Topics on geometric Hardy inequalities

Gerassimos Barbatis
National and Kapodistrian University of Athens

Ghent University
August 31 - September 1, 2023

Contents:

Introduction.

Hardy inequality.

Let $1 < p < +\infty$ and let $\alpha > 0$. Then

$$\int_0^\alpha |u'(t)|^p dt \geq \left(\frac{p-1}{p}\right)^p \int_0^\alpha \frac{|u|^p}{t^p} dt\ , \quad \text{for all } u \in C_c^\infty(0,\alpha)$$

- The power t^p is optimal
- The constant $\left(\frac{p-1}{p}\right)^p$ is optimal.
- There is no minimizer in $W_0^{1,p}(0,\alpha)$

These remain true if $(0,\alpha)$ is replaced by $(0,+\infty)$.

For $p = 2$ the inequality reads

$$\int_0^\alpha u'^2 dt \geq \frac{1}{4} \int_0^\alpha \frac{u^2}{t^2} dt\ , \qquad u \in C_c^\infty(0,\alpha)$$

Proof of the Hardy inequality:

Given $u \in C_c^\infty(0,\alpha)$ we have

$$\begin{aligned} 0 &\le \int_0^\alpha \left(u' - \frac{1}{2t}u\right)^2 dt \\ &= \int_0^\alpha u'^2 dt + \frac{1}{4}\int_0^\alpha \frac{u^2}{t^2}dt - \frac{1}{2}\int_0^\alpha \frac{(u^2)'}{t}dt \\ &= \int_0^\alpha u'^2 dt + \frac{1}{4}\int_0^\alpha \frac{u^2}{t^2}dt - \frac{1}{2}\int_0^\alpha \frac{u^2}{t^2}dt \end{aligned}$$

and therefore

$$\int_0^\alpha u'^2 dt \ge \frac{1}{4}\int_0^\alpha \frac{u^2}{t^2}dt \; .$$

Equality holds if and only if $u(t) = ct^{1/2}$, which does not belong in $H_0^1(0,\alpha)$.

To prove the optimality of the exponent and the constant use the sequence

$$u_\epsilon(t) = t^{\frac{1}{2}+\epsilon}\phi(t) \; , \quad \epsilon > 0,$$

where $\phi(t)$ is a smooth function with $\phi(t) = 1$ near $t = 0$ and $\phi(t) = 0$ near $t = \alpha$.

Higher dimensional analogues.

There are two main generalizations of the Hardy inequality in higher dimensions.

A. Hardy inequalities involving distance to a point: let $\Omega \subset \mathbb{R}^n$ be a domain containing the origin and assume that $p < n$. There holds

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \ , \qquad u \in C_c^\infty(\Omega)$$

In case $p > n$ the same inequality remains true, the constant now being $\left|\frac{n-p}{p}\right|^p$. But in this case the inequality is valid for $u \in C_c^\infty(\Omega \setminus \{0\})$.

In both cases the constant $\left|\frac{n-p}{p}\right|^p$ is sharp. This is seen using the sequence

$$u_\epsilon(x) = |x|^{\frac{p-n}{p}+\epsilon}\phi(x) \ , \quad \epsilon > 0,$$

where ϕ is a function in $C_c^\infty(\Omega)$ with $\phi(x) = 1$ near the origin.

B. Hardy inequalities involving distance to the boundary: let $\Omega \subset \mathbb{R}^n$ be a domain and $d(x) = \mathrm{dist}(x, \partial\Omega)$.

We say that the (geometric) Hardy inequality is valid for the domain Ω if there exists $c > 0$ such that

$$\int_\Omega |\nabla u|^p dx \geq c \int_\Omega \frac{|u|^p}{d^p}\, dx \ , \qquad u \in C_c^\infty(\Omega)$$

This inequality is not always valid: the geometry of Ω plays a role.

We shall restrict attention to the case $p = 2$. We define

$$H(\Omega) = \inf_{u \in C_c^\infty(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx}{\displaystyle\int_\Omega \frac{u^2}{d^2}\, dx} \ ,$$

the *Hardy constant* of the domain Ω.

It is easy to see that

- The Hardy constant is invariant under dilations, that is $H(\Omega) = H(\lambda\Omega)$, $\lambda > 0$
- There is no domain monotonicity for the Hardy constant

Q: When is the Hardy inequality valid? What can we say about $H(\Omega)$?

