $b \in S(M^{-1},\varphi,\Phi)$ such that

$$b(x,D)a(x,D) = I + S_1,\ \ a(x,D)b(x,D) = I + S_2,$$

on $\mathcal{S}$ and $\mathcal{S}'$, where S_1 and S_2 are regularizing operators which means they map $\mathcal{S}'$ into $\mathcal{S}$.
A continuous operator $A : H_1 \to H_2$ is called Fredholm if $\ker A$ and $\operatorname{coker} A = H_2/A(H_1)$ are finite dimensional; $\operatorname{ind} A = \dim\ker A - \dim\operatorname{coker} A$.
Recall, If T is Fredholm and K is compact (or has a sufficiently small norm) then $T+K$ is Fredholm with the same index.
Let $T \in \mathcal{B}(H_1,H_2)$ and $S_j \in \mathcal{B}(H_2,H_1), j=1,2$. If $TS_2 = I_2 + K_2,\ S_1T = I_1 + K_1$, where K_j are compact, then T, S_1, S_2 are Fredholm and $indT = -indS_j, j=1,2$.

ΨDOEllipticity and Fredholmness

For $a \in \mathcal{S}(\mathbb{R}^{2n})$, the Weyl quantisation of a is:

$$a^w \varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a((x+y)/2,\xi)\varphi(y)dyd\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n);$$

$a^w : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous (in fact, it extends to a continuous mapping $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$)

Examples

With $D_x = \frac{1}{2\pi i}\partial_x$,

$$(x \cdot \xi)^w = (x \cdot D_x + D_x \cdot x)/2$$

$$L = t_1 \cdot x + t_2 \cdot \xi \Rightarrow L^w u(x) = \int_{\mathbb{R}^{2n}} e^{2i\pi(x-y)\cdot\xi}(t_1(x+y)/2 + t_2\xi)u(y)dyd\xi =$$

$$(t_1 x + t_2 D_x)u(x)$$

$$L^w L^w = (L^2)^w, \;\; L^w a^w = (La + \frac{1}{4\pi i}\{L, a\})^w,$$

$$\{L, a\} = H_L(a) e^{iL^w} = (e^{iL})^w$$

ΨDOEllipticity and Fredholmness

- (the Shubin classes) $a \in \Gamma^m_\rho$ $(0 < \rho \leq 1)$ if

$$|D^\alpha_\xi D^\beta_x a(x,\xi)| \leq C_{\alpha,\beta} \langle (x,\xi)\rangle^{m-\rho(|\alpha|+|\beta|)}, \ \forall (x,\xi) \in \mathbb{R}^{2n};$$

- Let $h, m > 0$. Then, $\Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m)$ the (B)-space of all $a \in C^\infty(\mathbb{R}^{2d})$ for which the norm

$$\sup_{\alpha,\beta\in\mathbb{N}^d} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \frac{\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha|+|\beta|} A_\alpha A_\beta}$$

is finite. Several papers are written within this class....Hypoellipticity, Complex powers, Weyl formula (counting function)...

Symbols

- (the Beals-Fefferman calculus) $a \in S(M; \varphi, \Phi)$ if

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_{\alpha,\beta} M(x,\xi)\varphi(x,\xi)^{-|\beta|}\Phi(x,\xi)^{-|\alpha|}, \ \forall (x,\xi) \in \mathbb{R}^{2n}.$$

  The Shubin calculus when $\varphi(x,\xi) = \Phi(x,\xi) = \langle (x,\xi)\rangle^\rho$, $M(x,\xi) = \langle (x,\xi)\rangle^m$.
  The Hörmander $S_{\rho,\delta}$-calculus, when $\varphi(x,\xi) = \langle \xi \rangle^{-\delta}$ and $\Phi(x,\xi) = \langle \xi \rangle^\rho$, $M(x,\xi) = \langle \xi \rangle^m$.
  The SG-calculus (scattering calculus), when $\varphi(x,\xi) = \langle x \rangle^\rho$ and $\Phi(x,\xi) = \langle \xi \rangle^\rho$, $M(x,\xi) = \langle x \rangle^s \langle \xi \rangle^t$.

Ellipticity and Sobolev spaces

- The ΨDO a^w is called elliptic if $cM(x,\xi) \leq |a(x,\xi)| \leq CM(x,\xi)$ outside of a compact neighbourhood of the origin.
- If the calculus satisfies the strong uncertainty principle, i.e. $\varphi(x,\xi)\Phi(x,\xi) \geq c\langle(x,\xi)\rangle^{\varepsilon}$, $\varepsilon > 0$, (the Shubin calculus, the *SG*-calculus), then elliptic operators have parametrices; i.e. there exists b such that $b^w a^w = \mathrm{Id} + R$, where $R : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ (regularising operator).
- The Sobolev space $H(M) = \{u \in \mathcal{S}'(\mathbb{R}^n) |\, a^w u \in L^2\}$, where a^w is elliptic operator of order M; For the Shubin calculus when $M = \langle(x,\xi)\rangle^m$, $m \in \mathbb{Z}_+$,

$$H(M) = \{u \in \mathcal{S}'(\mathbb{R}^n) |\, x^\beta D^\alpha u \in L^2(\mathbb{R}^n), \text{ for all } |\alpha| + |\beta| \leq m\}.$$

Ellipticity $\Rightarrow$ Fredholmness

- A consequence of the existence of parametrices is that every elliptic operator a^w of order M restricts to a Fredholm mapping $H(M_1) \to H(M_1/M)$, for any M_1 and its index is independent of M_1
- Is the converse true?
  This is true for a number of specific instances of the Weyl-Hörmander calculus (cf. Cordes, Beals and Fefferman, Schrohe ...)

Spectral invariance

- Let a be a 0-order symbol, i.e. bounded by a constant times $M(x,\xi)^0 = 1$. If a^w is bijective operator on $L^2(\mathbb{R}^d)$, is the inverse again a ΨDO?
  A result of Bony and Chemin verifies this for the Weyl-Hörmander calculus (under certain technical assumptions).
- This property of the calculus is called spectral invariance.
- If $\lambda \mapsto a_\lambda$ is $\mathcal{C}^k$-mapping ($0 \leq k \leq \infty$) of 0-order symbols such that each a_λ^w is invertible on $L^2(\mathbb{R}^d)$, is the same true for the mapping of the inverses $\lambda \mapsto b_\lambda$? ($b_\lambda^w a_\lambda^w = \mathrm{Id} = a_\lambda^w b_\lambda^w$)

Hörmander metric

V-an n dimensional real vector space with V' its dual;
$W = V \times V'$ is symplectic with the symplectic form $[(x,\xi),(y,\eta)] = \langle \xi, y\rangle - \langle \eta, x\rangle$ (the phase space).
We denote the points in W with capital letters $X, Y, Z, \ldots$.
Let $X \mapsto g_X$ be a Borel measurable symmetric covariant 2-tensor field on W that is positive definite at every point; we employ the notation $g_X(T) = g_X(T,T)$, $T \in T_X W$.

