Elliptic boundary problems for edge-degenerate pseudodifferential operators

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Plan of the lectures

What is singular analysis?

In my lecture, I will provide an introduction to singular analysis by discussing one substantial example.

- Singular analysis is the analysis of PDEs on spaces (or spacetimes) with geometric singularities using methods from microlocal analysis.
- Typical singularities are conic points, edges, and corners, but also boundaries.
- We will meet (special classes of) both cone-degenerate and edge-degenerate pseudodifferential operators.

Some history of singular analysis and references

- Kondratiev (1967).
- Melrose-Mendoza (preprint, 1983), Schulze (1991), Melrose (1993): Cone-degenerate pseudodifferential operators, cone calculus/b-calculus.
- Schulze (1988): Edge-degenerate pseudodifferential operators, edge calculus.
- Schulze (1994, 1998), Harutyunyan-Schulze (2008).

We will work on a $\mathscr{C}^\infty$ compact manifold X with non-empty boundary, $Y = \partial X$.

- The local model near a boundary point is the closed half-space $\overline{\mathbb{R}}^{1+d} = \{(x, y) \mid x \geq 0,\ y \in \mathbb{R}^d\}$.
- On X, we can think of x as a boundary defining function and $y \in Y$ as a generic point in ∂X.

Edge-degenerate operators

Definition

A differential operator $A \in \mathrm{Diff}^m(X \setminus Y)$ is said to be edge-degenerate if, near the boundary, it takes the form

$$A = x^{-m} a(x, y, xD_x, xD_y),$$

where $a = a(x, y, \xi, \eta)$ is $\mathscr{C}^\infty$ up to $x = 0$.

Exercise

Show that differential operators A in $\mathrm{Diff}^m(X)$, i.e., with coefficients $\mathscr{C}^\infty$ up to $x = 0$, are edge-degenerate.

We will generalize edge-degenerate differential operators to edge-degenerate pseudodifferential operators.

Important examples include:

- The spectrally defined fractional Laplacian $(-\Delta)^s$ in a $\mathscr{C}^\infty$ bounded domain $\Omega \subset \mathbb{R}^n$ (ongoing project with N. Popivanov (Sofia) and Z.-P. Ruan).
- Consider the Zaremba problem

$$\Delta u = f(x) \ \text{in } \Omega, \quad u\big|_{\Gamma_D} = g(y), \quad \frac{\partial u}{\partial \nu}\Big|_{\Gamma_N} = h(y),$$

where $\partial\Omega = \Gamma_D \sqcup \Gamma_N \sqcup \Sigma$ and the interface $\Sigma = \partial\Gamma_D = \partial\Gamma_N$ is smooth. Boundary reduction to Γ_N (by comparising the Zaremba problem to the Dirichlet problem) leads to an an elliptic edge-degenerate operator on $\Gamma_N \sqcup \Sigma$ (D.-C. Chang, N. Habal, and B.-W. Schulze, JPDOA 2014).

Main objective

Want to understand the analytic structure of elliptic boundary value problems for edge-degenerate differential (and pseudodifferential) operators on X.

- Boutet de Monvel's calculus for handling elliptic pseudodifferential boundary value problems (with coefficients $\mathscr{C}^\infty$ up to the boundary) serves as a blueprint. I will recall stuff around this calculus shortly.
- The transmission property will play a crucial role.
- The transmission property is reflected by an appropriate choice of conormal symbols. More on this later.

In our analysis, we need to perform the following steps:

- Understand the asymptotics of solutions to elliptic edge-degenerate problems as $x \to 0$.
- Introduce a class of Sobolev spaces – henceforth referred to as Sobolev spaces with asymptotics – that incorporate such asymptotic information.
- Specify conditions on edge-degenerate operators – i.e., generalizations of the transmission property – that guarantee that operators act continuously between Sobolev spaces with prescribed asymptotics.
- Finally, develop a corresponding pseudodifferential calculus with all the necessary elements, as there are
  - a composition staying in the calculus,
  - a principal symbol map,
  - a parametrix construction for the elliptic elements, and
  - ellipticity being equivalent to the Fredholm property.

Some history again

- Calculus for pseudodifferential boundary problems: L. Boutet de Monvel (1971). See also the monographs by S. Rempel and B.-W. Schulze (1985), G. Grubb (1996), B.-W. Schulze (1998).
- Edge-degenerate pseudodifferential operators: S. Rempel and B.-W. Schulze (1989), B.-W. Schulze (1988, 1998).
- Construction of the conormal symbols: I. Witt (2002, 2007).
- Corresponding cone calculus ($\dim X = 1$): X.-C. Liu (2000), X.-C. Liu and I. Witt (2004).
- Function spaces $H^{s,\delta}_{P,\theta}(X)$: Z.-P. Ruan and I. Witt (preprint 2021).

An example

Let $\Omega \subset \mathbb{R}^3$ be a bounded $\mathscr{C}^\infty$ domain. We consider the Dirichlet problem

$$-\Delta u = f(x) \text{ in } \Omega, \quad \gamma_0 u = u\big|_{\partial\Omega} = g(y).$$

This can be written as $\mathcal{A}u = \binom{f}{g}$, where

$$\mathcal{A} = \begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix} : \mathscr{C}^\infty(\overline{\Omega}) \to \begin{matrix} \mathscr{C}^\infty(\overline{\Omega}) \\ \oplus \\ \mathscr{C}^\infty(\partial\Omega) \end{matrix}.$$

It is well-known that $\mathcal{A}$ is invertible, the inverse being of the form

$$\mathcal{A}^{-1} = \begin{pmatrix} P_+ + G & K \end{pmatrix}.$$

- P is the Newton potential, i.e., $Pf = \frac{1}{4\pi|x|} * f$. Note that $P \in \Psi_{\mathrm{cl}}^{-2}(\mathbb{R}^3)$ with principal symbol $\sigma_\psi^{-2}(P)(x,\xi) = |\xi|^{-2}$.
- $P_+ = r_+ P e_+$, where
  - $e_+ \colon L^2(\Omega) \to L^2(\mathbb{R}^3)$ is extension (from Ω to $\mathbb{R}^3$) by zero,
  - $r_+ \colon L^2(\mathbb{R}^3) \to L^2(\Omega)$ is restriction (from $\mathbb{R}^3$) to Ω.

  Hence,

$$P_+ f(z) = \frac{1}{4\pi} \int_\Omega \frac{f(x')}{|x - x'|}\,\mathrm{d}x', \quad x \in \Omega.$$

- $K \colon \mathscr{C}^\infty(\partial\Omega) \to \mathscr{C}^\infty(\overline{\Omega})$ is a potential operator.
- $G \in \Psi^{-\infty}(\Omega)$ is a (singular) Green operator.
  - G is the boundary correction to P_+ (we need $Gf\big|_{\partial\Omega} = -Pf\big|_{\partial\Omega}$).
  - G is essentially of the form $K \circ T$, where K is potential operator and $T \colon \mathscr{C}^\infty(\overline{\Omega}) \to \mathscr{C}^\infty(\partial\Omega)$ is a trace operator (such as γ_0).

Altogether, we expect a calculus $\Psi^{m;d}(X; E, F)$ with operators like

$$\begin{pmatrix} A_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} \mathscr{C}^\infty(X) \\ \oplus \\ \mathscr{C}^\infty(Y; E) \end{matrix} \to \begin{matrix} \mathscr{C}^\infty(X) \\ \oplus \\ \mathscr{C}^\infty(Y; F) \end{matrix},$$

where E, F are $\mathscr{C}^\infty$ vector bundles over Y and S is a pseudodifferential operator in the boundary.

Some history and references

- Vishik and Eskin (1964-68), Eskin (1973).
- Boutet de Monvel (1971): added the transmission condition.
- Rempel and Schulze (1982), Grubb (1996).
- Schulze (1998): Boutet de Monvel's calculus reformulated as an edge calculus.