In some sense $H(\Omega)$ is more rigid than the first Dirichlet eigenvalue.

Motivation.

G.H. Hardy

"I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world."

Heat equation in $\mathbb{R}^n$

$$\begin{cases} u_t = \Delta u, & t > 0,\ x \in \mathbb{R}^n, \\ u = u_0, & t = 0,\ x \in \mathbb{R}^n. \end{cases}$$

Solution

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{|x-y|^2}{4t}} u_0(y)dy.$$

The function

$$h(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{|x-y|^2}{4t}}$$

is the heat kernel (fundamental solution).

Consider know the heat equation on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} u_t = \Delta u, & t > 0,\ x \in \mathbb{R}^n, \\ u = u_0, & t = 0,\ x \in \mathbb{R}^n, \\ u = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

The solution is again represented as

$$u(x,t) = \int_\Omega h(t,x,y) u_0(y) dy.$$

The heat kernel $h(t,x,y)$ cannot be written explicitly. Given a subdomain $V \subset\subset \Omega$ and $T > 0$ there exist $C > 0$ such that

$$C^{-1} t^{-\frac{n}{2}} e^{-C \frac{|x-y|^2}{t}} < h(t,x,y) < C t^{-\frac{n}{2}} e^{-C^{-1} \frac{|x-y|^2}{t}}$$

for all $0 < t < T$ and $x, y \in V$.

Assume now the additional presence of a potential $V(x)$, $x \in \Omega$.

$$\begin{cases} u_t = \Delta u + V(x), & t > 0,\ x \in \Omega, \\ u = u_0, & t = 0,\ x \in \Omega, \\ u = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

If V is bounded then the same estimate as above is valid. But what is V is unbounded? In particular we are interest in the case where $\sup_\Omega V(x) = +\infty$.

Assume for simplicity that

$$V(x) \geq 0 \,, \qquad x \in \Omega.$$

The behaviour of the heat kernel depends on 'how bad' the potential $V(x)$ is.

Consider potentials of the specific form

$$V(x) = \lambda \frac{1}{|x|^\beta}$$

where Ω contains the origin and $\beta,\ \lambda > 0$.

Then the following is true:

- If $0 < \beta < 2$ then the same estimate as above is valid:

  $$C^{-1}t^{-\frac{n}{2}}e^{-C\frac{|x-y|^2}{t}} < h(t,x,y) < Ct^{-\frac{n}{2}}e^{-C^{-1}\frac{|x-y|^2}{t}}$$

  for all $0 < t < T$ and $x, y \in V$.

- For $\beta > 2$ the problem is ill-posed.

In this sense the potential

$$V(x) = \lambda\frac{1}{|x|^2}$$

is a critical potential.

Let $V(x) = \lambda \frac{1}{|x|^2}$. Then for $0 < t < T$ and $x, y \in V$ we have:

- If $0 < \lambda \leq (\frac{n-2}{2})^2$ then

$$C^{-1} t^{-\frac{n}{2}} |x|^{-\alpha} |y|^{-\alpha} e^{-C \frac{|x-y|^2}{t}} < h(t,x,y) < C t^{-\frac{n}{2}} |x|^{-\alpha} |y|^{-\alpha} e^{-C^{-1} \frac{|x-y|^2}{t}}$$

where $\alpha > 0$ is the smallest solution of the equation $\alpha(n-2-\alpha) = \lambda$.

- If $\lambda > (\frac{n-2}{2})^2$ then the problem is ill-posed.

The above are intrically related to the Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx \,, \qquad u \in C_c^\infty(\Omega)$$

and its critical nature. Analogous results are valid for the geometric Hardy inequality (distance to the boundary).

I. The Hardy inequality in $\mathbb{R}^n$.

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{d^2}\, dx\,, \qquad u \in C_c^\infty(\Omega)$$

Sufficient conditions for the Hardy inequality.

Boundary regularity.

If $\partial\Omega$ is bounded, then some boundary regularity is enough.

Let $\Omega \subset \mathbb{R}^n$ be bounded. Assume that for each $y \in \partial\Omega$ there exists an open neighbourhood V_y of y and $c_y > 0$ such that

$$\int_{V_y} |\nabla u|^2 dx \geq c_y \int_{V_y} \frac{u^2}{d^2}\, dx\,, \qquad u \in C_c^\infty(V_y).$$

Then the Hardy inequality is valid in Ω.