$g_X^\sigma(T) = \sup_{S \in W\setminus\{0\}} [T,S]^2/g_X(S)$ is called the symplectic dual of g.

Hörmander metric

$X \mapsto g_X$ is a Hörmander metric if:

(i) (slow variation) there exist $C \geq 1$ and $r > 0$ such that for all $X, Y, T \in W$

$$g_X(X - Y) \leq r^2 \Rightarrow C^{-1} g_Y(T) \leq g_X(T) \leq C g_Y(T);$$

(ii) (temperance) there exist $C \geq 1$, $N \in \mathbb{N}$ such that for all $X, Y, T \in W$

$$(g_X(T)/g_Y(T))^{\pm 1} \leq C(1 + g_X^\sigma(X - Y))^N;$$

(iii) (the uncertainty principle) $g_X(T) \leq g_X^\sigma(T)$, for all $X, T \in W$.

Denote $\lambda_g(X) = \inf_{T \in W \setminus \{0\}} (g_X^\sigma(T)/g_X(T))^{1/2}$; it is Borel measurable and $\lambda_g(X) \geq 1$, $\forall X \in W$.

Admissible weights. Symbol classes

A positive Borel measurable function M on W is said to be g-admissible if there are $C \geq 1$, $r > 0$ and $N \in \mathbb{N}$ such that for all $X, Y \in W$

$$g_X(X - Y) \leq r^2 \Rightarrow C^{-1}M(Y) \leq M(X) \leq CM(Y);$$
$$(M(X)/M(Y))^{\pm 1} \leq C(1 + g_X^\sigma(X - Y))^N.$$

$S(M, g)$ is the space of all $a \in \mathcal{C}^\infty(W)$ for which

$$\|a\|_{S(M,g)}^{(k)} = \sup_{l \leq k} \sup_{\substack{X \in W \\ T_1,\ldots,T_l \in W \setminus \{0\}}} \frac{|a^{(l)}(X; T_1, \ldots, T_l)|}{M(X) \prod_{j=1}^{l} g_X(T_j)^{1/2}} < \infty, \quad \forall k \in \mathbb{N}.$$

$S(M, g)$ is an (F)-space.

Examples

Hörmander metric

$$g_{x,\xi} = \frac{|dx|^2}{< \xi >^{-2\delta}} + \frac{|d\xi|^2}{< \xi >^{2\rho}}, g_{x,\xi}(t,\tau) = \frac{|t|^2}{< \xi >^{-2\delta}} + \frac{|\tau|^2}{< \xi >^{2\rho}}$$

gives $S^m_{\rho,\delta}$. When

$$g_{x,\xi} = \frac{|dx|^2}{\varphi^2(x,\xi)} + \frac{|d\xi|^2}{\Phi^2(x,\xi)},$$

$S(M,g)$ reduces to the Beals-Fefferman classes;in this case

$$g^\sigma_{x,\xi} = \Phi^2(x,\xi)|dx|^2 + \varphi^2(x,\xi)|d\xi|^2$$

and $\lambda_g(X) = \varphi(X)\Phi(X)$.

Examples

$$g_{x,\xi} = \frac{|dx|^2}{\Phi(x,\xi)^2} + \frac{|d\xi|^2}{\Psi(x,\xi)^2},\ g_{x,\xi}(t,\tau) = \frac{|t|^2}{\Phi(x,\xi)^2} + \frac{|\tau|^2}{\Psi(x,\xi)^2},$$

If $|\alpha+\beta| = 1$

$$|\triangledown a(X)\cdot T| \leq Cg_X(T)^{1/2} \Leftrightarrow \partial_\xi^\alpha \partial_x^\beta a \leq C_{\alpha,\beta}\Psi^{-\alpha}\Phi^{-\beta}$$

$$\omega(X,r) = \int_{\mathbb{R}^{2n}} \kappa_0(r^{-2}g_Y(X-Y))|g_Y|^{1/2}$$

ΨDOs with symbols in $S(M, g)$

When $a \in S(M, g)$, a^w is continuous operator on $\mathcal{S}(V)$ and it extends to a continuous operator on $\mathcal{S}'(V)$.

The composition $a^w b^w$ is the ΨDO $(a\#b)^w$ where

$$a\#b(X) = \frac{1}{\pi^{2n}} \int_W \int_W e^{-2i[X-Y_1, X-Y_2]} a(Y_1) b(Y_2) dY_1 dY_2.$$

The mapping $\# : S(M_1, g) \times S(M_2, g) \to S(M_1 M_2, g)$ is continuous.

When E is a Hausdorff locally compact topological space $\mathcal{C}(E; S(1, g))$ becomes a unital algebra (with unity $\mathbf{f}(\lambda) = 1$).

When E is a smooth manifold, $\mathcal{C}^k(E; S(1, g))$, $0 \le k \le \infty$, becomes a unital algebra. Furthermore, the smooth vector fields on E are derivations of the unital algebra $\mathcal{C}^\infty(E; S(1, g))$, i.e.

$$X(\mathbf{f}_1 \# \mathbf{f}_2) = X\mathbf{f}_1 \# \mathbf{f}_2 + \mathbf{f}_1 \# X\mathbf{f}_2$$

for $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{C}^\infty(E; S(1, g))$, X a smooth vector field on E.

The Sobolev space $H(M,g)$

Let M be an admissible weight. There exist $a \in S(M,g)$ and $b \in S(1/M,g)$ such that $a\#b = 1 = b\#a$.

The Sobolev space $H(M,g)$ is defined as

$$H(M,g) = \{u \in \mathcal{S}'(V) |\, a^w u \in L^2(V)\}.$$

It is a Hilbert space with inner product $(u,v)_{H(M,g)} = (a^w u, a^w v)_{L^2(V)}$.

$H(1,g) = L^2(V)$.

Additional hypothesis for spectral invariance

The Hörmander metric g is said to be geodesically temperate if there exist $C \geq 1$ and $N \in \mathbb{N}$ such that

$$g_X(T) \leq C g_Y(T)(1 + d(X,Y))^N, \ \ \forall X, Y, T \in W,$$

where $d(\cdot, \cdot)$ stands for the geodesic distance on W induced by the symplectic intermediate $g^\#$.

The metrics of all of the frequently used calculi are geodesically temperate.

Inverse smoothness in $S(1, g)$

Theorem

Assume that g is a geodesically temperate Hörmander metric. Let E be a Hausdorff topological space and $\mathbf{f} : E \to S(1, g)$ a continuous mapping. If for each $\lambda \in E$, $\mathbf{f}(\lambda)^w$ is invertible operator on $L^2(V)$, then there exists a unique continuous mapping $\tilde{\mathbf{f}} : E \to S(1, g)$ such that

$$\tilde{\mathbf{f}}(\lambda)\#\mathbf{f}(\lambda) = \mathbf{f}(\lambda)\#\tilde{\mathbf{f}}(\lambda) = 1, \quad \forall\lambda \in E. \tag{36}$$

If E is a smooth manifold without boundary and $\mathbf{f} : E \to S(1, g)$ is of class $\mathcal{C}^N$, $0 \leq N \leq \infty$, then $\tilde{\mathbf{f}} : E \to S(1, g)$ is also of class $\mathcal{C}^N$.