Definition

A pseudodifferential operator $A \in \Psi^m_{\mathrm{cl}}(\mathbb{R}^{1+d})$ is said to have the transmission property with respect to $x = +0$ if

$$A_+ : \mathscr{C}^\infty_{\mathrm{b}}(\overline{\mathbb{R}}^{1+d}_+) \to \mathscr{C}^\infty_{\mathrm{b}}(\overline{\mathbb{R}}^{1+d}_+).$$

Theorem

Let $A = a(x, D) \in \Psi^m_{\mathrm{cl}}(\mathbb{R}^{1+d})$, where $m \in \mathbb{R}$. Then A possesses the transmission property if and only if, for all $j \in \mathbb{N}_0$ and $(\alpha, \beta) \in \mathbb{N}_0^{2+2d}$,

$$\partial^\alpha_{x,y}\partial^\beta_{\xi,\eta} a_{(m-j)}(0, y, -1, 0) = \mathrm{e}^{\mathrm{i}\pi(m-j-|\beta|)}\partial^\alpha_{x,y}\partial^\beta_{\xi,\eta} a_{(m-j)}(0, y, 1, 0).$$

Recall that edge-degenerate operators are of the form

$$A = x^{-m} a(x, y, xD_x, xD_y).$$

The principal symbolic structure is as follows: We have

- the principal (pseudodifferential or inner) symbol

$$\sigma_\psi^m(A)(x, y, \xi, \eta) = a_{(m)}(x, y, \xi, \eta), \quad (x, y, \xi, \eta) \in T^*X \setminus 0,$$

- the principal boundary symbol

$$\sigma_\partial^m(A)(y, \eta) = x^{-m} a(0, y, xD_x, x\eta), \quad (y, \eta) \in T^*Y \setminus 0.$$

  Notice that $\sigma_\partial^m(A)$ takes values in the cone-degenerate operators along the inner normal.

There is a compatibility condition between both principal symbols in the sense that the (pointwise taken) principal symbol of $\sigma_\partial^m(A)(y, \eta)$ equals $a_{(m)}(0, y, \xi, 0)$.

Furthermore, we have the full sequence $\{\sigma_{\mathrm{c}}^{m-j}(A)\}_{j\in\mathbb{N}_0}$ of conormal symbols, where

$$\sigma_{\mathrm{c}}^{m-j}(A)(y,z,\eta)=\frac{1}{j!}\,\frac{\partial^j}{\partial x^j}\,[a(x,y,\mathrm{i}z,x\eta)]\big|_{x=0}.$$

Notice that $\sigma_{\mathrm{c}}^{m-j}(A)(y,z,\eta)$ is $\mathscr{C}_{\mathrm{b}}^{\infty}$ in $y\in\mathbb{R}^d$, meromorphic in $z\in\mathbb{C}$ (by construction), and polynomial of degree j in η.
The sequence of conormal symbols controls the way in which asymptotics is mapped by A.

Proposition

Suppose that A possesses the transmission property with respect to $x=+0$. Then A_+ is edge-degenerate. In addition,

$$\sigma_{\mathrm{c}}^{m-j}(A_+)(y,z,\eta)=\frac{\Gamma(z+m-j)}{\Gamma(z)}\sum_{|\alpha|\le j}a_{j,\alpha}(y)\eta^{\alpha},$$

where $a_{j,\alpha}\in\mathscr{C}_{\mathrm{b}}^{\infty}(\mathbb{R}^d)$.

In pseudodifferential analysis, the Mellin transform replaces the Fourier transform near the singularities.

The Mellin transform of $u \in \mathscr{C}_c^\infty(\mathbb{R}_+)$ is

$$\mathcal{M}u(z) = \widetilde{u}(z) = \int_0^\infty x^{z-1} u(x)\,\mathrm{d}x, \quad z \in \mathbb{C}.$$

$\mathcal{M}$ is then extended to certain classes of functions and distributions on $\mathbb{R}_+ = (0, \infty)$.

Properties

- The inverse Mellin transform is

  $$\mathcal{M}^{-1}v(x) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\beta} x^{-z} v(z)\,\mathrm{d}z,$$

  for a suitably chosen $\beta \in \mathbb{R}$, where $\Gamma_\beta = \{z \in \mathbb{C} \mid \Re z = \beta\}$.
- $\mathcal{M}\,(-x\partial_x) = z\,\mathcal{M}$.
- $\mathcal{M} \colon x^\gamma L^2(\mathbb{R}_+, \mathrm{d}x) \to L^2\left(\Gamma_{1/2-\gamma}, (2\pi\mathrm{i})^{-1}\,\mathrm{d}z\right)$ is unitary.

Important Asymptotic expansions as $x \to 0$ of conormal type, i.e.,

$$u(x) \sim \sum_{p,k} \frac{(-1)^k}{k!} x^{-p} \log^k x \, u_{pk} \quad \text{as } x \to 0,$$

are reflected after Mellin transform in a pattern of poles.

Exercise

Show that we have the following dictionary for the behavior of asymptotic terms under the Mellin transform:

$$\frac{(-1)^k}{k!} x^{-p} \log^k x \, u_{pk} \quad \overset{1-1}{\longleftrightarrow} \quad \frac{u_{pk}}{(z-p)^{k+1}} + O(1) \;\text{ as } z \to p.$$

Hint.

- Compute $\int_0^1 x^{z-1} x^{-p} \, dx$. Then differentiate with respect to p.
- Why does $\int_1^\infty \ldots dx$ not contribute?

We expect (actually, consider only situations where) solutions u to elliptic problems to have (have) asymptotics of the form

$$u(x,y) \sim \sum_{p,k} \frac{(-1)^k}{k!} x^{-p} \log^k x \, u_{pk}(y) \quad \text{as } x \to 0,$$

where $\Re p \to -\infty$ as $|p| + k \to \infty$. These expansions are only formal. The coefficients u_{pk} are, in general, not smooth enough in order to write asymptotic terms as tensor products.

Example

Asymptotics resulting from a Taylor series expansion at $x = 0$ is

$$u(x,y) \sim \sum_{\ell \in \mathbb{N}_0} x^\ell u_\ell(y) \quad \text{as } x \to 0,$$

where $u_\ell \in H^{s-\ell-1/2}(\mathbb{R}^d)$ for $u \in H^s(\overline{\mathbb{R}}^{1+d}_+)$ (and s is large enough).

Collect the $(p, k) \in \mathbb{C} \times \mathbb{N}_0$ occuring in the asymptotic expansions into asymptotic types.

Definition

For $\delta \in \mathbb{R}$, an asymptotic type $P \in \underline{\mathrm{As}}^\delta$ is given by a discrete set $\pi_{\mathbb{C}} P \subset \mathbb{C}$ and a sequence $\{m_p\}_{p \in \pi_{\mathbb{C}} P} \subset \mathbb{N}$ such that

- $\pi_{\mathbb{C}} P \subset \{z \in \mathbb{C} \mid \Re p < 1/2 - \delta\}$,
- $\Re p \to -\infty$ as $p \in \pi_{\mathbb{C}} P$, $|p| \to \infty$,
- $p \in \pi_{\mathbb{C}} P$ implies $p - 1 \in \pi_{\mathbb{C}} P$ and $m_{p-1} \geq m_p$ (needed for coordinate invariance).

It is often convenient to write P as a set, i.e.,

$$P = \{(p, k) \in \pi_{\mathbb{C}} P \times \mathbb{N}_0 \mid k < m_p\}.$$

Example

$P_0 = \{(-\ell, 0) \mid \ell \in \mathbb{N}_0\}$ is the type for Taylor asymptotics.

Definition

For $s \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$, the space $\mathcal{H}^{s,\gamma}(\mathbb{R}^{1+d}_+)$ consist of all u such that

$$x^{-\gamma}(xD_x)^j D_y^\alpha u \in L^2(\mathbb{R}^{1+d}_+), \quad j + |\alpha| \leq s.$$

For general $s, \gamma \in \mathbb{R}$, these spaces are defined by duality and interpolation.

Lemma

Let $s \geq 0$, $\gamma \in \mathbb{R}$. Then $u \in \mathcal{H}^{s,\gamma}(\mathbb{R}^{1+d}_+)$ if and only if

$$\frac{1}{2\pi \mathrm{i}} \int_{\Re z = 1/2-\gamma} \Big(\|\widetilde{u}(z,\cdot)\|^2_{H^s(\mathbb{R}^d)} + (1+|z|^2)^s \|\widetilde{u}(z,\cdot)\|^2_{L^2(\mathbb{R}^d)} \Big)\, \mathrm{d}z < \infty,$$

where $\widetilde{u}(z,\cdot)$ is the Mellin transform of $u(x,\cdot)$ with respect to x.

