Beyond the Hodge Theorem: curl and asymmetric pseudodifferential projections

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Joint work with Matteo Capoferri. Work in progress.

Playing field: (M, g), a connected oriented closed Riemannian manifold of dimension d.

Object of study: $\Omega^k(M)$, the Hilbert space of real-valued k-forms.

Hodge's Theorem: $\Omega^k(M)$ decomposes into a direct sum of three orthogonal closed subspaces

$$\Omega^k(M) = d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M) \oplus \mathcal{H}^k(M),$$

where $d\Omega^{k-1}(M)$, $\delta\Omega^{k+1}(M)$, $\mathcal{H}^k(M)$ are the Hilbert subspaces of exact, coexact and harmonic k-forms, respectively.

Important special case: $d = 3$

Hodge's Theorem:

$$\Omega^1(M) = d\Omega^0(M) \oplus \delta\Omega^2(M) \oplus \mathcal{H}^1(M)\,.$$

Main result of this talk: we can decompose $\delta\Omega^2(M)$ further

$$\delta\Omega^2(M) = \delta\Omega^2_+(M) \oplus \delta\Omega^2_-(M)$$

and we have a very detailed algorithm for this decomposition.

A very special operator in dimension 3

The operator curl acts on 1-forms and is defined by the formula

$$\mathrm{curl} := *d.$$

Why is the operator curl interesting?

1 Homogeneous vacuum Maxwell equations reduce to

$$\left(-i\frac{\partial}{\partial t} + \operatorname{curl}\right) u = 0,$$

$$\operatorname{div} u = 0,$$

where $u := E + iH$.

2 The spectrum of the operator curl

$$\operatorname{curl} u = \lambda u$$

is asymmetric. Eigenvalues λ accumulate to $+\infty$ and to $-\infty$, but in an asymmetric fashion. Operator feels difference between right-handedness and left-handedness.

3 Very few publications on the spectrum of the operator curl. Paper by Jason Lotay (2012) and another by Christian Bär (2019).

Why is the study of the operator curl challenging?

Operator is not elliptic.

Definition of ellipticity for a formally self-adjoint system?

Leave only higher order derivatives and replace each $\partial/\partial x^\alpha$ by $i\xi_\alpha$. Operator turns into a matrix-function on T^*M. This matrix-function is called *principal symbol* of the operator.

Determinant of principal symbol should not vanish on $T^*M \setminus \{0\}$.

Determinant of principal symbol of curl is identically zero.

Eigenvalues of the (principal) symbol of the operator curl read

$$0 \qquad \text{and} \qquad \pm\sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta}\,.$$

Proper definition of curl and its basic properties

1 curl is an operator in the Hilbert space of real-valued coexact 1-forms $\delta\Omega^2(M)$.

2 The domain of curl is $(\delta\Omega^2 \cap H^1)(M)$, where $H^1(M)$ is the Sobolev space of 1-forms which are square integrable together with their first partial derivatives.

3 curl is self-adjoint and has discrete spectrum.

4 Zero is not an eigenvalue of curl.

5 curl^{-1} is a bounded operator from $(\delta\Omega^2 \cap H^s)(M)$ to $(\delta\Omega^2(M) \cap H^{s+1})(M)$ for all $s = 0, 1, \ldots$.

Spectral problem for curl:

$$\operatorname{curl} u_k = \lambda_k u_k, \qquad k \in \mathbb{Z}.$$

Can study the distribution of positive and negative eigenvalues of curl. Initial results already obtained by Christian Bär (2019).

But one can do something far more interesting with curl ...

Asymmetric projection operators

$$P_\pm := \theta(\pm\,\mathrm{curl}),$$

where

$$\theta(x) := \begin{cases} 1, & x > 0 \\ 0, & x \le 0, \end{cases}$$

is the Heaviside step function.