Equivalence of ellipticity and the Fredholm property

Lemma

Let g be a Hörmander metric satisfying $\lambda_g \to \infty$ and M a g-admissible weight. If $a \in S(M, g)$ is elliptic than for any g-admissible weight M_1, a^w restricts to a Fredholm operator from $H(M_1, g)$ into $H(M_1/M, g)$ and its index is independent of M_1.

Theorem

Let g be a geodesically temperate Hörmander metric satisfying $\lambda_g \to \infty$ and M and M_1 two g-admissible weights. If $a \in S(M, g)$ is such that a^w restricts to a Fredholm operator from $H(M_1, g)$ into $H(M_1/M, g)$ then a is elliptic.

Fedosov-Hörmander integral formula for the index

- All of the above results hold equally well for matrix valued symbols, i.e. for symbols in $S(M, g; \mathcal{L}(\mathbb{C}^\nu))$, $\nu \in \mathbb{Z}_+$.

If the Hörmander metric g satisfies the strong uncertainty principle, i.e. there are $C, \delta > 0$ such that $\lambda_g(X) \geq C(1 + g_0(X))^\delta$, $\forall X \in W$, and $a \in S(1, g; \mathcal{L}(\mathbb{C}^\nu))$ is elliptic, then $\mathrm{ind}\, a^w$ can be given by the Fedosov-Hörmander integral formula. As a consequence of this result, we derive that the same is true if a is an elliptic symbol in $S(M, g; \mathcal{L}(\mathbb{C}^\nu))$.

Remark

If we fix a basis for V and take the dual basis for V', the orientation on W is given by the nonvanishing $2n$-form $d\xi_1 \wedge dx^1 \wedge \ldots \wedge d\xi_n \wedge dx^n$.

Fedosov-Hörmander integral formula for the index

Proposition

Assume that the Hörmander metric g satisfies the strong uncertainty principle and let a be an elliptic symbol in $S(M, g; \mathcal{L}(\mathbb{C}^\nu))$ for some g-admissible weight M. Let D be any compact properly embedded codimension-0 submanifold with boundary in W which contains in its interior the set where a is not invertible. Then

$$\operatorname{ind} a^w = -\frac{(n-1)!}{(2n-1)!(2\pi i)^n} \int_{\partial D} \operatorname{tr}(a^{-1} da)^{2n-1}. \tag{37}$$

The orientation of D is the one induced by W, where the latter has the orientation induced by the symplectic form.

An illustrative example

Consider the operator

$$a^w = -\Delta + \langle x\rangle^{-2s}, \;\; 0 < s < 1.$$

with Weyl symbol $a(x,\xi) = |\xi|^2 + \langle x\rangle^{-2s}$.

- a^w is not elliptic in any of the “classical” symbolic calculi, but ... ;
- a^w is elliptic in the Weyl-Hörmander calculus for an appropriate choice of the metric, namely a is elliptic in $S(M,g)$ with $g_{x,\xi} = \langle x\rangle^{-2}|dx|^2 + \langle x\rangle^{2s}\langle\xi\rangle^{-2}|d\xi|^2$ and $M = a$ (one can prove that g is a Hörmander metric and M is g-admissible);
- the above results imply $a^w : H(M_1,g) \to H(M_1/M,g)$ is Fredholm, for every g-admissible weight M_1 and its index is independent of M_1. In fact, the Fedosov-Hörmander formula gives $\operatorname{ind} a^w = 0$;
- one easily verifies that $\ker a^w \subseteq \mathcal{S}(\mathbb{R}^n)$ and $(a^w\varphi,\varphi)_{L^2} > 0$, $\forall\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$; consequently (as $\operatorname{ind} a^w = 0$) $a^w : H(M_1,g) \to H(M_1/M,g)$ is an isomorphism, for any g-admissible weight M_1;
- one can easily prove that the latter implies that a^w also restricts to a topological isomorphism on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ as well.

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Gelfand-Shilov spaces

Let V be a finite dimensional real vector space and $s, s' \geq 1$. Given a norm $|\cdot|$ on V and $h > 0$ we define the Banach space $\mathcal{S}^{s;h}_{s';h}(V)$ as the space of all $\varphi \in C^\infty(V)$ for which the following norm is finite

$$\sup_{k\in\mathbb{N}^d} \sup_{x\in V} \sup_{t_1,\ldots,t_k\in V} \frac{h^k|\varphi^{(k)}(x;t_1,\ldots,t_k)|e^{|hx|^{1/s'}}}{|t_1|\ldots|t_k|k!^s}.$$

The Gelfand-Shilov spaces of Beurling and Roumieu type are defined as

$$\Sigma^s_{s'}(V) = \varprojlim_{h\to\infty} \mathcal{S}^{s;h}_{s';h}(V) \text{ and } \mathcal{S}^s_{s'}(V) = \varinjlim_{h\to 0^+} \mathcal{S}^{s;h}_{s';h}(V)$$

respectively. Although the definition of $\mathcal{S}^{s;h}_{s';h}(V)$ depends on the choice of the norm $|\cdot|$ on V, the spaces $\Sigma^s_{s'}(V)$ and $\mathcal{S}^s_{s'}(V)$ are independent of $|\cdot|$. Their strong duals $\Sigma'^s_{s'}(V)$ and $\mathcal{S}'^s_{s'}(V)$ are the spaces of tempered ultradistributions of Beurling and Roumieu type respectively.

Metrics on phase space

Let g be a Riemannian metric on the symplectic space or the phase space $W = T^*V$, where V is a d-dimensional real vector space. Then we recall that g is called *slowly varying* if there are positive constants r and C such that

$$g_X(X-Y) \le r^2 \quad \Rightarrow g_X \le Cg_Y, \quad X, Y \in W.$$

By replacing the constant r in suitable way, it follows that

$$g_X(X-Y) \le r^2 \quad \Rightarrow C^{-1}g_Y \le g_X \le Cg_Y, \quad X, Y \in W. \tag{38}$$

We also recall that g is called σ-temperated if it is slowly varying and there are positive constants C and N such that

$$g_X(Z) \le Cg_Y(Z)(1+g_X^\sigma(X-Y))^N, \quad X, Y, Z \in W. \tag{39}$$

If (39) holds, then it follows by duality that for some new choices of $C > 0$ and $N > 0$ we have

$$g_X(Z) \le Cg_Y(Z)(1+g_Y^\sigma(X-Y))^N, \quad X, Y, Z \in W. \tag{40}$$

Metrics

For any Riemannian metric g on W, let

$$\mathfrak{D}_1(g; X, Y) \equiv \min(g_X(X-Y), g_Y(X-Y)), \quad X, Y \in W. \tag{41}$$

Evidently, $\mathfrak{D}_1(g; X, Y)$ is symmetric in X and Y. By modifying the constants C and N in (39), it follows that g is if σ-temperated, then

$$g_X(Z) \le C g_Y(Z)\big(1 + \mathfrak{D}_1(g^\sigma; X, Y)\big)^N, \quad X, Y, Z \in W. \tag{42}$$

Here g^σ is the symplectic dual metric of g, i. e.

$$g_X^\sigma(Z) \equiv \sup_{g_X(Y)\le 1} \sigma(X, Y)^2,$$

where σ is the symplectic form of W. A common condition on g is that it should fullfil *uncertainty principle*, i. e. .