This is the case, in particular, if Ω has a Lipschitz boundary.

Davies's mean distance function method.

Let $\Omega \subset \mathbb{R}^n$. For $\omega \in S^{n-1}$ and $x \in \Omega$ denote

$$L_\omega(x) = \Omega \cap \{x + s\omega : s \in \mathbb{R}\}$$

and

$$d_\omega(x) = \min\{s > 0 : \; x + s\omega \notin \Omega \text{ or } x - s\omega \notin \Omega\}$$

Let $u \in C_c^\infty(\Omega)$. By the Hardy inequality in one dimension we have

$$\int_{L_\omega} (\nabla u \cdot \omega)^2 ds \geq \frac{1}{4} \int_{L_\omega} \frac{u^2}{d_\omega^2} ds.$$

Intagrating over all directions perpendicular to ω we obtain

$$\int_\Omega (\nabla u \cdot \omega)^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d_\omega^2} dx.$$

[REP] $$\int_\Omega (\nabla u\cdot\omega)^2 dx \ge \frac{1}{4}\int_\Omega \frac{u^2}{d_\omega^2}dx.$$

Now average over all $\omega\in S^{n-1}$. Get

$$⨍_{S^{n-1}}\int_\Omega |\nabla u\cdot\omega|^2 dx\, dS(\omega) \ge \frac{1}{4}⨍_{S^{n-1}}\int_\Omega \frac{u^2}{d_\omega^2}dx\, dS(\omega)\,.$$

But for any $p\in\mathbb{R}^n$, $⨍_{S^{n-1}}|p\cdot\omega|^2 dS(\omega)=\frac{1}{n}|p|^2$, hence

$$\frac{1}{n}\int_\Omega |\nabla u|^2 dx \ge \frac{1}{4}\int_\Omega \frac{u^2}{d_{av}^2}dx$$

where

$$\frac{1}{d_{av}^2(x)} = ⨍_{S^{n-1}}\frac{u^2}{d_\omega^2(x)}dS(\omega)\,, \quad x\in\Omega,$$

defines the *mean distance function* $d_{av}(x)$, $x\in\Omega$.

So: if the exists $c>0$ such that

$$d_{av}(x)\le c d(x)\,, \quad x\in\Omega,$$

then the Hardy inequality is valid on Ω.

Note. The mean distance function is useful also for L^p and higher-order Hardy inequalities.

Remark. If Ω is unbounded then boundary regularity is not enough to guarantee the validity of the Hardy inequality.

Assume for example that $\Omega = \mathbb{R}^2 \setminus D(1)$. Then, by the scale invariance of the Hardy constant $(H(\Omega) = H(\lambda\Omega),\ \lambda > 0)$

$$H(\mathbb{R}^2 \setminus D(1)) = H(\mathbb{R}^2 \setminus D(r)) = \lim_{r \to 0} H(\mathbb{R}^2 \setminus D(r)) = H(\mathbb{R}^2 \setminus \{0\}) = 0 ,$$

since the inequality

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx , \quad u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$$

is sharp also for $n = 2$.

Note. In what follows we shall be considering (mostly) Lipschitz domains.

The importance of $H(\Omega)$ being 1/4

Theorem. If some part of $\partial\Omega$ is C^2 then $H(\Omega) \leq 1/4$.

Proof. For $\epsilon > 0$ define

$$u_\epsilon(x) = d(x)^{\frac{1}{2}+\epsilon}\phi(x)$$

where ϕ is supported in a small enough neighbourhood where $\partial\Omega$ is C^2. Then $u_\epsilon \in H_0^1(\Omega)$ and

$$\frac{\displaystyle\int_\Omega |\nabla u_\epsilon|^2 dx}{\displaystyle\int_\Omega \frac{u_\epsilon^2}{d^2}\, dx} = \frac{(\frac{1}{2}+\epsilon)^2 \int_\Omega d^{-1+2\epsilon}\phi\, dx + O(1)}{\int_\Omega d^{-1+2\epsilon}\phi\, dx} \longrightarrow \frac{1}{4},$$

as $\epsilon \to 0$.