Let $\varphi \in \mathscr{C}^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function, i.e., $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $x \le 1/2$, and $\varphi(x) = 0$ for $x \ge 1$.

Definition

For $s, \gamma \in \mathbb{R}$, we set

$$\mathcal{K}^{s,\gamma}(\mathbb{R}^{1+d}_+) = \{u \mid \varphi u \in \mathcal{H}^{s,\gamma}(\mathbb{R}^{1+d}_+),\ (1-\varphi)\, u \in H^s(\mathbb{R}^{1+d}_+)\}.$$

Notation

Fix $\delta \in \mathbb{R}$. In the sequel, $\mathcal{K}^{0,\delta}(\mathbb{R}^{1+d}_+)$ will be our reference Hilbert space. Write $\langle u, v\rangle$ for the inner product and $\|u\| = \sqrt{\langle u, u\rangle}$ for the norm in $\mathcal{K}^{0,\delta}(\mathbb{R}^{1+d}_+)$. Note that

$$\langle u, v\rangle = \int_{\mathbb{R}^{1+d}_+} u(x,y)\overline{v(x,y)}\,\psi^{-2-2\delta}(x)\,dxdy,$$

where $\psi \in \mathscr{C}^\infty(\overline{\mathbb{R}}_+)$, $\psi(x) = x$ for $x \le 1/2$, and $\psi(x) = 1$ for $x \ge 1$.

- For $s \in \mathbb{R}$, $k \in \mathbb{Z}$, the space $H^{s,\langle k \rangle}(\mathbb{R}^d)$ consists of all $w = w(y)$ such that $\langle \eta \rangle^s \log^k \langle \eta \rangle \, \widehat{w}(\eta) \in L^2(\mathbb{R}^d)$. Here, $\langle \eta \rangle = (4 + |\eta|^2)^{1/2}$.
- For $(p, k) \in \mathbb{C} \times \mathbb{N}_0$, we set

$$(\Gamma_{pk} w)(x, y) = \frac{(-1)^k}{k!} \mathcal{F}^{-1} \left\{ \varphi(x \langle \eta \rangle) \widehat{w}(\eta) \right\} x^{-p} \log^k x.$$

This is the prototypical example of a potential operator.

Lemma

Let $(p, k) \in \mathbb{C} \times \mathbb{N}_0$, $w \in H^{s,\langle k \rangle}(\mathbb{R}^d)$. *Then*

$$\Gamma_{pk} w \in \bigcap_{\epsilon > 0} \mathcal{K}^{s+\epsilon, 1/2 - \Re p - \epsilon}(\mathbb{R}^{1+d}_+),$$

while $\Gamma_{pk} w - \varphi(x) \frac{(-1)^k}{k!} x^{-p} \log^k x \, w(y) \in \bigcap_{\epsilon > 0} \mathcal{K}^{s-\epsilon, 1/2 - \Re p + \epsilon}(\mathbb{R}^{1+d}_+)$.

Definition

Let $s \in \mathbb{R}$, $P \in \underline{\mathrm{As}}^{\delta}$, and $\theta \geq 0$. For $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\delta-\theta} = \emptyset$, the space $H^{s,\delta}_{P,\theta}(\overline{\mathbb{R}}^{1+d}_{+})$ consists of all $u \in \mathcal{K}^{s,\delta}(\mathbb{R}^{1+d}_{+})$ for which that there are $u_{pk} \in H^{s+\Re p+\delta-1/2,\langle k\rangle}(\mathbb{R}^d)$ for $(p,k) \in P$, $\Re p > 1/2 - \delta - \theta$ such that

$$u(x,y) - \sum_{\substack{(p,k)\in P,\\ \Re p > 1/2-\delta-\theta}} (\Gamma_{pk} u_{pk})(x,y) \in \mathcal{K}^{s-\theta,\delta+\theta}(\mathbb{R}^{1+d}_{+}).$$

For general $\theta \geq 0$, these spaces are defined by interpolation.

We will write $\gamma_{pk} u = u_{pk}$. The γ_{pk} are prototypical examples of trace operators.

The case most often needed is $s \geq 0$ and $\theta = s$. In this case, we will use the notation

$$H^{s,\delta}_{P}(\overline{\mathbb{R}}^{1+d}_{+}) = H^{s,\delta}_{P,s}(\overline{\mathbb{R}}^{1+d}_{+}).$$

Proposition

For $s \geq 0$,

- $H^s(\mathbb{R}^{1+d}_+) = H^{s,0}_{P_0}(\overline{\mathbb{R}}^{1+d}_+)$,
- $H^s_0(\overline{\mathbb{R}}^{1+d}_+) = H^{s,0}_{\mathcal{O}}(\overline{\mathbb{R}}^{1+d}_+)$.

Here,

$$H^s_0(\overline{\mathbb{R}}^{1+d}_+) = \{u \in H^s(\mathbb{R}^{1+d}) \mid \operatorname{supp} u \subseteq \overline{\mathbb{R}}^{1+d}_+\}$$

and $\mathcal{O}$ is the empty asymptotic type characterized by $\pi_{\mathbb{C}}\mathcal{O} = \emptyset$.

Remark

The spaces $H^{s,\delta}_{P,\theta}(\overline{\mathbb{R}}^{1+d}_+)$ are coordinate invariant. Consequently, we also have the spaces $H^{s,\delta}_{P,\theta}(X)$. Recall that X is a $\mathscr{C}^\infty$ compact manifold with boundary.

Exercise

Show that $u \in \mathscr{S}_P(\overline{\mathbb{R}}_+)$ if and only if

- $\tilde{u}(z)$ is meromorphic on $\mathbb{C}$ with poles given by P, i.e., $\tilde{u}(z)$ has a pole at $z = p$ for $p \in \pi_{\mathbb{C}} R$ of order at most m_p and it has no other poles,
- $\chi(z)\tilde{u}(z) \in \mathscr{C}^\infty(\mathbb{R}_\beta; \mathscr{S}(\mathbb{R}_\tau))$, where $z = \beta + \mathrm{i}\tau$ with $\beta, \tau \in \mathbb{R}$ and $\chi \in \mathscr{C}^\infty(\mathbb{C})$ is such that $\chi(z) = 0$ for $\operatorname{dist}(z, \pi_{\mathbb{C}} P) \le 1/2$ and $\chi(z) = 1$ for $\operatorname{dist}(z, \pi_{\mathbb{C}} P) \ge 1$.

Example

We have $\mathrm{e}^{-x} \in \mathscr{S}(\overline{\mathbb{R}}_+) = \mathscr{S}_{P_0}(\overline{\mathbb{R}}_+)$. The Mellin transform of e^{-x} is the Gamma function $\Gamma(z)$ which has simple poles at the non-positive integers.

Exercise

Show that

$$\psi(x)^\rho\, H^{s,\gamma}_{P,\theta}(\overline{\mathbb{R}}^{1+d}_+) = H^{s,\gamma+\rho}_{T^{-\rho}P,\theta}(\overline{\mathbb{R}}^{1+d}_+),$$

where $T^{-\rho}P = \{(p-\rho,k) \mid (p,k) \in P\}$.

Hint. Let us check that the regularity of the traces is okay.
So, let $u \in H^{s,\gamma}_{P,\theta}(\overline{\mathbb{R}}^{1+d}_+)$, $v = \psi^\rho\, u$, and $(p,k) \in P$, $1/2 - \delta - \theta < \Re p$.

- We have $\gamma_{pk} u \in H^{s+\Re p+\gamma-1/2,\langle k\rangle}(\mathbb{R}^d)$.
- It follows that $\gamma_{p-\rho,k} v = \gamma_{pk} u \in H^{s+(\Re p-\rho)+(\gamma+\rho)-1/2,\langle k\rangle}(\mathbb{R}^d)$ as required.

Cone-degenerate pseudo-differential operators on the half-line $\overline{\mathbb{R}}_+ = [0, \infty)$ are of the form

$$A = x^{-m} a(x, xD_x),$$

where $m \in \mathbb{R}$ is the order of A and $a(x, \xi)$ is smooth up to $x = 0$. The principal symbolic structure is as follows:

- The principal symbol $\sigma_\psi^m(A)(x, \xi) = a_{(m)}(x, \xi)$,
- The principal conormal symbol $\sigma_c^m(A)(z) = a(0, iz)$, where $z \in \mathbb{C}$.