Metrics

$$h_g(X) \equiv \sup_{g_X^\sigma(Z)\leq 1} g_X(Z)^{\frac{1}{2}}, \qquad X \in W. \tag{43}$$

In what follows we recall the various levels of feasibility for Riemannian metrics on phase space.

Definition

Let g be a Riemannian metric on W.

1. *g is called feasible if g is slowly varying and $h_g \leq 1$ everywhere;*
2. *g is called strongly feasible if g is feasible and σ-temperate.*

Remark

Let g be slowly varying on W and let r be the same as in (38). *Then it follows from Theorem 1.4.10 in [**?**] that there is a constant $r_0 > 0$, an integer $N_0 \geq 0$ and a sequence $\{Y_j\}_{j\in\mathbf{N}}$ in W such that the following is true:*

1. *there is a positive number ε such that $g_{Y_j}(Y_j - Y_k) \geq r_0^2$ for every $j, k \in \mathbf{N}$ such that $j \neq k$;*
2. *$W = \bigcup_{j\in\mathbf{N}} B_{g,r}(Y_j)$, where $B_{g,r}(Y)$ is the g_Y-ball $X \in W g_Y(X - Y) < r^2$;*
3. *the intersection of more than N_0 balls U_j is empty.*

In the following definition we introduce the ultra-distributional analogue of σ-temperated metrics. Here and in what follows we let

$$Z_{g,r,Y} \equiv (1 + r^{-2} g_Y(Z))^{\frac{1}{2}}, \qquad Z \in W$$

when g is a Riemannian metric on W, $r > 0$ and $Y \in W$.

Definition

Let $s \geq \frac{1}{2}$, g be a feasible metric on the phase space W and let $C > 0$ and $r > 0$ be chosen such that (38) *holds. Then $\chi \in C_0^\infty(\mathbf{R}; [0,1])$ is called Gevrey-Roumieu adapted (Gevrey-Beurling adapted) with respect to g and order s, if the following conditions are fulfilled:*

1. *χ_0 is non-increasing on $[0,1]$, $\chi(t) = 0$ when $t \geq 1$;*
2. *for some choice of $\{Y_j\}_{j \in \mathbf{N}}$ in Remark 3.1, some $C_0 \geq 1$ and $h > 0$ it holds*

$$\sup_{Y \in W} |\chi_0^{(k)}(r^{-2} g_Y(X-Y))| \leq C_0 h^k k!^s \text{ and } \sum_{j \in \mathbf{N}} \chi_0(r^{-2} g_{Y_j}(X - Y_j)) \geq C_0^{-1},$$

when $X \in W$. Here N is the same as in (40).

Remark

If $s > 1$, then we may always find χ_0 which fullfils the desired properties properties in Definition 3.2. We also observe that if $\{Y_j\}_{j\in\mathbf{N}}$ are the same as in Remark 3.1 and for some $h > 0$ (every $h > 0$), there is a $C_0 > 0$ such that (2) holds, then by the support properties of χ_0, it follows that for some $h > 0$ (every $h > 0$),

$$\sum_{j\in\mathbf{N}} \left(\sup_{Y\in B_{g,r}(Y_j)} |\chi_0^{(k)}(r^{-2}g_Y(X-Y))X - Y_{g,r,Y}^{2k}| \right) \leq C_0 h^k k!^s \tag{44}$$

holds for some choice of $C_0 > 0$.

Let $s \geq 0$, m be a weight function and g be a Riemannian metric on W. For any $a \in C^\infty(W)$, let

$$|a|_k^g(X) \equiv \sup\left(|a^{(k)}(X; T_1, \dots, T_k)|\right), \qquad k \in \mathbf{N}, \tag{45}$$

where the supremum is taken over all $T_1, \dots, T_k \in W$ such that

$$g_X(Y_j) \leq 1, \qquad j = 1, \dots, k. \tag{46}$$

Also let

$$aS_{s;h}(m, g) \equiv \sup_{k \in \mathbf{N}} \sup_{X \in W} \left(\frac{|a|_k^g(X)}{h^k k!^s m(X)}\right). \tag{47}$$

Definition

Let g be a Riemannian metric on W and let $s \geq \frac{1}{2}$.

1. *The set $S_{s;h}(m,g)$ consists of all $a \in C^\infty(W)$ such that $aS_{s;h}(m,g)$ is finite;*
2. *The sets $S_s(m,g)$ and $S_{0,s}(m,g)$ are the inductive respective projective limits of $S_{s;h}(m,g)$ with respect to $h > 0$.*

It is clear that $S_{s;h}(m,g)$ in Definition 3.3 is a Banach space which increases with g, h and m, and that

$$S_s(m,g) = \cup_{h>0} S_{s;h}(m,g) \quad \text{and} \quad S_{0,s}(m,g) = \cap_{h>0} S_{s;h}(m,g). \tag{48}$$

We notice that a bounded set in $S_s(m, g)$ is a bounded set in $S_{s;h}(m, g)$ for some choice of $h > 0$.

Remark

Let g, Y_j and r be as in Remark 3.1. Then there is a bounded set $\{\phi_j\}$ of $S(1, g)$ such that

$$\sum_{j \in N} \phi_j = 1 \quad \text{and} \quad \operatorname{supp} \phi_j \subseteq B_{g,r}(Y_j).$$

*(See e.g.[**?**, ??])*

The following lemma shows that $S_s(m, g)$ and $S_{0,s}(m, g)$ possess suitable algebraic properties.

Lemma

Let $s > 0$, g be a Riemannian metric on the phase space W, m and m_j be a weight on W such that and let $a_j \in S_s(m_j, g)$ ($a_j \in S_{0,s}(m_j, g)$) be such that $a_0 \gtrsim m_0$, $j = 0, 1, 2$. Then the following is true:

1. $a_1 a_2 \in S_s(m_1 m_2, g)$ ($a_1 a_2 \in S_{0,s}(m_1 m_2, g)$);
2. *if in addition* $s \geq \frac{1}{2}$ ($s > \frac{1}{2}$), *then* $\frac{1}{a_0} \in S_s(\frac{1}{m_0}, g)$ ($\frac{1}{a_0} \in S_{0,s}(\frac{1}{m_0}, g)$).

An essential part of our analysis concerns sets of symbols which are bounded with respect to certain families of symbol classes. More precisely, Let g above be fixed, Ω is a set of weight functions and

$$\{a_m\}_{m\in\Omega} \subseteq C^\infty(W)$$

be a set of smooth functions on W indexed by Ω. Then $\{a_m\}_{m\in\Omega}$ is said to be bounded with respect to $\{S_s(m,g)\}_{m\in\Omega}$, if

$$\sup_{m\in\Omega} a_m S_{s;h}(m,g) < \infty \tag{49}$$

holds for some $h > 0$. In the same way, $\{a_m\}_{m\in\Omega}$ is said to be bounded with respect to $\{S_{0,s}(m,g)\}_{m\in\Omega}$, if (49) holds for every $h > 0$.