A dichotomy for the associated minimization problem:

$$H(\Omega) = \inf_{H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{d^2} dx}$$

Theorem. Assume that Ω is bounded with C^2 boundary. Then the infimum is attained if and only if

$$H(\Omega) < \frac{1}{4} .$$

Moreover, if a minimizer $u \in H_0^1(\Omega)$ exists, then

$$u(x) \asymp d(x)^\alpha , \qquad x \in \Omega ,$$

where α is the largest solution of the equation

$$\alpha(1-\alpha) = H(\Omega)$$

Proof. (only for the existence or non-existence of a minimizer) For $\beta > 0$ define

$$\Omega_\beta = \{x \in \Omega \ : \ d(x) < \beta\}$$

($\Leftarrow$) It can be shown that there exists $\beta > 0$ so that

$$\int_{\Omega_\beta} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\Omega_\beta} \frac{u^2}{d^2} dx , \qquad u \in C_c^\infty(\Omega_\beta).$$

Let $(w_k) \subset H_0^1(\Omega)$ be a minimizing sequence for the Hardy quotient normalized so that $\int_\Omega \frac{w_k^2}{d^2}dx = 1$; hence

$$\int_\Omega |\nabla w_k|^2 dx \longrightarrow H(\Omega), \quad \text{as } k \to \infty.$$

The sequence (w_k) has a weakly convergent in $H_0^1(\Omega)$ subsequence, $w_k \rightharpoonup w_0 \in H_0^1(\Omega)$. Let $v_k = w_k - w_0$. Then

$$\int_\Omega |\nabla w_k|^2 dx = \int_\Omega |\nabla v_k|^2 dx + \int_\Omega |\nabla w_0|^2 dx + o(1)$$

and

$$\int_\Omega \frac{w_k^2}{d^2} dx = \int_\Omega \frac{v_k^2}{d^2} dx + \int_\Omega \frac{w_0^2}{d^2} dx + o(1)\,.$$

Let $\phi \in C^\infty(\Omega)$ be a function such that $0 \le \phi \le 1$, $\phi(x) = 1$ if $d(x) < \beta/2$ and $\phi(x) = 0$, if $d(x) > \beta$. Writing $v_k = \phi v_k + (1-\phi)v_k$ and integrating by parts we find

$$\begin{aligned}\int_\Omega |\nabla v_k|^2 dx &= \int_\Omega |\nabla(\phi v_k)|^2 dx + \int_\Omega |\nabla((1-\phi)v_k)|^2 dx \\ &\quad +2\int_\Omega \phi(1-\phi)|\nabla v_k|^2 dx + \int_\Omega (2\phi-1)\Delta\phi\, v_k^2\, dx\,.\end{aligned}$$

The last integral converges to zero by the compactness of the imbedding $H_0^1(\Omega) \subset L^2(\Omega)$.

It follows that

$$\begin{aligned}\int_\Omega |\nabla v_k|^2 dx &\geq \int_\Omega |\nabla(\phi v_k)|^2 dx + o(1)\\ &\geq \frac{1}{4}\int_\Omega \frac{\phi^2 v_k^2}{d^2} dx + o(1)\\ &= \frac{1}{4}\int_\Omega \frac{v_k^2}{d^2} dx + \frac{1}{4}\int_\Omega \frac{(\phi^2-1)v_k^2}{d^2} dx + o(1)\\ &= \frac{1}{4}\int_\Omega \frac{v_k^2}{d^2} dx + o(1),\end{aligned}$$

by compactness and the fact that $\phi = 1$ near $\partial\Omega$.
Combining the above we obtain

$$\begin{aligned}H(\Omega) &= \int_\Omega |\nabla v_k|^2 dx + \int_\Omega |\nabla w_0|^2 dx + o(1)\\ &\geq \frac{1}{4}\int_\Omega \frac{v_k^2}{d^2} dx + H(\Omega)\int_\Omega \frac{w_0^2}{d^2} dx + o(1)\\ &= \frac{1}{4}\Big(1 - \int_\Omega \frac{w_0^2}{d^2} dx\Big) + H(\Omega)\int_\Omega \frac{w_0^2}{d^2} dx + o(1),\end{aligned}$$

and hence

$$\Big(H(\Omega) - \frac{1}{4}\Big)\Big(1 - \int_\Omega \frac{w_0^2}{d^2} dx\Big) \geq 0.$$

$$\text{[REP]} \qquad \Big(H(\Omega) - \frac{1}{4}\Big)\Big(1 - \int_\Omega \frac{w_0^2}{d^2}\,dx\Big) \geq 0.$$

Since $H(\Omega) < 1/4$, we conclude that $\int_\Omega w_0^2/d^2\,dx \geq 1$. By weak lower semicontinuity we conlude that

$$H(\Omega) \geq \int_\Omega |\nabla w_0|^2 dx \geq H(\Omega) \int_\Omega \frac{w_0^2}{d^2} dx \geq H(\Omega).$$

Hence w_0 is a minimizer.