Example

Let $A = x^{-m} \sum_{0 \le j \le m} a_j(x)(xD_x)^j$ be a cone-degenerate differential operator. Then $\sigma_c^m(A)(z) = \sum_{0 \le j \le m} a_j(0)(iz)^j$ is a polynomial in z of degree m.

In general, $\sigma_c^m(A)$ belongs to the space $\mathcal{M}_R^m$ for some asymptotic type R. (As before, $\pi_{\mathbb{C}}R$ is a discrete subset of $\mathbb{C}$, but now we allow $|\Re r| \to \infty$ as $r \in \pi_{\mathbb{C}}R$, $|r| \to \infty$.)

Definition

The function h is said to belong to $\mathcal{M}_R^m$ if

- $h = h(z)$ is meromorphic on $\mathbb{C}$ with poles given by R, i.e., h has a pole at $z = r$ for $r \in \pi_{\mathbb{C}}R$ of order at most m_r and it has no other poles,
- $\chi(z)h(z) \in \mathscr{C}^\infty(\mathbb{R}_\beta; S_{\mathrm{cl}}^m(\mathbb{R}_\tau))$, where $z = \beta + i\tau$ with $\beta, \tau \in \mathbb{R}$ and $\chi \in \mathscr{C}^\infty(\mathbb{C})$ is such that $\chi(z) = 0$ for $\operatorname{dist}(z, \pi_{\mathbb{C}}R) \leq 1/2$ and $\chi(z) = 1$ for $\operatorname{dist}(z, \pi_{\mathbb{C}}R) \geq 1$.

We also write $\mathcal{M}_{\mathrm{as}}^m = \bigcup_R \mathcal{M}_R^m$.

Exercise

Let $h \in \mathcal{M}^m_{\mathrm{as}}$. Recall that $\chi(z)h(z) \in \mathscr{C}^\infty(\mathbb{R}_\beta; S^m_{\mathrm{cl}}(\mathbb{R}_\tau))$, where $z = \beta + \mathrm{i}\tau$. Prove that $\sigma^m_\psi((\chi h)(\beta + \mathrm{i}\,\cdot)) \in S^{(m)}(\mathbb{R}_\tau \setminus 0)$ is independent of $\beta \in \mathbb{R}$.

Hint.

- It holds $\mathcal{M}^m_{\mathrm{as}} = \mathcal{M}^m_{\mathcal{O}} + \mathcal{M}^{-\infty}_{\mathrm{as}}$.
- Thus, we can assume that $h \in \mathcal{M}^m_{\mathcal{O}}$. Then the assertion is a consequence of the Cauchy-Riemann equations.

Proposition

Suppose that $h \in \mathcal{M}^m_{\mathrm{as}}$ *is elliptic in the sense that* $\sigma^m_\psi(h)(\tau) \neq 0$ *for all* $\tau \in \mathbb{R} \setminus 0$. *Then* $h^{-1}(z) \in \mathcal{M}^{-m}_{\mathrm{as}}$.

Why conormal symbols?

We have the sequence $\{\sigma_c^{m-j}(A)\}_{j\geq 0}$ of conormal symbols defined by

$$\sigma_c^{m-j}(A)(z) = \partial_x^j a(0, iz)/j!.$$

This sequence tells us the way in which asymptotics are mapped by A.

This is because, formally,

$$A \sim \sum_{j\geq 0} x^{-m+j}\, h_j(xD_x)$$

by a Taylor series expansion at $x = 0$, where $h_j(z) = \sigma_c^{m-j}(A)(z)$, and

$$h(xD_x)\left[x^{-p}\right] = h(ip)\, x^{-p},$$

$h(xD_x)\,[x^{-p}\log x] = h(ip)\, x^{-p}\log x - ih'(ip)x^{-p}$, etc.

The Mellin translation product

Exercise

Prove that

$$\sigma_{\mathrm{c}}^{m+m'-l}(AB)(z) = \sum_{j+k=l} \sigma_{\mathrm{c}}^{m-j}(z+m'-k)\sigma_{\mathrm{c}}^{m'-k}(B)(z)$$

for cone-degenerate operators A, B of orders m, m'.

Proposition

There is a family $\{\Gamma_P(z)\}_{P \in \underline{\mathrm{As}}^\delta}$ *of meromorphic functions on* $\mathbb{C}$ *with the following properties*:

- $\Gamma_P(z)$ *has poles exactly as given by* P*,*
- $(1/\Gamma_P)(z)$ *is an entire function,*
- $\Gamma_P(z+m)/\Gamma_Q(z) \in \mathcal{M}^m_{\mathrm{as}}$ *for all* $P, Q \in \underline{\mathrm{As}}^\delta$ *and all* $m \in \mathbb{R}$*.*

Moreover, the $\Gamma_P(z)$ *are unique up to multiplication by an elliptic* $h(z) \in \mathcal{M}^0_{\mathcal{O}}$ *such that* $(1/h)(z) \in \mathcal{M}^0_{\mathcal{O}}$*.*

Example

$\Gamma(z) = \Gamma_{P_0}(z)$.

Definition (preliminary)

For $m \in \mathbb{R}$, $P, Q \in \underline{\mathrm{As}}^\delta$, the class $\Psi^{m,\delta}_{P,Q}(\overline{\mathbb{R}}_+)$ consists of all cone-degenerate pseudodifferential operators (as defined by Schulze) such that

$$\sigma_{\mathrm{c}}^{m-j}(A)(z) = a_j(z)\,\frac{\Gamma_Q(z+m-j)}{\Gamma_P(z)}, \quad j \geq 0,$$

for certain $a_j \in \mathcal{M}^j_{\mathcal{O}}$.

Still missing: Conditions on the remainder terms. Those remainder terms are the Green operators.

Exercise

Verify by direct computation that $A \in \Psi^{m,\delta}_{Q,R}(\overline{\mathbb{R}}_+)$, $B \in \Psi^{m',\delta}_{P,Q}(\overline{\mathbb{R}}_+)$ implies $AB \in \Psi^{m+m',\delta}_{P,R}(\overline{\mathbb{R}}_+)$.

For the sake of simplicity, we shall assume that

$$(1/2 - \delta - \Re\pi_{\mathbb{C}} P) \cap \mathbb{N} = \emptyset.$$

Let

$$d_0 = \min\{d \in \mathbb{N} \mid 1/2 - \delta - \Re p < d \text{ for some } p \in \pi_{\mathbb{C}} P\}.$$

For $d \in \mathbb{N}$, $d \geq d_0$, we then set

$$s_d = \max\{1/2 - \delta - \Re p \mid p \in \pi_{\mathbb{C}} P,\ 1/2 - \delta - \Re p < d\}.$$

We further set

$$\zeta_P^d u(x) = u(x) - \varphi(x) \sum_{\substack{(p,k) \in P, \\ \Re p < 1/2 - \delta - d}} \frac{(-1)^k}{k!}\, x^{-p} \log^k x \, \gamma_{pk} u$$

for $u \in H_P^{s_d+0,\delta}(\overline{\mathbb{R}}_+)$ if $d \geq d_0$ and $u \in \mathcal{K}^{0,\delta}(\mathbb{R}_+)$ otherwise.

Definition

The class $\Psi^{\delta;d}_{G;P,Q}(\overline{\mathbb{R}}_+; N_-, N_+)$ for $d \in \mathbb{N}$, $N_-, N_+ \in \mathbb{N}_0$ consists of all operators

$$\mathscr{G} = \begin{pmatrix} G & K \\ T & S \end{pmatrix} : \begin{matrix} \mathscr{S}_P(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} \mathscr{S}_Q(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix},$$

where

$$Gu = \operatorname{op}_G(g)\zeta_P^d u + \sum_{\substack{(p,k)\in P, \\ \Re p < 1/2-\delta-d}} d_{pk}(x)\gamma_{pk} u$$

for certain $g \in \mathscr{S}_Q(\overline{\mathbb{R}}_+)\hat{\otimes}\mathscr{S}^{\delta}_{\mathrm{as}}(\overline{\mathbb{R}}_+)$, $d_{pk} \in \mathscr{S}_Q(\overline{\mathbb{R}}_+)$,

Here, $\operatorname{op}_G(g)u(x) = \int_0^\infty g(x,x')u(x')\psi^{-2-2\delta}(x')\,dx'$.