We have the following proposition. Here $|g_Y|$ is the determinant of g_Y.

Proposition

Let $s > 1$ and g be a slowly varying metric on W. Then there is $r_0 > 0$ such that for every $r \in (0, r_0]$, there exists a family $\{\varphi_Y\}$, $Y \in W$ of functions uniformly bounded in $S_{0,s}(1, g)$ and such that $\operatorname{supp} \varphi_Y \subseteq B_{r,g}(Y)$ and

$$\int_W \varphi_Y(X)|g_Y|^{\frac{1}{2}}\, dY = 1. \tag{50}$$

Confinement. Continuity on ΨDOs

Let γ be a positive-definite quadratic form on W which satisfies $\gamma \leq \gamma^\sigma$. The function $a \in C^\infty(W)$ is said to be $\Sigma^s_{s'}$-(γ, U)-confined (resp., $\mathcal{S}^s_{s'}$-(γ, U)-confined) in $U \subseteq W$ if the seminorms

$$\|a\|^{s,s';h}_{\gamma,U} := \sup_{k \in \mathbf{N}} \sup_{X \in W} \frac{h^k |a|^\gamma_k(X) e^{h^{1/s'} \gamma^\sigma (X-U)^{1/(2s')}}}{k!^s} \tag{51}$$

are finite for all $h > 0$ (resp., for some $h > 0$). Notice that when U is bounded the set of all $\Sigma^s_{s'}$-(γ, U)-confined and $\mathcal{S}^s_{s'}$-(γ, U)-confined functions coincides with $\Sigma^s_{s'}(W)$ and $\mathcal{S}^s_{s'}(W)$ respectively; furthermore, the set of all $\mathcal{S}^{s;h}_{s';h}$-(γ, U)-confined functions coincides with $\mathcal{S}^{s;h}_{s';h}(W)$ defined via the norm $\gamma^{1/2}$. We start with the following simple result which will prove useful in what follows.

Lemma

Assume $s' \geq s \geq 1$. Let γ be a positive-definite quadratic form on W which satisfies $\gamma \leq \gamma^\sigma$ and U a γ-ball with radius at most 1. For each $T \in W$ denote by $L_T : W \to \mathbf{R}$ the linear form $X \mapsto [X, T]$. There exists $c_1 > 1$ such that for every $h > 0$ there exists $C_1 > 0$ such that for every $\mathcal{S}^{s;h}_{s';h}$-(γ, U)-confined function a and $S_1, \dots, S_k, T_1, \dots, T_p \in W$, $k, p \in \mathbf{N}$, the following holds

$$\frac{(h/c_1)^{p+k} \| L_{S_1} \dots L_{S_k} \partial_{T_1} \dots \partial_{T_p} a \|^{s,s';h/c_1}_{\gamma,U}}{k!^{s'} p!^{s} \prod_{j=1}^{p} \gamma(T_j)^{1/2} \prod_{j=1}^{k} \gamma^\sigma(S_j)^{1/2}} \leq C_1 \|a\|^{s,s';h}_{\gamma,U}.$$

Theorem

Let γ_1 and γ_2 be two positive-definite quadratic forms on W which satisfy $\gamma_j \leq \gamma_j^\sigma$ and U_j, $j = 1, 2$, γ_j-ball with radius at most 1. Assume $s' \geq s \geq 1$.

(i) *(Beurling case) For every $h > 0$ there exist $h_1, C > 0$ which depend only on h, s, s' and $\dim W$ (but not on γ_j and U_j, $j = 1, 2$) such that for all $\Sigma_{s'}^{s}$-(γ_j, U_j)-confined functions a_j, $j = 1, 2$, the following holds*

$$\frac{h^p |(a_1 \# a_2)^{(p)}(X) T^p|}{p!^s} \leq C \|a_1\|_{\gamma_1, U_1}^{s,s';h_1} \|a_2\|_{\gamma_2, U_2}^{s,s';h_1} \sum_{l=0}^{p} \gamma_1(T)^{(p-l)/2} \gamma_2(T)^{l/2}$$
$$\cdot \exp\left(-h^{1/s'}((\gamma_1^\sigma \wedge \gamma_2^\sigma)(X - U_1) + (\gamma_1^\sigma \wedge \gamma_2^\sigma)(X - U_2))^{1/(2s')}\right), \quad (52)$$

for all $T, X \in W$, $p \in \mathbf{N}$.

(ii) *(Roumieu case) For every $h_1 > 0$ there exist $h, C > 0$ which depend only on h_1, s, s' and $\dim W$ (but not on γ_j and U_j, $j = 1, 2$) such that for all $\mathcal{S}_{s';h_1}^{s;h_1}$-$(\gamma_j, U_j)$-confined functions a_j, $j = 1, 2$, (52) holds true.*

As a consequence of this theorem and [**?**, Lemma 4.2.3, p. 302] we have the following result.

Corollary

Under the same conditions as in Theorem 3.1, the following holds.

(*i*) *(Beurling case) For every $h>0$ there exist $h_1, C>0$ which depend only on h, s, s' and $\dim W$ (but not on γ_j and U_j, $j=1,2$) such that for all $\Sigma^s_{s'}$-(γ_j, U_j)-confined functions a_j, $j=1,2$, the following holds*

$$\frac{h^p|a_1\#a_2|_p^{\gamma_1}(X)}{p!^s} \leq C\|a_1\|^{s,s';h_1}_{\gamma_1,U_1}\|a_2\|^{s,s';h_1}_{\gamma_2,U_2}\sup_{l\leq p}\sup_{T\in W}(\gamma_2(T)/\gamma_1(T))^{l/2}$$
$$\cdot\exp\left(-h^{1/s'}((\gamma_1^\sigma\wedge\gamma_2^\sigma)(X-U_1)+(\gamma_1^\sigma\wedge\gamma_2^\sigma)(X-U_2))^{1/(2s')}\right),\ \forall X\in W,\ p\in\mathbb{N}. \tag{53}$$

(*ii*) *(Roumieu case) For every $h_1>0$ there exist $h, C>0$ which depend only on h, s, s' and $\dim W$ (but not on γ_j and U_j, $j=1,2$) such that for all $\mathcal{S}^{s;h_1}_{s';h_1}$-$(\gamma_j, U_j)$-confined functions a_j, $j=1,2$,* (53) *holds true.*

Let now g be a strongly feasible metric on W. Assume that for each $Y \in W$ we are given an $\psi_Y \in \Sigma^s_{s'}(W)$ (resp., $\psi_Y \in \mathcal{S}^s_{s'}(W)$). We say that the family $\{\psi_Y\}_{Y\in W}$ is uniformly $\Sigma^s_{s'}$-$(g_Y, B_{g,r}(Y))$-confined (resp., uniformly $\mathcal{S}^s_{s'}$-$(g_Y, B_{g,r}(Y))$-confined) if

$$\sup_{Y\in W} \|\psi_Y\|^{s,s';h}_{g_Y,B_{g,r}(Y)} < \infty,\ \forall h > 0,\ \text{(resp. for some } h > 0).$$

