SUMMER SCHOOL "Modern Problems in PDEs and Applications"
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Ghent Analysis & PDE Center
Ghent University

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Abstracts of the minicourses

Fibrewise Geometric Analysis

Simon Scott (King's College London, UK)

We shall describe the extension of some standard constructions of spectral geometric invariants via operator traces on compact manifolds to the case of fibrations of manifolds over a parameter space. We shall describe how some of this is rather well understood while other areas remain to be elucidated or are conjectural, with parallel questions for functorial QFT, which we hope may provide interesting questions for researchers.

Analysis on manifolds with conical singularities

Elmar Schrohe (Leibniz University Hanover, Germany)

In the three lectures I will try to develop the ideas of the Mellin pseudodifferential calculus on manifolds with conical singularities from the basics up to applications to nonlinear evolution equations.

Lecture 1. Manifolds with conical singularities, conically degenerate differential operators, cone Sobolev spaces, asymptotics, the algebra of cone pseudodifferential operators.

Lecture 2. Symbols, ellipticity, parametrix construction, closed extensions of cone operators.

Lecture 3. Maximal regularity and H_∞-calculus, the Clément-Li Theorem, the porous medium equation on manifolds with conical singularities.

Cordes condition, Campanato nearness and beyond

Nikos Yannakakis (National Technical University of Athens, Greece)

Let X be a Banach space and $b \in X$. Many problems in mathematics lead to the problem of existence and uniqueness of solutions of the equation

$$Ax = b\,,$$

where A is a suitable operator from a set V into X. One common way to deal with this problem is to guarantee that A is, in some appropriate sense, near an invertible operator B. The subject of this mini course is Campanato nearness whose origins may be traced back to the so called Cordes condition.

A detour on Sobolev inequalities in the Heisenberg group and applications to critical subelliptic problems

Patrizia Pucci (University of Perugia, Italy)

Thanks to important applications to geometric control theory and PDEs, great attention has been recently devoted to the study of geometric inequalities in the context of stratified Lie groups. In the first part of the course we present some classical results and open problems related to the Sobolev inequalities in the Heisenberg group. In the second part, we apply these inequalities to obtain existence results for several nonlinear critical problems involving subelliptic operators in the Heisenberg group, even with non–standard growth conditions, also known as subelliptic (p, q) operators.

Maximal regularity as a tool for partial differential equations

Sylvie Monniaux (Aix-Marseille University, France)

In the last decades, a lot of progress has been made on the subject of maximal regularity. The property of maximal L^p property is an a priori estimate and reads as follows:

For A the negative generator of an analytic semigroup on a Banach space X, for $1 < p < \infty$, for $0 < T <= \infty$, does there exist $C_p > 0$ a constant such that for all $f \in L^p(0, T; X)$, there exists a unique solution $u \in L^p(0, T; D(A)) \cap W^{1,p}(0, T; X)$ of the equation $\partial_t(u) + Au = f$ with the estimate $||\partial_t(u)||_{L^p(X)} + ||Au||_{L^p(X)} \leq C_p||f||_{L^p(X)}$?

It started with a paper by De Simon in 1964 in which the author proved maximal L^p regularity for negative generators of analytic semigroups in Hilbert spaces. The next big step has been made by Dore and Venni in their 1987 paper on operators admitting bounded imaginary powers. The final result on maximal regularity was done by Weis in his 2001 paper where he gave a characterisation of negative generators of analytic semigroups in Banach spaces which have the maximal regularity property. These results are all about linear theory of unbounded operators in Banach spaces and will be the subject of my first lecture.

I will then show in the other two lectures how to use this property to find solutions or to prove uniqueness of solutions of semi linear partial differential equations of parabolic type such as: the non linear heat equation, Navier-Stokes equations, the magneto-hydrodynamical system (MHD), Keller-Segel Navier-Stokes system, etc…

Applications of hypercomplex analysis in the theory of reproducing kernel spaces and White noise analysis

Uwe Kähler (University of Aveiro, Portugal)

In these lectures we will discuss the theory of reproducing kernel spaces over non-commutative structures in the context of hypercomplex analysis. Afterwards we will apply it to develop the theory of white noise analysis and a Malliavin/Itô calculus. If time permits we are going to discuss how to construct the theory in general cases like grey noise spaces and ternary Grassmann algebras.

Introduction to hypercomplex analysis

Paula Carejeiras (University of Aveiro, Portugal)

In these lectures we will give an introduction to hypercomplex analysis, in particular its functional analytic and function theoretical foundations. As it is well known there is no function theory in higher dimensions which incorporates all tools from complex analysis. Here we will use an approach where the field of complex numbers is replaced by Clifford algebras. The main advantage of this approach consists in the possibility to combine much of the algebraical and geometrical properties of complex numbers. After a short introduction, we develop the necessary tools from a functional analytic and function theoretical point of view.