Definition (continued)

Where $T = (T_1, \dots, T_{N_+})$,

$$T_j u = \langle \zeta_P^\delta u, b_j \rangle + \sum_{\substack{(p,k)\in P, \\ \Re p < 1/2-\delta-d}} \beta_{jpk} \gamma_{pk} u$$

for certain $b_j \in \mathscr{S}_{\mathrm{as}}^\delta(\overline{\mathbb{R}}_+)$, $\beta_{jpk} \in \mathbb{C}$,

$$Kc = \sum_{l=1}^{N_-} c_l k_l(x), \quad c = (c_1, \dots, c_{N_-}) \in \mathbb{C}^{N_-}$$

for certain $k_l \in \mathscr{S}_Q(\overline{\mathbb{R}}_+)$, and S is an $N_+ \times N_-$ matrix.

Definition

For $m \in \mathbb{R}$, the class $S^m_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Gamma_{1/2-\delta})_{P,Q}$ consists of $h \in S^m_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Gamma_{1/2-\delta})$ such that, for $j \in \mathbb{N}_0$,

$$\partial_x^j h(0,z) = a_j(z)\,\frac{\Gamma_Q(z+m-j)}{\Gamma_P(z)}, \tag{$*$}$$

where $a_j \in \mathcal{M}^j_{\mathcal{O}}$.

Remark

The requirement $(*)$ fixes the conormal symbols.

Definition

The class $\Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}_+; N_-, N_+)$ consists of all operators

$$\mathscr{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathscr{G} : \begin{matrix} \mathscr{S}_P(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} \mathscr{S}_Q(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix},$$

where

$$A = \varphi \operatorname{op}_M(h)\varphi_0 + (1-\varphi) \operatorname{op}_\psi(p)(1-\varphi_1)$$

for certain $h \in S^m_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Gamma_{1/2-\delta})_{P,Q}$ and $p \in S^m_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R})$ with a suitable behavior as $x \to \infty$. Moreover, the cut-off functions $\varphi, \varphi_0, \varphi_1$ satisfy $\varphi\varphi_0 = \varphi$, $\varphi\varphi_1 = \varphi_1$.

Here, $\operatorname{op}_M(h) = \mathcal{M}^{-1} h(x,z)\mathcal{M}$ and $\operatorname{op}_\psi(p) = \mathcal{F}^{-1} p(x,\xi)\mathcal{F}$.

Theorem (Mapping properties)

Let $\mathscr{A} \in \Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}_+; N_-, N_+)$. *Then*

$$\mathscr{A} : \begin{array}{c} H^{s,\delta}_P(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} H^{s-m,\delta}_Q(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

continuously for each $s \geq m^+$ *such that* $s > s_d$ *if* $d \geq d_0$.

Theorem (Composition)

Let $\mathscr{A} \in \Psi^{m,\delta;d}_{Q,R}(\overline{\mathbb{R}}_+; N_0, N_+)$, $\mathscr{B} \in \Psi^{m',\delta;d'}_{P,Q}(\overline{\mathbb{R}}_+; N_-, N_0)$. *Then*

$$\mathscr{A}\mathscr{B} \in \Psi^{m+m',\delta;d''}_{P,R}(\overline{\mathbb{R}}_+; N_-, N_+).$$

Here, $d'' = \max\{d + m', d'\}$ *if* $m' \in \mathbb{Z}$.

Definition

The principal symbol space $\mathfrak{S}^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}_+)$ consists of all $(a_{(m)}, h)$, where $a_{(m)} \in S^{(m)}(\overline{\mathbb{R}}_+ \times (\mathbb{R} \setminus 0))$, $h \in \mathcal{M}^m_{T^{-m}Q-P}$, and $a_{(m)}(0,\xi) = \sigma^m_\psi(h)(\tau)\big|_{\tau=-\xi}$.

Here, $T^{-m}Q = \{(q-m,k) \mid (q,k) \in Q\}$.

Theorem

The principal symbol map fits into a short exact sequence

$$0 \longrightarrow \Psi^{m-1,\delta;d}_{P,Q}(\dots) \longrightarrow \Psi^{m,\delta;d}_{P,Q}(\dots) \xrightarrow{(\sigma^m_\psi,\sigma^m_c)} \mathfrak{S}^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}_+) \longrightarrow 0$$

which splits (both algebraically and topologically).

Important observation

Let

$$\mathscr{A} = \begin{pmatrix} A+G & K \\ T & S \end{pmatrix} \in \Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}_+; N_-, N_+).$$

Then the invertibility of $\mathscr{A}$ in the calculus up to a remainder in the Green class only depends on A.

Definition

The operator $\mathscr{A}$ is said to be elliptic if

- $\sigma^m_\psi(A)(x,\xi) \neq 0$ for all $(x,\xi) \in \overline{\mathbb{R}}_+ \times (\mathbb{R} \setminus 0)$,
- $\sigma^m_c(A)(z) = a_0(z)\,\Gamma_Q(z+m)/\Gamma_P(z)$ with $a_0(z) \in \mathcal{M}^0_{\mathcal{O}}$ has $a_0^{-1}(z) \in \mathcal{M}^0_{\mathcal{O}}$,

and a suitable condition as $x \to \infty$.

Theorem

For $\mathscr{A} \in \Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}_+; N_-, N_+)$, the following are equivalent:

- *$\mathscr{A}$ is elliptic,*
- *$\mathscr{A} : \begin{matrix} H^{s,\delta}_P(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} H^{s-m,\delta}_Q(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}$ is a Fredholm operator for some (and then for all) $s \geq m^+$ satisfying $s > s_d$ if $d \geq d_0$,*
- *$\mathscr{A}$ admits a parametrix in the calculus, i.e., there is a $\mathscr{P} \in \Psi^{-m,\delta;0}_{Q,P}(\overline{\mathbb{R}}_+; N_+, N_-)$ such that*

$$\mathscr{P}\mathscr{A} - 1 \in \Psi^{\delta;\max\{m,d\}}_{G;P}(\dots), \quad \mathscr{A}\mathscr{P} - 1 \in \Psi^{\delta;(d-m)^+}_{G;Q}(\dots).$$

Corollary

Suppose $\mathscr{A}$ is invertible. Then $\mathscr{A}^{-1} \in \Psi^{-m,\delta;(d-m)^+}_{Q,P}(\overline{\mathbb{R}}_+; N_+, N_-)$.

Theorem

The class $\Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}_+; N_-, N_+)$ *is coordinate invariant. More precisely, let* $\chi\colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *be* $\mathscr{C}^\infty$ *diffeomorphism (suitably behaved as* $x \to \infty$*). Then*

$$\chi_* A \in \Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}_+; N_-, N_+)$$

for $A \in \Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}_+; N_-, N_+)$.

- Edge philosophy means that certain strongly continuous group actions get involved.
- The local model near an edge is a cone bundle over that edge.
- Locally, we have streching along the cone axes.

Let E be a Hilbert space and $\{\kappa_\lambda\}_{\lambda>0}$ be a strongly continuous representation of $\mathbb{R}_+$ on E. We shall write $\kappa(\eta)$ in place of $\kappa_{\langle\eta\rangle}$.

Definition

For $s \in \mathbb{R}$, the abstract edge Sobolev space $\mathcal{W}^s(\mathbb{R}^d; E)$ consists of all $u \in \mathscr{S}'(\mathbb{R}^d; E)$ such that

$$\|u\|^2_{\mathcal{W}^s(\mathbb{R}^d;E)} = \int_{\mathbb{R}^d} \langle\eta\rangle^{2s} \|\kappa(\eta)^{-1}\mathscr{F}u(\eta)\|^2_E \, d\eta < \infty.$$

Proposition

For $s \in \mathbb{R}$,

$$H^s(\mathbb{R}^{m+d}) = \mathcal{W}^s(\mathbb{R}^d; H^s(\mathbb{R}^m)),$$

where $(\kappa_\lambda v)(x) = \lambda^{m/2} v(\lambda x)$ for $\lambda > 0$.