In the Roumieu case, when we want to emphasise the particular h for which this holds we will say that $\{\psi_Y\}_{Y\in W}$ is $\mathcal{S}^{s;h}_{s';h}$-$(g_Y, B_{g,r}(Y))$-confined. When $s' \geq s > 1$, if $\operatorname{supp}\psi_Y \subseteq B_{g,r}(Y)$, $\forall Y \in W$, then $\{\psi_Y\}_Y \in W$ is bounded in $S_{0,s}(1, g)$ if and only if $\{\psi_Y\}_{Y\in W}$ is uniformly $\Sigma^s_{s'}$-$(g_Y, B_{g,r}(Y))$-confined and $\{\psi_Y\}_{Y\in W}$ is bounded in $S_s(1, g)$ if and only if $\{\psi_Y\}_{Y\in W}$ is uniformly $\mathcal{S}^s_{s'}$-$(g_Y, B_{g,r}(Y))$-confined (in the Roumieu case, a subset of $S_s(m, g)$ is bounded if and only if it is contained in some $S_{s;h}(m, g)$ and bounded there since $S_s(m, g)$ is a regular (LF)-space).

Furthermore, $\sup_{Y\in W}\|\psi_Y\|_{S_{s;h}(1,g)} = \sup_{Y\in W}\|\psi_Y\|^{s,s';h}_{g_Y,B_{g,r}(Y)}$. A particular (and for what follows, a very important) instance of such family is the one constructed in Proposition 3.1.

Our next goal is to prove the continuity on $\Sigma^s_{s'}(V)$ and $\mathcal{S}^s_{s'}(V)$ of pseudodifferential operators with symbols in $S_{0,s}(m,g)$ and $S_s(m,g)$ respectively. For this purpose, we impose the following assumptions on the metric g and the weight m.

Definition

Let g be a strongly feasible metric and $s > 1$. A positive and Borel measurable function m on W is said to be slowly varying weight with respect to g if there exists $C \geq 1$ and $0 < r < 1$ such that

$$g_X(X - Y) \leq r^2 \Rightarrow C^{-1}m(Y) \leq m(X) \leq Cm(Y), \ \forall X, Y \in W.$$

We say m is Σ_s^s-temperate (resp., $\mathcal{S}_s^s$-temperate) with respect to g if there exists $C, h > 0$ (resp., for every $h > 0$ there exists $C > 0$) such that

$$m(X) \leq Cm(Y)e^{h^{1/s}(g_X^\sigma \wedge g_Y^\sigma (X-Y))^{1/(2s)}}, \ \ \forall X, Y \in W. \tag{54}$$

Finally, we say m is Σ_s^s-admissible (resp., $\mathcal{S}_s^s$-admissible) if it is slowly varying and Σ_s^s-temperate (resp., $\mathcal{S}_s^s$-temperate) with respect to g.

Remark

*Given a strongly feasible metric g, every admissible weight m for g (in Hörmander sense) is both Σ^s_s-admissible and $\mathcal{S}^s_s$-admissible for any $s > 1$; this is a direct consequence of [**?**, Remark 2.2.16, p. 76].*

Lemma

Let g be a strongly feasible metric, m a slowly varying weight with respect to g and $s > 1$, $s' > 0$. Assume $\{\varphi_Y\}_{Y \in W}$ is a uniformly $\Sigma^s_{s'}$-$(g_Y, B_{g,r}(Y))$-confined (resp., uniformly $\mathcal{S}^{s;h}_{s';h}$-$(g_Y, B_{g,r}(Y))$-confined, $h > 0$) family such that $\operatorname{supp} \varphi_Y \subseteq B_{g,r}(Y)$, $\forall Y \in W$. There exist $c_1 > 1$ which depends only on the structure constants of the metric such that for any $a \in S_{0,s}(m, g)$ (resp., $a \in S_{s;h}(m, g)$) and any $T_1, \dots, T_k \in W$, $k \in \mathbf{N}$, the following holds

$$\frac{(h/c_1)^k \|\partial_{T_1} \dots \partial_{T_k}(a\varphi_Y)\|^{s,s';h/c_1}_{g_Y, B_{g,r}(Y)}}{k!^s m(Y) \prod_{j=1}^k g_Y(T_j)^{1/2}} \leq C_1 \|a\|_{S_{s;h}(m,g)} \sup_{Y \in W} \|\varphi_Y\|^{s,s';h}_{g_Y, B_{g,r}(Y)}, \ \forall Y \in W,$$

where $C_1 > 0$ is the constant from the slow variation of m.

In what follows, we will frequently use the following function on $W \times W$. Let g be a strongly feasible metric and $0 < r_0 < 1$ the constant from the slow variation of g. For $0 < r \leq r_0$, define $\delta_r : W \times W \to [1, \infty)$ by

$$\delta_r(X, Y) = 1 + (g^\sigma_X \wedge g^\sigma_Y)(B_{g,r}(X) - B_{g,r}(Y)).$$

We refer to [**?**, Section 2.2.6] for its properties; here we just mention that it is Borel measurable on $W \times W$.

Definition

Let g be a strongly feasible metric on W and $s' \geq s > 1$. We say that g is (s, s')-strongly feasible if it satisfies the following assumption: there exists $C_1 \geq 1$ such that

$$\frac{g_X(T)}{g_Y(T)} \leq C_1 \lambda_g(X)^2 \lambda_g(Y)^2 \left(1 + g^\sigma_X \wedge g^\sigma_Y(X - Y)\right)^{(s'-s)/s'}, \quad \textit{for all}\, X, Y, T \in W. \quad (55)$$

It is good to have in mind that if g is (s, s')-strongly feasible metric on W, the condition (55) immediately implies

$$\frac{g_Y^\sigma(T)}{g_X^\sigma(T)} \leq C_1 \lambda_g(X)^2 \lambda_g(Y)^2 \left(1 + g_X^\sigma \wedge g_Y^\sigma(X - Y)\right)^{(s'-s)/s'}, \quad \text{for all } X, Y, T \in W. \tag{56}$$

Lemma

Let g be a (s, s')-strongly feasible metric on W, $s' \geq s > 1$. There exists $C_1 > 1$ such that for all $0 < r \leq r_0$ the following estimate holds:

$$\frac{g_X}{g_Y} \leq C_1 \lambda_g(X)^2 \lambda_g(Y)^2 \delta_r(X, Y)^{(s'-s)/s}, \quad \textit{for all } X, Y \in W.$$