A more explicit definition:

$$P_+ = \sum_{\lambda_k > 0} \langle u_k, \cdot \rangle u_k, \qquad P_- = \sum_{\lambda_k < 0} \langle u_k, \cdot \rangle u_k.$$

Operators $P_\pm$ are determined by the Riemannian manifold (M, g) and its orientation, and they constitute an orthonormal pair of projections which decompose the Hilbert space of real-valued coexact 1-forms $\delta\Omega^2(M)$ into two orthogonal subspaces.

Theorem 1 We can construct the projection operators $P_\pm$ explicitly, modulo integral operators with smooth integral kernels.

The result is expressed in terms of metric, Christoffel symbols, curvature and covariant derivatives of curvature.

Key words: pseudodifferential operators, microlocal analysis.

Notion of pseudodifferential operator

Pseudodifferential operator P of order s acting on 1-forms over a manifold of dimension d. Representation in local coordinates

$$u_\mu(x) \mapsto \frac{1}{(2\pi)^d} \int p_\mu{}^\nu(x,\xi)\, e^{i(x-y)^\alpha \xi_\alpha}\, u_\nu(y)\, dy\, d\xi\,.$$

In our case $d = 3$ and $s = 0$.

Here $p_\mu{}^\nu(x,\xi)$ is the *symbol* which admits an asymptotic expansion into components positively homogeneous in momentum ξ:

$$p_\mu{}^\nu(x,\xi) \sim [p_s]_\mu{}^\nu(x,\xi) + [p_{s-1}]_\mu{}^\nu(x,\xi) + \dots .$$

The subscripts $s, s-1, \dots,$ indicate the degree of homogeneity.

Notions of principal and subprincipal symbol

Principal symbol of a pseudodifferential operator P:

$$[P_{\mathrm{prin}}]_\mu{}^\nu := [p_s]_\mu{}^\nu .$$

Subprincipal symbol of a pseudodifferential operator P:

$$\begin{aligned}[P_{\mathrm{sub}}]_\mu{}^\nu := [p_{s-1}]_\mu{}^\nu + \frac{i}{2}\frac{\partial^2 [p_s]_\mu{}^\nu}{\partial x^\gamma \partial \xi_\gamma} \\ + \frac{i}{2}\left(\Gamma^\alpha{}_{\gamma\alpha}\frac{\partial [p_s]_\mu{}^\nu}{\partial \xi_\gamma} - \Gamma^\alpha{}_{\gamma\mu}\frac{\partial [p_s]_\alpha{}^\nu}{\partial \xi_\gamma} - \Gamma^\nu{}_{\gamma\alpha}\frac{\partial [p_s]_\mu{}^\alpha}{\partial \xi_\gamma}\right).\end{aligned}$$

Both $[P_{\mathrm{prin}}]_\mu{}^\nu(x,\xi)$ and $[P_{\mathrm{sub}}]_\mu{}^\nu(x,\xi)$ are invariantly defined objects on $T^*M \setminus \{0\}$.

Original definition of subprincipal symbol is due to Duistermaat and Hörmander (1972). For scalar operators acting on half-densities.

Principal and subprincipal symbols of our $P_\pm$

$$[(P_\pm)_{\mathrm{prin}}]_\alpha{}^\beta(x,\xi) = \frac{1}{2}\Big[\delta_\alpha{}^\beta - (g^{\mu\nu}(x)\,\xi_\mu\xi_\nu)^{-1}\,\xi_\alpha\, g^{\beta\gamma}(x)\,\xi_\gamma \pm i\,(g^{\mu\nu}(x)\,\xi_\mu\xi_\nu)^{-1/2}\,E_\alpha{}^{\gamma\beta}(x)\,\xi_\gamma\Big],$$

where

$$E_{\alpha\beta\gamma}(x) := \rho(x)\,\varepsilon_{\alpha\beta\gamma}\,,$$

$\rho(x) := \sqrt{\det g_{\alpha\beta}(x)}$ is the Riemannian density and ε is the totally antisymmetric symbol, $\varepsilon_{123} := +1$.

$$[(P_\pm)_{\mathrm{sub}}]_\alpha{}^\beta(x,\xi) = 0.$$

We actually went much deeper and calculated components of the symbols of our operators $P_\pm$ of degree of homogeneity -2 and -3.