($\Rightarrow$) Assume for contradiction that $H(\Omega) = 1/4$ and $u_0 \in H_0^1(\Omega)$ is a minimizer.

There exists $\beta > 0$ such that for any $0 < \eta < 1/4$ the function

$$v = d^{\frac{1}{2}+\eta} + d$$

is a weak subsolution to the Euler equation in Ω_β, that is

$$\Delta v + \frac{1}{4d^2} v \geq 0\,, \qquad \text{in } \Omega_\beta.$$

Let $C > 0$ be such that

$$v < Cu_0 \ , \ \text{on } \{d(x) = \frac{\beta}{2}\}.$$

Then the function $w = (v - Cu_0)_+$ is a subsolution in the set $\Omega_{\beta/2}$ which vanishes in a neighbourhood of $\{d(x) = \beta/2\}$.

So

$$\int_{\Omega_{\beta/2}} \Big(\nabla w \cdot \nabla \phi - H(\Omega)\frac{w\phi}{d^2}\Big) dx \leq 0$$

for any non-negative $\phi \in C_c^\infty(\Omega_{\beta/2})$. Taking $\phi = \psi^2 w$ where $\psi \in C_c^\infty(\Omega)$ we obtain

$$\int_{\Omega_{\beta/2}} \Big(\nabla w \cdot \nabla(\psi^2 w) - \frac{\psi^2 w^2}{4d^2}\Big) dx \leq 0.$$

Using a simple identity for $\nabla(\psi^2 w)$ this is written

$$\int_{\Omega_{\beta/2}} \Big(|\nabla(\psi w)|^2 - \frac{\psi^2 w^2}{4d^2}\Big) dx \leq \int_{\Omega_{\beta/2}} w^2 |\nabla \psi|^2 dx \, .$$

[REP] $$\int_{\Omega_{\beta/2}} \Big(|\nabla(\psi w)|^2 - \frac{\psi^2 w^2}{4d^2} \Big) dx \le \int_{\Omega_{\beta/2}} w^2 |\nabla \psi|^2 dx$$

Also, since u_0 is a positive solution,

$$\int_{\Omega_{\beta/2}} \Big(\nabla u_0 \cdot \nabla (\frac{\psi^2 w^2}{u_0}) - \frac{1}{4} u_0 \frac{\psi^2 w^2}{u_0 d^2} \Big) dx = 0,$$

i.e.

$$\int_{\Omega_{\beta/2}} u_0^2 |\nabla (\frac{\psi w}{u_0})|^2 dx = \int_{\Omega_{\beta/2}} \Big(|\nabla(\psi w)|^2 - \frac{\psi^2 w^2}{4d^2} \Big) dx\,.$$

So

$$\int_{\Omega_{\beta/2}} u_0^2 |\nabla (\frac{\psi w}{u_0})|^2 dx \le \int_{\Omega_{\beta/2}} w^2 |\nabla \psi|^2 dx$$

Now take $\psi = \psi_\epsilon$ where

$$\psi_\epsilon = 0, \text{ in } \Omega_{\epsilon/2}\,, \quad \psi_\epsilon = 1, \text{ in } \Omega \setminus \Omega_\epsilon\,, \quad |\nabla \psi_\epsilon| \le \frac{c}{\epsilon}.$$

We obtain

$$\int_{\Omega_{\beta/2}} u_0^2 |\nabla (\frac{\psi_\epsilon w}{u_0})|^2 dx \le \frac{c}{\epsilon^2} \int_{\{\frac{\epsilon}{2} < d(x) < \epsilon\}} w^2 dx$$

[REP] $$\int_{\Omega_{\beta/2}} u_0^2 \left|\nabla\left(\frac{\psi_\epsilon w}{u_0}\right)\right|^2 dx \le \frac{c}{\epsilon^2}\int_{\{\frac{\epsilon}{2}<d(x)<\epsilon\}} w^2 dx$$