Proof.

First note that $\kappa_{1/\lambda}\mathcal{F}_{x\to\xi} = \mathcal{F}_{x\to\xi}\kappa_\lambda$. Therefore,

$$\|u\|^2_{H^s(\mathbb{R}^{m+d})} = \int_{\mathbb{R}^{m+d}} \langle \xi, \eta\rangle^{2s} |\widehat{u}(\xi,\eta)|^2 \, \mathrm{d}\xi \mathrm{d}\eta$$

$$= \int_{\mathbb{R}^{m+d}} \langle \xi/\langle\eta\rangle\rangle^{2s} \langle\eta\rangle^{2s} |\widehat{u}(\xi,\eta)|^2 \, \mathrm{d}\xi \mathrm{d}\eta = \int_{\mathbb{R}^d} \langle\eta\rangle^{2s} \left(\int_{\mathbb{R}^m} \langle\xi\rangle^{2s} |\langle\eta\rangle^{m/2} \widehat{u}(\xi\langle\eta\rangle, \eta)|^2 \, \mathrm{d}\xi \right) \mathrm{d}\eta$$

$$= \int_{\mathbb{R}^d} \langle\eta\rangle^{2s} \left(\int_{\mathbb{R}^m} \langle\xi\rangle^{2s} \left| \left(\kappa(\eta)\, \widehat{u}\right)(\xi,\eta)\right|^2 \mathrm{d}\xi \right) \mathrm{d}\eta = \int_{\mathbb{R}^d} \langle\eta\rangle^{2s} \|\kappa(\eta)^{-1}\widehat{u}(\cdot,\eta)\|^2_{H^s(\mathbb{R}^m)} \, \mathrm{d}\eta. \qquad \square$$

Here are a few more examples:

Proposition

For $s \in \mathbb{R}$,

$$H^s(\mathbb{R}_+^{1+d}) = \mathcal{W}^s(\mathbb{R}^d; H^s(\mathbb{R}_+))$$

and

$$H_0^s(\overline{\mathbb{R}}_+^{1+d}) = \mathcal{W}^s(\mathbb{R}^d; H_0^s(\overline{\mathbb{R}}_+)).$$

Proposition

Change the definition of $H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+d})$ *by demanding* $(1-\varphi)\,u \in x^\delta H^s(\mathbb{R}_+^{1+d})$ *(instead of* $(1-\varphi)\,u \in H^s(\mathbb{R}_+^{1+d})$*) and possibly enlarge the trace space according to the next proposition. Then*

$$H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+d}) = \mathcal{W}^s(\mathbb{R}^d; H_P^{s,\delta}(\overline{\mathbb{R}}_+)),$$

where $(\kappa_\lambda^{(\delta)} v)(x) = \lambda^{1/2-\delta} v(\lambda x)$ *for* $\lambda > 0$.

Proof.

It suffices to consider the case $P = \mathcal{O}$. So, we want to derive the result for $H_{\mathcal{O}}^{s,\delta}(\overline{\mathbb{R}}_+^{1+d}) = x^\delta H_{\mathcal{O}}^{s,0}(\overline{\mathbb{R}}_+^{1+d}) = x^\delta H_0^s(\overline{\mathbb{R}}_+^{1+d})$. We use the facts that

- the result is known for $\delta = 0$, i.e., for the space $H_0^s(\overline{\mathbb{R}}_+^{1+d})$,
- $\|\kappa_\lambda^{(\delta)} v\|_{H_{\mathcal{O}}^{s,\delta}(\overline{\mathbb{R}}_+)} \eqsim \|\kappa_\lambda^{(0)}(x^{-\delta} v)\|_{H_0^s(\overline{\mathbb{R}}_+)}$.

Then $\|u\|^2_{H_{\mathcal{O}}^{s,\delta}(\overline{\mathbb{R}}_+^{1+d})} \eqsim \|x^{-\delta} u\|^2_{H_0^s(\overline{\mathbb{R}}_+^{1+d})} \eqsim \int_{\mathbb{R}^d} \langle \eta \rangle^{2s} \|\kappa^{(0)}(\eta)^{-1}(x^{-\delta}\widehat{u}(\cdot,\eta))\|^2_{H_0^s(\overline{\mathbb{R}}_+)} d\eta$
$\eqsim \int_{\mathbb{R}^d} \langle \eta \rangle^{2s} \|\kappa^{(\delta)}(\eta)^{-1}\widehat{u}(\cdot,\eta)\|^2_{H_{\mathcal{O}}^{s,\delta}(\overline{\mathbb{R}}_+)} d\eta = \|u\|^2_{\mathcal{W}^s(\mathbb{R}^d; H_{\mathcal{O}}^{s,\delta}(\overline{\mathbb{R}}_+))}$. □

Let $m \in \mathbb{N}$ and introduce $\pi_\lambda \in \mathcal{L}(\mathbb{C}^m)$ for $\lambda > 0$ as the $m \times m$ upper triangular matrix with

$$(\pi_\lambda)_{jk} = \frac{(-1)^{k-j}}{(k-j)!} \log^{k-j} \lambda, \quad j \leq k.$$

Proposition

The trace space for $\gamma_p = (\gamma_{p0}, \gamma_{p1}, \ldots, \gamma_{p,m_p-1})$ *with* $p \in \pi_{\mathbb{C}} P$ *acting on* $\mathcal{W}^s(\mathbb{R}^d; H_P^{s,\delta}(\overline{\mathbb{R}}_+))$ *is*

$$H_\pi^{s'}(\mathbb{R}^d; \mathbb{C}^{m_p}) = \mathcal{W}^{s'}\left(\mathbb{R}^d; (\mathbb{C}^{m_p}, \{\pi_\lambda\}_{\lambda>0})\right),$$

where $s' = s + \Re p + \delta - 1/2$.

Proof.

First observe that the map $\mathcal{W}^s(\mathbb{R}^d; E) \to H^s(\mathbb{R}^d; E)$, $u \mapsto \mathcal{F}^{-1}\left(\kappa(\eta)^{-1}\widehat{u}(\eta)\right)$ is an isomorphism.

So, each $u \in \mathcal{W}^s(\mathbb{R}^d; H_P^{s,\delta}(\overline{\mathbb{R}}_+))$ is given as

$$\widehat{u}(x,\eta) = \varphi(x\langle\eta\rangle)x^{-p}\sum_{k=0}^{m_p-1}\frac{(-1)^k}{k!}\log^k(x\langle\eta\rangle)\widehat{v}_{pk}(\eta) + \dots$$

for some uniquely determined $v_{pk} \in H^{s'}(\mathbb{R}^d)$. Using the binomial formula $\log^k(x\langle\eta\rangle) = (\log x + \log\langle\eta\rangle)^k = \sum_{l=0}^k \binom{k}{l}\log^l x \, \log^{k-l}\langle\eta\rangle$, we find that

$$\left(\widehat{u}_{p0}(\eta), \dots, \widehat{u}_{p,m_p-1}(\eta)\right)^{\mathrm{T}} = \pi(\eta)\left(\widehat{v}_{p0}(\eta), \dots, \widehat{v}_{p,m_p-1}(\eta)\right)^{\mathrm{T}},$$

which yields the assertion. □

Exercise

Let

$$J = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

be the Jordan block of size m with eigenvalue 0. Show that

$$\pi_\lambda = \lambda^{-J}, \quad \lambda > 0.$$

Exercise

Show that $H^s(\mathbb{R}^d) \oplus H^{s,\langle 1 \rangle}(\mathbb{R}^d) \oplus \dots \oplus H^{s,\langle m-1 \rangle}(\mathbb{R}^d) \subseteq H^s_\pi(\mathbb{R}^d; \mathbb{C}^m)$.