Explicit algorithm leading to the determination of full symbols of pseudodifferential projections. Algorithm is global and does not use local coordinates. Magic!

Capoferri and Vassiliev, *Invariant subspaces of elliptic systems I: pseudodifferential projections*, Journal of Functional Analysis, 2022.

Capoferri and Vassiliev, *Invariant subspaces of elliptic systems II: spectral theory*, Journal of Spectral Theory, 2022.

Any (OK, almost any) system of m (pseudo)differential equations can be associated with m almost invariant almost orthogonal subspaces. Projections onto these subspaces can be constructed explicitly, modulo integral operators with smooth integral kernels.

Implementation of 'magic' algorithm benefits from the use of the computer algebra package Mathematica©.

Where did this algorithm come from?

Spectral theory of elliptic systems. Second Weyl coefficient.

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2020: Matteo Capoferri and I realised that we have been looking at elliptic systems the wrong way. Should look for almost invariant subspaces and pseudodifferential projections. Benefit of hindsight.

Using projections $P_\pm$ for the study of spectral asymmetry

We have

$$P_\pm : \; u_\mu(x) \mapsto \int_M [K_\pm]_\mu{}^\nu(x,y)\, u_\nu(y)\, \rho(y)\, dy\,,$$

where $K_\pm$ is the (distributional) integral kernel.

Want to define the trace of $P_+ - P_-$

$$\int_M \big([K_+]_\mu{}^\mu(y,y) - [K_-]_\mu{}^\mu(y,y)\big)\rho(y)\, dy$$
$$= \sum_k \theta(\lambda_k) = \sum_{\lambda_k>0} 1 - \sum_{\lambda_k<0} 1.$$

Problem: operator $P_+ - P_-$ is not of trace class.

Spectrum of $P_+ - P_-$ consists of two points $+1$ and -1, both eigenvalues of infinite multiplicity.

Idea: split the process of calculating trace into two separate steps.

- Take matrix trace first, which would give a scalar operator.
- Calculate the trace of the scalar operator the usual way, by taking the value of the integral kernel on the diagonal $x = y$ and integrating over the manifold M.

We define the scalar *asymmetry operator* as

$$A: \ f(x) \mapsto \int_M \left([K_+]_\mu{}^\mu(x, y) - [K_-]_\mu{}^\mu(x, y) \right) f(y) \, \rho(y) \, dy \, .$$

Slight problem: the lower and upper tensor indices in $[K_\pm]_\mu{}^\nu(x, y)$ live at different points, x and y. To make the definition of the asymmetry operator A invariant need to perform parallel transport along geodesic connecting x and y. Concept of two-point tensor.

Theorem 2 The scalar asymmetry operator is a pseudodifferential operator of order -3 and its principal symbol reads

$$A_{\mathrm{prin}}(x,\xi) = -\frac{1}{2}(g^{\mu\nu}(x)\,\xi_\mu\xi_\nu)^{-5/2}\,E^{\alpha\beta\gamma}(x)\,\nabla_\alpha Ric_\beta{}^\delta(x)\,\xi_\gamma\,\xi_\delta\,,$$

where

$$E_{\alpha\beta\gamma}(x) := \rho(x)\,\varepsilon_{\alpha\beta\gamma}\,,$$

$\rho(x) := \sqrt{\det g_{\alpha\beta}(x)}$ is the Riemannian density and ε is the totally antisymmetric symbol, $\varepsilon_{123} := +1$.

The order of the operator A is -3, dimension of the manifold is 3.

Good news: operator A is *almost* trace class.

We have

$$A:\ f(x) \mapsto \int_M K(x,y)\, f(y)\, \rho(y)\, dy\,,$$

where K is the integral kernel. No need for "distributional".

Denote by $\mathbb{S}_\epsilon(x)$ the geodesic sphere of radius $\epsilon > 0$ centred at the point $x \in M$.