Remark

In the same way one can proof the following. Let g be a strongly feasible metric with $r_0 < 1$ *and* $C_0 > 1$ *being the constants from the slow variation and m a slowly varying weight with* $r' < 1$ *and* $C' > 1$ *the constants from its slow variation. If m satisfies* (54) *with* $h, C > 0$ *then*

$$\frac{m(X)}{m(Y)} \leq CC'^2 e^{(hC_0)^{1/s}\delta_r(X,Y)^{1/(2s)}}, \quad \textit{for all } X, Y \in W, \ r \leq \min\{r_0, r'\}.$$

Remark

If the strongly feasible metric g on $\mathbf{R}^{2n}$ *is of the form*

$$g_{x,\xi} = \varphi(x,\xi)^{-2}|dx|^2 + \Phi(x,\xi)^{-2}|d\xi|^2 \tag{57}$$

where both φ *and* Φ *are bounded from below by a constant* $c > 0$ *then g is* (s,s)*-strongly feasible for any* $s > 1$ *and consequently it is* (s,s')*-strongly feasible for any* $s' \geq s > 1$. *This is straightforward to verify, one just needs to keep in mind that* $\lambda_g = \varphi\Phi$. *Conversely, if g is a* (s,s)*-strongly feasible metric on W then* (56) *implies* $g^\sigma_Y(T) \leq C_1 g^\sigma_X(T)\lambda_g(X)^2\lambda_g(Y)^2$, *for all* $X, Y, T \in W$. *Consequently*

$$g_Y(T) \leq \lambda_g(Y)^{-2} g^\sigma_Y(T) \leq C_1\lambda_g(0)^2 g^\sigma_0(T), \ \forall Y, T \in W,$$

i.e. the metric is bounded by a single positive-definite quadratic form on W. If g is a (s,s)*-strongly feasible metric on* $\mathbf{R}^{2n}$ *of the form* (57) *then both* φ *and* Φ *are bounded from below by a constant.*

Remark

(This remark is for us) I know that the condition (55) *may seem awkward and ad-hoc but I think that something additional (besides the strong feasibility of g) has to be assumed on the metric so the* Ψ*DOs will be continuous on* $\Sigma_{s'}^{s'}(V)$ *and* $\mathcal{S}_{s'}^{s'}(V)$*. As far as I see, the symbols have to be more regular than the regularity of* $\Sigma_{s'}^{s'}(V)$ *and* $\mathcal{S}_{s'}^{s'}(V)$ *and this is closely connected with how much the metric "grows with respect to its symplectic dual". In the condition* (55) *this is the factor* $(1 + g_X^{\sigma} \wedge g_Y^{\sigma}(X - Y))^{(s'-s)/s'}$*, where s is the order of regularity of the symbols; namely the symbol classes which will generate continuous* Ψ*DOs on* $\Sigma_{s'}^{s'}(V)$ *and* $\mathcal{S}_{s'}^{s'}(V)$ *will be* $S_{0,s}(m,g)$ *and* $S_s(m,g)$ *respectively (see the theorem below).*

As far as I see, a typical place where this difference of regularity will appear is when g is the metric of the Hörmander $S_{\rho,\delta}$-calculus, i.e. $g_{x,\xi} = \langle\xi\rangle^{2\delta}|dx|^2 + \langle\xi\rangle^{-2\rho}|d\xi|^2$, with $0 \le \delta \le \rho \le 1$, $\delta < 1$. If one takes a Gevrey regular symbol in the $S_{\rho,\delta}$-classes, namely $a \in C^\infty(\mathbf{R}^{2n})$ which satisfies

$$|D_x^\beta D_\xi^\alpha a(x,\xi)| \le Ch^{|\alpha|+|\beta|}\alpha!^s\beta!^s m(x,\xi)\langle\xi\rangle^{\delta|\beta|-\rho|\alpha|},$$

then as far as I see for $a(x,D)$ to be continuous on $\mathcal{S}_{s'}^{s'}(\mathbf{R}^{2n})$ the condition $s + \delta s' \le s'$ (i.e. $\delta \le (s'-s)/s'$) appears when one wants to estimate

$$\frac{1}{(2\pi)^n}\int e^{ix\xi}D_x^\alpha a(x,\xi)\mathcal{F}u(\xi)d\xi$$

by "oscillatory integral" techniques; the above factor appears as a summand in $D_x^\alpha(a(x,D)u(x))$. This "problem" appears due to the regularity of a and not because of the growth $m(x,\xi)$; in fact the same happens when a is a "0-order symbol", i.e. $m(x,\xi)=1$. This assumption on additional regularity of the symbol with respect to the regularity of the test space is not needed in the calculus that me and Stevan did (in this case g is the Shubin metric), in the calculus of Marco Cappiello (in this case g is the scattering metric (the SG-metric), i.e the $g_{x,\xi}=\langle x\rangle^{-2\rho}|dx|^2+\langle\xi\rangle^{-2\rho}|d\xi|^2$) and in the paper of Joachim (in this case g is the Euclidean metric).

This is because in all of these cases the metric satisfies (55) for $s' = s$ because it is bounded by a single positive-definite quadratic form (see Remark 3.6). I spent quite some time trying to find a geometric condition which will express the condition $\delta \leq (s' - s)/s'$ in the $S_{\rho,\delta}$-calculus and (55) is the closes I've got so that the ΨDOs will be continuous; maybe we should try to find a better/ less restrictive one. In my opinion (55) looks good because of the following reasons. When $s' = s$ it forces the metric to be bounded by a single positive-definite quadratic form which feels like it is necessary for the continuity of the ΨDOs; in fact because of Remark 3.6 this is "almost equivalent" to the metric being bounded by a positive-definite quadratic form (at least when $g_{x,\xi} = \varphi(x,\xi)^{-2}|dx|^2 + \Phi(x,\xi)^{-2}|d\xi|^2$).

Secondly, it is geometric (it does not depend on local coordinates) and is symmetric with respect to X and Y. However, in the case of the $S_{\rho,\delta}$-calculus (56) is equivalent to $\delta/(1-\delta) \le (s'-s)/s'$ and not to $\delta \le (s'-s)/s'$ (which are “close” when $\delta \to 0^+$ but very different when $\delta \to 1^-$). This follows from the following claim.
Claim. Let $g_{x,\xi} = \langle\xi\rangle^{2\delta}|dx|^2 + \langle\xi\rangle^{-2\rho}|d\xi|^2$ with $0 \le \delta \le \rho \le 1$, $\delta < 1$. Then g satisfies (55) if and only if $\delta/(1-\delta) \le (s'-s)/s'$.