Definition

Let $E, \widetilde{E}$ be Hilbert spaces with strongly continuous group actions $\{\kappa_\lambda\}$ and $\{\widetilde{\kappa}_\lambda\}$. Further let $a\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(E, \widetilde{E})$ be $\mathscr{C}^\infty$ for the strong operator topology. Then a is said to belong to $S^m(\mathbb{R}^d \times \mathbb{R}^d; E, \widetilde{E})$ if, for any $(\alpha, \beta) \in \mathbb{N}_0^{2d}$,

$$\left\|\widetilde{\kappa}(\eta)^{-1}(\partial_y^\alpha \partial_\eta^\beta a)(y, \eta)\kappa(\eta)\right\|_{E \to \widetilde{E}} \lesssim \langle\eta\rangle^{m-|\beta|}$$

Analogously, we say that $a\colon \mathbb{R}^d \times (\mathbb{R}^d \setminus 0) \to \mathcal{L}(E, \widetilde{E})$ that is $\mathscr{C}^\infty$ for the strong operator topology belongs to $S^{(m)}(\mathbb{R}^d \times (\mathbb{R}^d \setminus 0); E, \widetilde{E})$ if

$$a(y, \lambda\eta) = \lambda^m \widetilde{\kappa}_\lambda a(y, \eta)\kappa_\lambda^{-1}, \quad \lambda > 0.$$

Then $\chi(\eta)a \in S^m(\mathbb{R}^d \times \mathbb{R}^d; E, \widetilde{E})$, where $\chi \in S^0(\mathbb{R}^d)$, $\chi(\eta) = 0$ for $|\eta| \lesssim 1$.

Lemma

For any $p \in \pi_{\mathbb{C}} P$,

$$\gamma_p = \lambda^{1/2-p-\delta} \pi_\lambda \gamma_p \kappa_\lambda^{-1}, \quad \lambda > 0.$$

In particular, $\gamma_p \in S^{1/2-\Re p-\delta}(\mathbb{R}^d \times \mathbb{R}^d; H_P^{s,\delta}(\overline{\mathbb{R}}_+), \mathbb{C}^{m_p})$ provided that $s > 1/2 - \Re p - \delta$.

Proof.

We proceed formally. Write $v(x) = \varphi(x) \sum_{k=0}^{m_p-1} \frac{(-1)^k}{k!} x^{-p} \log^k x \, v_{pk} + \dots$ Then

$$\kappa_\lambda^{-1} v(x) = \lambda^{p+\delta-1/2} \varphi(\lambda^{-1} x) \sum_{k=0}^{m_p-1} \frac{(-1)^k}{k!} x^{-p} \log^k(\lambda^{-1} x) \, v_{pk} + \dots$$

and

$$\gamma_p \kappa_\lambda^{-1} v = \lambda^{p+\delta-1/2} \times \begin{pmatrix} 1 & \log\lambda & \frac{1}{2}\log^2\lambda & \dots & \frac{1}{(m_p-2)!}\log^{m_p-2}\lambda & \frac{1}{(m_p-1)!}\log^{m_p-1}\lambda \\ 0 & 1 & \log\lambda & \dots & \frac{1}{(m_p-3)!}\log^{m_p-3}\lambda & \frac{1}{(m_p-2)!}\log^{m_p-2}\lambda \\ 0 & 0 & 1 & \dots & \frac{1}{(m_p-4)!}\log^{m_p-4}\lambda & \frac{1}{(m_p-3)!}\log^{m_p-3}\lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \log\lambda \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} v_{p0} \\ v_{p1} \\ v_{p2} \\ \vdots \\ v_{p,m_p-2} \\ v_{p,m_p-1} \end{pmatrix}$$

$$= \lambda^{p+\delta-1/2} \pi_\lambda^{-1} \gamma_p v. \qquad \square$$

Proposition (Seiler, 1999)

Let $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d; E, \widetilde{E})$. Then

$$a(y, D_y)\colon \mathcal{W}^s(\mathbb{R}^d; E) \to \mathcal{W}^{s-m}(\mathbb{R}^d; \widetilde{E})$$

continuously for any $s \in \mathbb{R}$.

Corollary

Let $p \in \pi_{\mathbb{C}} P$ and $s > 1/2 - \Re p - \delta$. Then

$$\gamma_p\colon H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+d}) \to H_\pi^{s'}(\mathbb{R}^d; \mathbb{C}^{m_p}),$$

where $s' = s + \Re p + \delta - 1/2$.

In this last section, I will briefly indicate how to introduce the edge calculus $\Psi^{m,\delta;d}_{P,Q}(X;E,F)$. Here, E, F are vector bundles over $Y=\partial X$.

Operators $\mathscr{A}$ in the calculus have the form

$$\mathscr{A}=\begin{pmatrix} A+G & K \\ T & S \end{pmatrix} : \begin{matrix} \mathscr{C}^\infty_P(X) \\ \oplus \\ \mathscr{C}^\infty(Y;E) \end{matrix} \to \begin{matrix} \mathscr{C}^\infty_Q(X) \\ \oplus \\ \mathscr{C}^\infty(Y;F) \end{matrix},$$

where the components A, G, T, K, S have a similar meaning as before.

The principal symbolic structure is as follows: We have

- the principal pseudodifferential symbol $\sigma_\psi^m(\mathscr{A}) \in S^{(m)}(T^*X \setminus 0)$, where
$$\sigma_\psi^m(\mathscr{A})(x, y, \xi, \eta) = a_{(m)}(x, y, \xi, \eta)$$
for $A = x^{-m} a(x, y, xD_x, xD_y)$,
- the principal boundary symbol $\sigma_\partial^{m'}(\mathscr{A})$ which is an operator function indexed by points $(y, \eta) \in T^*Y \setminus 0$,
$$\sigma_\partial^{m'}(\mathscr{A})(y,\eta) \colon \begin{matrix} \mathscr{S}_P(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} \mathscr{S}_Q(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix},$$
N_- being the rank of E, N_+ being the rank of F, that takes values in $\Psi_{P,Q}^{m,\delta,d}(\overline{\mathbb{R}}_+; N_-, N_+)$. Moreover, $\sigma_\partial^{m'}(\mathscr{A})$ is of twisted homogeneity in the sense of the previous section.

In addition, $\sigma_\psi^m(\mathscr{A})$ and $\sigma_\partial^{m'}(\mathscr{A})$ are compatible in an obvious sense.

Especially important is the full sequence $\{\sigma_{\mathrm{c}}^{m-j}(A)\}_{j\geq 0}$ of conormal symbols.

Definition

Upon an appropriate choice of the Green operators, the class $\Psi_{P,Q}^{m,\delta;d}(\overline{\mathbb{R}}_+^{1+d})$ consists of all edge-degenerate pseudodifferential operators A (in the sense of Schulze) such that

$$\sigma_{\mathrm{c}}^{m-j}(A)(y,z,\eta)=\frac{\Gamma_Q(z+m-j)}{\Gamma_P(z)}\sum_{|\alpha|\leq j}a_{j\alpha}(y,z)\eta^\alpha,$$

where $a_{j\alpha}\in\mathscr{C}_{\mathrm{b}}^\infty(\mathbb{R}^d;\mathcal{M}_{\mathcal{O}}^0)$.

The Leibniz-Mellin translation product

Proposition

For edge-degenerate operators A, B of orders m, m', it holds

$$\sigma_{\mathrm{c}}^{m+m'-l}(AB)(y,z,\eta) = \sum_{\substack{j+k=l,\\ |\alpha|\le j}} \frac{1}{\alpha!}\,(\partial_\eta^\alpha \sigma_{\mathrm{c}}^{m-j}(A))(y,z+m'-k,\eta)(D_y^\alpha \sigma_{\mathrm{c}}^{m'-k}(B))(y,z,\eta)$$

Remark

The sum in the right-hand side is finite because $\sigma_{\mathrm{c}}^{m-j}(A)(y,z,\eta)$ is a polynomial of degree j in η.

Corollary

$A \in \Psi^{m,\delta;d}_{Q,R}(\overline{\mathbb{R}}^{1+d}_+)$, $B \in \Psi^{m',\delta;d}_{P,Q}(\overline{\mathbb{R}}^{1+d}_+)$ implies that $AB \in \Psi^{m+m',\delta;d''}_{P,R}(\overline{\mathbb{R}}^{1+d}_+)$, where $d'' = \max\{d+m', d'\}$.