Hence, for any $\eta > 0$ small,

$$\begin{aligned}
\int_{\Omega_{\beta/2}} u_0^2 \left|\nabla\left(\frac{w}{u_0}\right)\right|^2 dx &= \int_{\Omega_{\beta/2}} u_0^2 \lim_{\epsilon\to 0} \left|\nabla\left(\frac{\psi_\epsilon w}{u_0}\right)\right|^2 dx \\
&\le \liminf_{\epsilon\to 0} \int_{\Omega_{\beta/2}} u_0^2 \left|\nabla\left(\frac{\psi_\epsilon w}{u_0}\right)\right|^2 dx \\
&\le c \liminf_{\epsilon\to 0} \frac{c}{\epsilon^2} \int_{\{\frac{\epsilon}{2}<d(x)<\epsilon\}} w^2 dx \\
&\le c \liminf_{\epsilon\to 0} c \int_{\{\frac{\epsilon}{2}<d(x)<\epsilon\}} \frac{d^{1+2\eta}}{d^2} dx \\
&= 0.
\end{aligned}$$

It follows that there exists $c > 0$ such that $w = cu_0$ in $\Omega_{\beta/2}$. Hence $w = 0$ in $\Omega_{\beta/2}$.

Recalling that

$$w = (v - Cu_0)_+$$

we obtain

$$u_0 \geq \frac{1}{C} v \geq \frac{1}{C} d^{\frac{1}{2}+\eta}, \qquad \text{in } \Omega_{\beta/2}.$$

Letting $\eta \to 0$ we get

$$u_0 \geq \frac{1}{C} d^{\frac{1}{2}}.$$

Hence

$$\int_{\Omega_{\beta/2}} \frac{u_0^2}{d^2} dx \geq \frac{1}{C^2} \int_{\Omega_{\beta/2}} \frac{1}{d} dx = +\infty,$$

a contradiction. □

Q: Are there domains for which $H(\Omega) = 1/4$?

A: Yes. The simplest is the half-space $\mathbb{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$:

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \geq \int_{\mathbb{R}^{n-1}} \int_0^\infty u_{x_n}^2 dx_n dx' \geq \frac{1}{4} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{u^2}{x_n^2} dx_n dx' = \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{d^2} dx$$

Theorem. If Ω is convex then $H(\Omega) = 1/4$.

Proof. We have already seen that for any $\Omega \subset \mathbb{R}^n$ there holds

$$\int_\Omega |\nabla u|^2 dx \geq \frac{n}{4} \int_\Omega \frac{u^2}{d_{av}^2} dx$$

So it is enough to show that if Ω is convex then $d_{av}^{-2} \geq \frac{1}{n} d^{-2}$ in Ω.
Let $x \in \Omega$. Let $y \in \partial\Omega$ be such that $|y - x| = d(x)$ and let P_y be a supporting hyperplane at y. Let $\sigma_\omega(x)$ be the point of intersection of the half-line $x + t(y - x)$, $t > 0$, with P_y. as in the diagram. Then

$$\begin{aligned}
\frac{1}{d_{av}^2(x)} &= ⨍_{S^{n-1}} \frac{dS(\omega)}{d_\omega^2(x)} = 2 ⨍_{S_+} \frac{dS(\omega)}{d_\omega^2(x)} \\
&\geq 2 ⨍_{S_+} \frac{dS(\omega)}{\sigma_\omega^2(x)} \\
&= 2 ⨍_{S_+} \frac{\big((y - x)\cdot\omega)\big)^2}{d^4(x)} dS(\omega) \\
&= \frac{1}{d^4(x)} ⨍_{S^{n-1}} \big((y - x)\cdot\omega)\big)^2 dS(\omega) \\
&= \frac{1}{n\, d^2(x)}
\end{aligned}$$

□

The positive supersolution method.