COMMENT

Proof. Notice that $g^{\sigma}_{x,\xi} = \langle\xi\rangle^{2\rho}|dx|^2 + \langle\xi\rangle^{-2\delta}|d\xi|^2$ and

$$g^{\sigma}_{x,\xi} \wedge g^{\sigma}_{y,\eta}(x-y,\xi-\eta) = \frac{2\langle\xi\rangle^{2\rho}\langle\eta\rangle^{2\rho}}{\langle\xi\rangle^{2\rho}+\langle\eta\rangle^{2\rho}}|x-y|^2 + \frac{2}{\langle\xi\rangle^{2\delta}+\langle\eta\rangle^{2\delta}}|\xi-\eta|^2;$$

furthermore, $\lambda_g(x,\xi) = \langle\xi\rangle^{\rho-\delta}$. The metric g satisfies (55) if and only if

$$\frac{\langle\xi\rangle^{\rho}}{\langle\eta\rangle^{\rho}} \leq C\langle\xi\rangle^{\rho-\delta}\langle\eta\rangle^{\rho-\delta}\left(1+\frac{\langle\xi\rangle^{\rho}\langle\eta\rangle^{\rho}}{\langle\xi\rangle^{\rho}+\langle\eta\rangle^{\rho}}|x-y| + \frac{|\xi-\eta|}{\langle\xi\rangle^{\delta}+\langle\eta\rangle^{\delta}}\right)^{(s'-s)/s'} \tag{58}$$

$$\frac{\langle\xi\rangle^{\delta}}{\langle\eta\rangle^{\delta}} \leq C\langle\xi\rangle^{\rho-\delta}\langle\eta\rangle^{\rho-\delta}\left(1+\frac{\langle\xi\rangle^{\rho}\langle\eta\rangle^{\rho}}{\langle\xi\rangle^{\rho}+\langle\eta\rangle^{\rho}}|x-y| + \frac{|\xi-\eta|}{\langle\xi\rangle^{\delta}+\langle\eta\rangle^{\delta}}\right)^{(s'-s)/s'} \tag{59}$$

COMMENT

Assume $\delta/(1-\delta) \leq (s'-s)/s'$. To prove g satisfies (55), it is enough to only verify (58) for $|\xi| \geq 2|\eta|$ (when $|\xi| \leq 2|\eta|$ the left hand side is bounded by 2). Denoting by I the left hand side of (58) without the constant C, we infer

$$I \geq 4^{-\delta/(1-\delta)} \langle \xi \rangle^{\rho-\delta} \langle \eta \rangle^{\rho-\delta} \left(1 + \frac{|\xi|}{\langle \xi \rangle^{\delta}}\right)^{\delta/(1-\delta)} \geq 4^{-\delta/(1-\delta)} \langle \xi \rangle^{\rho}.$$

Assume now the g satisfies (58) and (59). Specialising $\eta = 0$ and $x = y$ in (58) we deduce

$$\langle \xi \rangle^{\rho} \leq C \langle \xi \rangle^{\rho-\delta} \left(1 + \frac{|\xi|}{\langle \xi \rangle^{\delta} + 1}\right)^{(s'-s)/s'} \leq 2^{(s'-s)/s'} C \langle \xi \rangle^{\rho-\delta} \langle \xi \rangle^{(1-\delta)(s'-s)/s'},$$

and thus $\delta/(1-\delta) \leq (s'-s)/s'$.

COMMENT

On a side note, I've tried a couple of different (geometric) conditions on the metric to capture the condition $\delta \leq (s' - s)/s'$ for the $S_{\rho,\delta}$-calculus which will also force the metric to be bounded by a single quadratic form when $s' = s$ and the ΨDOs will be continuous. The most promising were these two:

$$\begin{aligned} g_Y(T)/g_X(T) &\leq C\lambda_g(Y-X)^2 \left(1 + g_X^\sigma \wedge g_Y^\sigma(X-Y)\right)^{(s'-s)/s'}, \\ g_Y(T)/g_X(T) &\leq C\lambda_g(Y-X)^2 \left(1 + \lambda_g(Y-X) g_X^\# \wedge g_Y^\#(X-Y)\right)^{(s'-s)/s'}. \end{aligned}$$

However, both of them are essentially the same as (55) in the $S_{\rho,\delta}$-calculus; I went with (55) because it is symmetric with respect to X and Y (in general $\lambda_g(X-Y) \neq \lambda_g(Y-X)$).

Before we prove the continuity of ΨDOs on the test space we give the following result concerning the mapping properties of the operator $e^{it\langle D_x, D_\xi\rangle}$, $t \in \mathbf{R}\backslash\{0\}$.

Lemma

Let $|\cdot|$ be a norm on W and for $h > 0$ and $s' \geq s \geq 1$, define the space $\mathcal{S}^{s;h}_{s';h}(W)$ via $|\cdot|$. Then for every $t \in \mathbf{R}\backslash\{0\}$ there exists $c > 1$ such that for all $h > 0$, $e^{it\langle D_x, D_\xi\rangle} : \mathcal{S}^{s;h}_{s';h}(W) \to \mathcal{S}^{s;h/c}_{s';h/c}(W)$ is well defined and continuous.

CONTINUITY-MAIN

Assume g is an (s,s')-strongly feasible metric on W with $s' \geq s > 1$ and m a $\Sigma_{s'}^{s'}$-admissible (resp., $\mathcal{S}_{s'}^{s'}$-admissible) weight. Let $\{\varphi_Y\}_{Y\in W}$ be a uniformly $\Sigma_{s'}^{s}$-$(g_Y, B_{g,r}(Y))$-confined (resp., uniformly $\mathcal{S}_{s'}^{s}$-$(g_Y, B_{g,r}(Y))$-confined) family of nonnegative functions which satisfy the following three conditions:

- (i) $\operatorname{supp}\varphi_Y \subseteq B_{g,r}(Y)$, $\forall Y \in W$,
- (ii) the mapping $(X,Y) \mapsto \varphi_Y^{(p)}(X; T_1, \ldots; T_p)$, $W \times W \to \mathbf{C}$, is Borel measurable, for all $T_1, \ldots, T_p \in W$, $p \in \mathbf{N}$.
- (ii) $\int_W \varphi_Y(X)|g_Y|^{1/2} dY = 1$, $\forall X \in W$.

Then for each $a \in S_{0,s}(m,g)$ and $\psi \in \Sigma_{s'}^{s'}(V)$ (resp. $a \in S_s(m,g)$ and $\psi \in \mathcal{S}_{s'}^{s'}(V)$), the mapping $Y \mapsto L_1^w \ldots L_p^w (a\varphi_Y)^w \psi |g_Y|^{1/2}$, $W \mapsto L^2(V)$, is strongly Borel measurable and Bochner integrable on $L^2(V)$ for any linear forms $L_j : W \to \mathbf{R}$, $j = 1, \ldots, p$, $p \in \mathbf{N}$, and

$$L_1^w \ldots L_p^w a^w \psi = \int_W L_1^w \ldots L_p^w (a\varphi_Y)^w \psi |g_Y|^{1/2} dY.$$

Furthermore, $a^w\psi \in \Sigma_{s'}^{s'}(V)$ (resp., $a^w\psi \in \mathcal{S}_{s'}^{s'}(V)$) and the bilinear mapping

$$(a,\psi) \mapsto a^w\psi \quad , \quad S_{0,s}(m,g) \times \Sigma_{s'}^{s'}(V) \to \Sigma_{s'}^{s'}(V),$$

$$(\text{resp., } (a,\psi) \mapsto a^w\psi \quad , \quad S_s(m,g) \times \mathcal{S}_{s'}^{s'}(V) \to \mathcal{S}_{s'}^{s'}(V))$$

is continuous.