Tentative program:

1. Introduction to Clifford algebras

   Basics and definitions; main examples.

2. Basic functional analysis over Clifford algebra-valued linear spaces

   Right- and left-linear modules over Clifford algebras, Hilbert modules, Riesz representation theorem and Lax-Milgram lemma in Hilbert modules.

3. Introduction to hypercomplex function theory

   Dirac operators, Frank Sommen's approach to function theory: Sommen-Weyl relations, Euler and Gamma operators, homogeneous polynomials, monogenic homogeneous polynomials, Fischer inner product, Fischer decomposition, McDonald formula and connection with Fock space, CK-extension.

The Wodzicki residue in various disguises

Sylvie Paycha (University of Potsdam, Germany)

It is well-known that the Wodzicki residue is (up to a multiplicative factor) the only trace on the algebra of classical pseudodifferential operators on a closed manifold (of dimension larger than 1), an important property proved by M. Wodzicki in 1984, that we will discuss after having recalled the definition of the residue named after him. It is probably less known that the Wodzicki residue can serve as a tool to measure obstructions to the extendibility of the ordinary trace on trace-class pseudodifferential operators, to a linear form on the whole the algebra of classical pseudodifferential operators. We will briefly derive these obstructions from the so-called "defect formulae" we studied together with S. Scott (2005). On operators of order no larger than "minus" the dimension of the manifold, so those which are not trace-class, one can view the Wodzicki residue as a leading symbol trace in the spirit of work by S. Rosenberg and myself (2004). This is implicitly the point of view adopted in the recent work by E. van Erp and B. Yuncken (2015), as well as by B. Yuncken and N. Couchet (2022), who generalise the notion of Wodzicki residue to the groupoid setup for operators of order no larger than "minus" the dimension. This uses A. Connes' tangent groupoid, later revisited by C. Debord and G. Skandalis, which we shall briefly review before giving the main steps towards this groupoid version of the Wodzicki residue. The plan of the three lectures is as follows:

1. The Wodzicki residue as a useful tool to measure obstructions.
2. Two other useful tools: leading symbol traces and the tangent groupoid.
3. The Wodzicki residue generalised to a class of pseudodifferential operators on the tangent groupoid.

Perturbation of Dirichlet forms

Boguslaw Zegarlinski (Paul Sabatier University , France)

Coercive inequalities and applications.

Dynamics of concentrated vorticities in 2 and 3 dimensional Euler flows

Manuel Del Pino (University of Bath, UK)

We consider the Euler equations for an incompressible, inviscid fluid

$$\begin{cases} v_t + (v\cdot\nabla)v = -\nabla p \quad \text{in } \mathbb{R}^n \times (0,T) \\ \operatorname{div} v = 0 \end{cases} \tag{1}$$

where $n = 2, 3$. The $\omega = \operatorname{curl} v$ is called the vorticity of the solution. In these lectures we will analyze solutions with *highly concentrated vorticity*. More precisely with vorticity concentrated around a finite number of moving points (vortices) when $n = 2$, and around curves when $n = 3$. When $n = 2$, the vorticity-stream formulation of (1) is the scalar problem

$$\begin{cases} \omega_t + (v\cdot\nabla)\omega = 0 \quad \text{in } \mathbb{R}^2 \times (0,T) \\ v(x,t) = \displaystyle\int_{\mathbb{R}^2} K(x-y)\,\omega(y,t)\,dy, \quad K(x) = \frac{1}{2\pi}\frac{1}{|x|^2}(x_2,-x_1). \end{cases} \tag{2}$$

We discuss the construction of solutions to Problem (2) of the form

$$\omega(x,t) = \sum_{j=1}^{N} \frac{\kappa_j}{\varepsilon^2} W\left(\frac{x-\xi_j(t)}{\varepsilon}\right) + o(1)$$

in a fixed interval $[0,T]$ and $o(1) \to 0$ as $\varepsilon \to 0$ and $W(y)$ is a positive, rapidly decaying profile. For special configurations, for instance *two vortex pairs* travelling in opposite directions, we find solutions of this type with $o(1) \to 0$ as

$t \to +\infty$. We discuss the highly nontrivial generalization of these constructions to the generalized SQG equation, $0 < s < 1$,

$$\begin{cases} \omega_t + (v \cdot \nabla)\omega = 0 \quad \text{in } \mathbb{R}^2 \times (0,T) \\ v(x,t) = \displaystyle\int_{\mathbb{R}^2} K_s(x-y)\,\omega(y,t)\,dy, \quad K_s(x) = \frac{c_s}{|x|^{3-2s}}(x_2, -x_1). \end{cases}$$

In the 3-dimensional case we explain the *vortex filament conjecture* in its connection with the binormal flow of curves and find its first mathematical proof in the helical case. Finally, we establish the first mathematical justification of the *leapfrogging vortex ring dynamics* first conjectured by Helmholtz in 1858.