Theorem 3 The integral kernel $K(x,y)$ is a bounded function, smooth outside the diagonal and discontinuous on the diagonal: for any $x \in M$ the limit $\lim_{y \to x} K(x,y)$ depends on the direction along which y tends to x. Furthermore, for any $x \in M$ the limit

$$\lim_{\epsilon \to 0^+} \frac{1}{4\pi\epsilon^2} \int_{\mathbb{S}_\epsilon(x)} K(x,y)\, dS_y$$

exists and defines a smooth scalar function $\mu(x)$, $\mu : M \to \mathbb{R}$.

We have defined a differential geometric invariant

$$\int_M \mu(x)\,\rho(x)\,dx\,,$$

a measure of the asymmetry of our Riemannian manifold under change of orientation.

Is this a new differential geometric invariant? Probably not.

People who contributed to this subject area: Michael Atiyah, Vijay Kumar Patodi, Isadore Singer, Nigel Hitchin, Peter Gilkey, Lev Pontryagin, Friedrich Hirzebruch, Shiing-Shen Chern, Jim Simons, Bob Seeley ...

Hirzebruch L-polynomials, Hirzebruch $\hat{A}$-polynomials, Pontryagin forms, Pontryagin classes ...

Expressing our geometric invariant in terms of known ones

Put

$$\eta_{\mathrm{curl}}(s) := \sum_k \frac{\operatorname{sgn} \lambda_k}{|\lambda_k|^s}\,.$$

Note: series converges absolutely for $\operatorname{Re} s > 3$.

Conjecture 1 The function $\eta_{\mathrm{curl}}(s)$ is holomorphic in the half-plane $\operatorname{Re} s > -2$.

Conjecture 2

$$\eta_{\mathrm{curl}}(0) = \int_M \mu(x)\,\rho(x)\,dx\,.$$

If we prove these two conjectures, we will express our differential geometric invariant in terms of known differential geometric invariants.

Summary of our results

1 Working in dimension 3, we have refined Hodge's Theorem

$$\Omega^1(M) = d\Omega^0(M) \oplus \underbrace{\delta\Omega^2(M)}_{\delta\Omega^2_+(M)\oplus\delta\Omega^2_-(M)} \oplus \mathcal{H}^1(M)\,.$$

2 We developed a new approach to the subject of spectral asymmetry.

3 Our approach to the subject of spectral asymmetry is, in a sense, more straightforward and logical. It does not involve dealing with analytic functions.

4 Our approach to the subject of spectral asymmetry provides more detailed information: not only it gives a number characterising asymmetry, it also gives a pseudodifferential operator characterising asymmetry.

Step 1. Given the m eigenprojections $P^{(j)}$ of A_{prin} , choose m arbitrary pseudodifferential operators $P_{j,0} \in \Psi^0$ satisfying $(P_{j,0})_{\mathrm{prin}} = P^{(j)}$.

Step 2. For $k = 1, 2, \ldots$ define

$$P_{j,k} := P_{j,0} + \sum_{n=1}^{k} X_{j,n}, \qquad X_{j,n} \in \Psi^{-n}.$$

Assuming we have determined the pseudodifferential operator $P_{j,k-1}$, compute, one after the other, the following quantities:

(a) $R_{j,k} := -((P_{j,k-1})^2 - P_{j,k-1})_{\mathrm{prin,k}}$,

(b) $S_{j,k} := -R_{j,k} + P^{(j)} R_{j,k} + R_{j,k} P^{(j)}$,

(c) $T_{j,k} := [P_{j,k-1}, A]_{\mathrm{prin},k-s} + [S_{j,k}, A_{\mathrm{prin}}]$.

Step 3. Choose a pseudodifferential operator $X_{j,k} \in \Psi^{-k}$ satisfying

$$(X_{j,k})_{\mathrm{prin}} = S_{j,k} + \sum_{l \neq j} \frac{P^{(j)} T_{j,k} P^{(l)} - P^{(l)} T_{j,k} P^{(j)}}{h^{(j)} - h^{(l)}}.$$

Step 4. Put

$$P_j \sim P_{j,0} + \sum_{n=1}^{+\infty} X_{j,n}\,.$$