Locally near the boundary, operators

$$\mathscr{A} = \begin{pmatrix} A+G & K \\ T & S \end{pmatrix} : \begin{matrix} \mathscr{C}^\infty(X) \\ \oplus \\ \mathscr{C}^\infty(Y;E) \end{matrix} \to \begin{matrix} \mathscr{C}^\infty(X) \\ \oplus \\ \mathscr{C}^\infty(Y;F) \end{matrix}$$

in the calculus are of the form $\mathscr{A} = \mathfrak{a}(y, D_y) + \mathscr{G}$, where

$$\mathfrak{a}(y,\eta) = \begin{pmatrix} a(y,\eta)+g(y,\eta) & k(y,\eta) \\ t(y,\eta) & q(t,\eta) \end{pmatrix}$$

is an operator function that takes values in $\Psi^{m,d,\delta}_{P,Q}(\overline{\mathbb{R}}_+; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$, and is such that

$$\mathfrak{a} \in S^m(\mathbb{R}^d \times \mathbb{R}^d; H^{s,\delta}_P(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_-}, H^{s-m,\delta}_Q(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_+})$$

for $s \geq m^+$ and $s > s_d$ if $d \geq d_0$. Moreover, $\mathscr{G} \in \Psi^{-\infty,\delta;d}_{G;P,Q}(X; E, F)$.

We show exemplary how to introduce the potential part $k(y,\eta)$.

Pick $p \in \pi_{\mathbb{C}} P$. We want that, for each $(y,\eta) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$k(y,\eta)\colon \mathbb{C}^{m_p} \to \mathscr{S}_Q(\overline{\mathbb{R}}_+).$$

Let $f \in S^{m-\delta+1/2}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m_p}, \mathbb{C}) \hat{\otimes} \mathscr{S}_Q(\overline{\mathbb{R}}_+)$ and set

$$k(y,\eta)c = f(y,\eta,x\langle\eta\rangle)c$$

for $c \in \mathbb{C}^{m_p}$. Then

$$\kappa(\eta)^{-1} f(y,\eta,x\langle\eta\rangle)\pi(\eta)c = \langle\eta\rangle^{\delta-1/2} f(y,\eta,x)\pi(\eta)c$$

so that

$$\|\kappa(\eta)^{-1} k(y,\eta)\pi(\eta)\|_{\mathbb{C}^{m_p} \to H_Q^{s-m,\delta}(\overline{\mathbb{R}}_+)} \lesssim \langle\eta\rangle^m,$$

similar for $\partial_y^\alpha \partial_\eta^\beta k(y,\eta)$.

Lemma

Let $p \in \pi_{\mathbb{C}} P$ and $b_0, b_1, \dots, b_{m_p-1} \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$. Then the operator

$$H_\pi^{s+\Re p+\delta-1/2}(\mathbb{R}^d; \mathbb{C}^{m_p}) \to H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+d}), \quad w \mapsto \sum_{0 \le k < m_p} \Gamma_{pk} \left(b_k(y, D_y) w_k \right),$$

can be written as $k(y, D_y)$, where $k(y, \eta) = f(x, \eta, x\langle\eta\rangle)$ as on the previous page with

$$f(y, \eta, x)c = \langle\eta\rangle^p \begin{pmatrix} \varphi(x)x^{-p} \\ -\varphi(x)x^{-p}\log x \\ \vdots \\ \frac{(-1)^{m_p-1}}{(m_p-1)!}\varphi(x)x^{-p}\log^{m_p-1} x \end{pmatrix} \cdot \pi(\eta)^{-1} \begin{pmatrix} b_0(y,\eta)c_0 \\ b_1(y,\eta)c_1 \\ \vdots \\ b_{m_p-1}(y,\eta)c_{m_p-1} \end{pmatrix}.$$

In particular, $f \in S^{\Re p}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m_p}, \mathbb{C}) \hat{\otimes} \mathscr{S}_P(\overline{\mathbb{R}}_+)$ as required.

Remark

The principal boundary symbol $\sigma_\partial^m(k) \in S^{(m)}(\mathbb{R}^d \times (\mathbb{R}^d \setminus 0); \mathbb{C}^{m_p}, \mathscr{S}_Q(\overline{\mathbb{R}}_+))$ of k is given by

$$\sigma_\partial^m(k)(y,\eta)c = f_{(m-\delta-1/2)}(y,\eta,x|\eta|)c, \quad c \in \mathbb{C}^{m_p},$$

for $(y,\eta) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$.

Theorem (Mapping properties)

Let $\mathscr{A} \in \Psi^{m,\delta;d}_{P,Q}(X; E, F)$. *Then*

$$\mathscr{A} : \begin{matrix} H^{s,\delta}_P(X) \\ \oplus \\ H^{s'}(Y; E) \end{matrix} \to \begin{matrix} H^{s-m,\delta}_Q(X) \\ \oplus \\ H^{s''}(Y; F) \end{matrix}$$

continuously for each $s \geq m^+$ *such that* $s > s_d$ *if* $d \geq d_0$.

Theorem (Composition)

Let $\mathscr{A} \in \Psi^{m,\delta;d}_{Q,R}(X; E', F)$, $\mathscr{B} \in \Psi^{m',\delta;d'}_{P,Q}(X; E, E')$. *Then*

$$\mathscr{A}\mathscr{B} \in \Psi^{m+m',\delta;d''}_{P,R}(X; E, F),$$

where $d'' = \max\{d + m', d'\}$.

Theorem

The principal symbol map fits into a short exact sequence

$$0 \longrightarrow \Psi^{m-1,\delta;d}_{P,Q}(\ldots) \longrightarrow \Psi^{m,\delta;d}_{P,Q}(\ldots) \xrightarrow{(\sigma^m_\psi,\sigma^{m'}_\partial)} \mathfrak{S}^{m,\delta;d}_{P,Q}(X;E,F) \longrightarrow 0$$

which splits (both algebraically and topologically).

Definition

The operator $\mathscr{A} \in \Psi^{m,\delta;d}_{P,Q}(X;E,F)$ is said to be elliptic if

- $\sigma^m_\psi(A)(x,y,\xi,\eta) \neq 0$ for all $(x,y,\xi,\eta) \in T^*X \setminus 0$,
- $$\sigma^{m'}_\partial(\mathscr{A})(y,\eta)\colon \begin{matrix} \mathscr{S}_P(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} \mathscr{S}_Q(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}$$

  is invertible for all $(y,\eta) \in T^*Y \setminus 0$.

Theorem

For $\mathscr{A} \in \Psi^{m,\delta;d}_{P,Q}(X; E, F)$, the following are equivalent:

- $\mathscr{A}$ *is elliptic,*
- $\mathscr{A} : \begin{matrix} H^{s,\delta}_P(X) \\ \oplus \\ H^{s'}(Y;E) \end{matrix} \to \begin{matrix} H^{s-m,\delta}_Q(X) \\ \oplus \\ H^{s'-m'}(Y;F) \end{matrix}$ *is a Fredholm operator for some (and then for all) $s \geq m^+$ satisfying $s > s_d$ if $d \geq d_0$,*
- $\mathscr{A}$ *admits a parametrix in the calculus, i.e., there is a $\mathscr{P} \in \Psi^{-m,\delta;(d-m)^+}_{Q,P}(X; F, E)$ such that*

$$\mathscr{P}\mathscr{A} - 1 \in \Psi^{\delta;\max\{m,d\}}_{G;P}(X; E), \quad \mathscr{A}\mathscr{P} - 1 \in \Psi^{\delta;(d-m)^+}_{G;Q}(X; F).$$

Corollary

Suppose $\mathscr{A}$ is invertible. Then $\mathscr{A}^{-1} \in \Psi^{-m,\delta;(d-m)^+}_{Q,P}(X; F, E)$.

Theorem

The class $\Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}^{1+d}_{+}; E, F)$ is coordinate invariant. More precisely, let $\chi\colon \overline{\mathbb{R}}^{1+d}_{+} \to \overline{\mathbb{R}}^{1+d}_{+}$ be $\mathscr{C}^\infty$ diffeomorphism with a suitable behavior as $x + |y| \to \infty$. Then

$$\chi_* A \in \Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}^{1+d}_{+}; (\chi^{-1})^* E, (\chi^{-1})^* F)$$

for $A \in \Psi^{m,\delta;d}_{P,Q}(\overline{\mathbb{R}}^{1+d}_{+}; E, F)$.

Proof.

This theorem has been proven if we fix a boundary defining function for the compact manifold X and use this global function as local coordinate $x \geq 0$ everywhere near the boundary.

The proof in the general case has not been completely finished yet, although I am very optimistic ... □