These results correspond to collaboration with Juan Davila (Bath), Antonio Fernandez (UAM, Madrid), Monica Musso (Bath), Shrish Parmeshwar (Imperial College London).

Elliptic systems of phase transition type

Panayotis Smyrnelis (University of Athens, Greece)

This mini course focuses on the vector Allen-Cahn equation which models coexistence of three or more phases of a substance. It is based on our recently published monograph: N. D. Alikakos, G. Fusco, P. Smyrnelis, Elliptic systems of phase transition type. Progress in Nonlinear Differential Equations and Their Applications Vol. 91, Springer-Birkhäuser (2019).

In 1978, De Giorgi suggested a striking analogy of the scalar Allen-Cahn PDE with minimal surface theory that led to significant developments in P.D.E. and the Calculus of Variations. In the vector case, the mathematical phenomena are considerably richer. Our main focus will be on the construction of entire solutions converging to the phases (i.e. the global minima of the potential), as the variable goes to infinity along certain directions.

More precisely, we will study:

1) The connecting orbit problem. In space dimension one, the vector Allen-Cahn equation reduces to a Hamiltonian system of O.D.E. Assuming that the zero level set of the potential is partitioned into two compact subsets, we shall establish the existence of heteroclinic, homoclinic and periodic orbits.

2) The Heteroclinic double layers problem. Heteroclinic double layers have initially been constructed by Alama-Bronsard-Gui (1997), and Schatzman (2002). They can also be obtained as heteroclinic orbits in a Hilbert space of functions, defined by the boundary conditions of the problem. It is an important result, providing the first examples of nontrivial minimal solutions for the Allen-Cahn system.

3) Symmetric structures. Symmetric solutions of the vector Allen-Cahn equation can be obtained in equivariant classes: (a) for general point groups, and (b) for general discrete reflection groups, thus establishing the existence of lattice solutions.

Boundary value problems for elliptic operators satisfying Carleson condition

Martin Dindos (The University of Edinburgh, UK)

The mini-course will present in concise form recent results, with illustrative proofs, on solvability of the L^p Dirichlet, Regularity and Neumann problems for scalar elliptic equations on Lipschitz domains with coefficients satisfying a naturally arising condition of Carleson type.

More precisely, with $L = \operatorname{div}(A\nabla)$, we assume that A variable coefficient matrix valued function such that A is uniformly elliptic and satisfies $|\nabla A(X)| \lesssim \operatorname{dist}(X, \partial\Omega)^{-1}$ and $|\nabla A|(X)^2 \operatorname{dist}(X, \partial\Omega)\, dX$ (or variants of this condition). We will show how condition like this one arises by considering a specific change of variables.

We will discuss two types of results, the first is the so-called "small Carleson" case where, for a given $1 < p < \infty$, we prove solvability of the three considered boundary value problems under assumption the Carleson norm of the coefficients and the Lipschitz constant of the considered domain is sufficiently small. The second type of results ("large Carleson") relaxes the constraints to any Lipschitz domain and to the assumption that the Carleson norm of the coefficients is merely bounded. In this case we have L^p solvability for a range of p's in a subinterval of $(1, \infty)$.

Finally, we will discuss possible extensions of these results to complex coefficients PDEs and PDE systems and interesting open problems.

Geometric Hardy inequalities

Gerassimos Barbatis (National and Kapodistrian University of Athens, Greece)

The topic of these lectures is the Hardy inequality (HI) with distance to the boundary, that is the inequality

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{d^2} dx \;, \qquad u \in C_c^\infty(\Omega),$$

where Ω is a Euclidean domain and $d(x) = \operatorname{dist}(x, \partial\Omega)$. We present an overview of various results related to this inequality aiming to convey some of the ideas and techniques that have been central in the study of these inequalities in the past few decades. In particular we shall address questions such as the validity of the HI, best constants, attainance of best constants and more. One of our aims will be to highlight the importance of geometric considarations in relation to the HI. We shall study in more detail the HI for planar domains and compute explicitly the best Hardy constant for certain classes of domains. The lectures will close with a brief mention of some open problems.

Some harmonic analysis in a general Gaussian setting

Valentina Casarino (University of Padova, Italy)

Given two $n \times n$ real matrices Q and B, such that Q is symmetric and positive definite and all the eigenvalues of B have negative real parts, we consider the Ornstein–Uhlenbeck semigroup $\big(T(t)\big)_{t>0}$, with covariance Q and drift B. This semigroup, generated by the Ornstein–Uhlenbeck operator $\mathcal{L}$, is usually nonsymmetric.

The lectures will begin with a survey of the main properties of the spectrum of $\mathcal{L}$. We shall later discuss the orthogonality of distinct generalized eigenspaces of $\mathcal{L}$, which cannot be guaranteed without selfadjointness.

Then we shall review some recent results in harmonic analysis, obtained in a joint work with Paolo Ciatti and Peter Sjögren, concerning maximal operators, variational bounds and multipliers associated to $\big(T(t)\big)_{t>0}$, with a focus on the differences between the symmetric and the nonsymmetric context.

Introduction to the restriction theory of the Fourier transform

Paolo Ciatti (University of Padova, Italy)

1. In the first lecture we plan to review some basic and standard result about the Fourier transform. The topics discussed will be: the definition of the Fourier transform of an integrable function and of a complex measure, the elementary properties of the transform, the inversion theorem, the Plancherel theorem and the definition of the Fourier transform in L^2, the uncertainty principle, the Fourier transform of an L^p function for $1 \leq p \leq 2$ and the Riesz-Thorin interpolation theorem, the Hausdorff-Young inequality.

2. In the second lecture we will give a brief introduction to the theory of oscillating integrals, focusing on the following topics: Van der Corput lemma, stationary phase principle, decay of the Fourier transform of a measure supported on a hypersurface and the role of the curvature.

3. Finally, in the third and last lecture we will discuss some classical results concerning the restriction problem: the restriction conjecture for the Fourier transform, the Knapp example, the restriction theorem in dimension two, the Tomas-Stein theorem.

Short talks

Inverse source problems for space-dependent sources in thermoelastic systems

Frederick Maes (Ghent University, Belgium)

Thermoelastic systems describe the interaction between changes of the shape of an object and the fluctuation in temperature. We consider isotropic thermoelastic systems of type-III in a bounded domain. Several uniqueness results for inverse source problems will be discussed under the assumption that either the heat source or the load source can be decomposed as a product of a given time-dependent and an unknown space-dependent function. The main goal is to find the spatial component of the source, given some suitable measurement(s). The measurements under consideration include final time and time-average measurements.

Symmetry equivalences and dynamic analysis of Euler-Bernoulli beams

Daulet Nurakhmetov (Institute of Mathematics and Mathematical Modelling and Nazarbayev University, Kazakhstan)

We will talk two problems:
1st problem. We will describe symmetry equivalences of eigenvalues and eigenfunctions of Euler- Bernoulli beams on the Winkler foundation [1].
2nd problem. We will show analysis of dynamic pull-in voltage for a micro-electro-mechanical oscillator of the platform type made of nonlinear materials [2].

References

[1]. D. Nurakhmetov, S. Jumabayev, A. Aniyarov, R. Kussainov, Symmetric properties of eigenvalues and eigenfunctions of uniform beams, Symmetry, Vol.12, 2020, 2097, pp.1–13.
[2]. P. Skrzypacz, D. Wei, D. Nurakhmetov et al, Analysis of dynamic pull-in voltage and response time for a micro-electro-mechanical oscillator made of power-law materials, Nonlinear Dynamics, Vol. 105, 2021, pp. 227-240

$S(m, g)$ calculus and applications to the analysis of PDEs

Julio Delgado (University of Valle, Colombia)

In this talk we present some applications of the $S(m, g)$ calculus to the analysis of PDEs. We establish first the well-posedness for a class of degenerate Schrödinger equations with irregular potentials and secondly some spectral properties for a class of anharmonic oscillators by mean of the membership to corresponding Schatten-von Neumann ideals.

Stein-Weiss-Adams inequality on Morrey spaces

Aidyn Kassymov (Institute of Mathematics and Mathematical Modelling, Kazakhstan)

We establish Adams type Stein-Weiss inequality on global Morrey spaces on general homogeneous groups. Special properties of homogeneous norms and some boundedness results on global Morrey spaces play key roles in our proofs. As consequence, we obtain fractional Hardy, Hardy-Sobolev, Rellich and Gagliardo-Nirenberg inequalities on Morrey spaces on stratified groups. While the results are obtained in the setting of general homogeneous groups, they are new already for the Euclidean space $\mathbb{R}^n$.