Let ϕ be a positive function on Ω. For any $u \in C_c^\infty(\Omega)$,

$$0 \le \int_\Omega |\nabla u - \frac{\nabla \phi}{\phi} u|^2 dx = \int_\Omega |\nabla u|^2 dx + \int_\Omega \frac{|\nabla \phi|^2}{\phi^2} u^2 dx - \int_\Omega \nabla u^2 \cdot \frac{\nabla \phi}{\phi} dx$$

$$\Longrightarrow \quad \int_\Omega |\nabla u|^2 dx \ge -\int_\Omega \frac{\Delta \phi}{\phi} u^2 dx\,.$$

Hence, if in addition

$$\Delta \phi + \frac{c}{d^2}\phi \le 0\,, \quad \text{in } \Omega\,,$$

then

$$\int_\Omega |\nabla u|^2 dx \ge c \int_\Omega \frac{u^2}{d^2} dx\,.$$

Conclusion: To prove the Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \ge c \int_\Omega \frac{u^2}{d^2} dx\,, \quad u \in C_c^\infty(\Omega).$$

it is enough to find a (weak) positive supersolution ϕ to the Euler-Lagrange equation,

$$\Delta \phi + \frac{c}{d^2}\phi \le 0\,.$$

Assume that the domain Ω is such that

$$\Delta d \leq 0, \qquad \text{in } \Omega.$$

Then for the function $\phi = d^{1/2}$ we compute

$$\Delta\phi + \frac{1}{4d^2}\phi = \mathrm{div}\left(\frac{1}{2}d^{-\frac{1}{2}}\nabla d\right) + \frac{1}{4}d^{-\frac{3}{2}} = \frac{1}{2}d^{-\frac{1}{2}}\Delta d \leq 0$$

Definition. A domain $\Omega \subset \mathbb{R}^n$ is weakly mean convex if $\Delta d \leq 0$ in Ω.

Hence we have

Theorem. If Ω is weakly mean convex then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\, dx\,, \qquad u \in C_c^\infty(\Omega)\,.$$

Remark. The condition $\Delta d \leq 0$ must be understood in the distributional sense.

Examples.

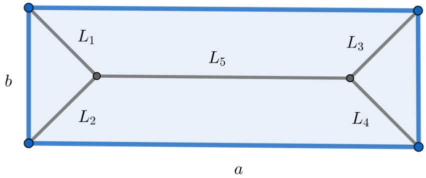
1. Suppose $\Omega = (0,1)$. Then

$$\int_0^1 d''(t)\phi(t)dt = -2\phi(\frac{1}{2})$$

2. Suppose $\Omega = B_1 \subset \mathbb{R}^n$, $n \geq 2$. Then

$$\Delta d = -\frac{n-1}{|x|}$$

3. Suppose $\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$ where $a > b > 0$.



Then

$$\int_\Omega \Delta d \phi \, dx = -\sqrt{2} \sum_{k=1}^{4} \int_{L_k} \phi \, ds - 2 \int_{L_5} \phi \, ds$$

Concerning weak mean convexity:

- Any convex domain is weakly mean convex.
- If Ω is bounded with C^2 boundary then weak mean convexity is equivalent to mean convexity, i.e. to the mean curvature being non-negative.
- In two dimensions and for C^2 boundary mean convexity is equivalent to convexity.
- In three of more dimensions there exist mean convex domains which are not convex.

Other domains for which $H(\Omega) = \frac{1}{4}$.

- Let $n \geq 3$. The annulus

$$\{x \in \mathbb{R}^n : r < |x| < R\}$$

has Hardy constant equal to $1/4$.

Proof. The function

$$\phi(x) = |x|^{\frac{n-1}{2}} d^{\frac{1}{2}}(x)$$

is a positive supersolution to the Euler equation

- Let $\Omega \subset \mathbb{R}^n$ be bounded and $D = \sup_\Omega d(x)$. There exists a constant c_n such that if each $y \in \partial\Omega$ admits an exterior ball of radius (at least) $c_n D$ then the Hardy constant of Ω is equal to $1/4$.

Q: What about domains with $H(\Omega) < \frac{1}{4}$?

II. The Hardy inequality in two dimensions

Theorem. If $\Omega \subset \mathbb{R}^2$ is simply connected then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{16} \int_\Omega \frac{u^2}{d^2}\, dx\ , \qquad u \in C_c^\infty(\Omega).$$

Proof. Makes use of Koebe's 1/4 theorem: If $g : \mathbb{D} \to g(\mathbb{D})$ is conformal, then

$$g(\mathbb{D}) \supseteq D\left(g(0), \frac{|g'(0)|}{4}\right)$$

In particular if $\Omega = g(\mathbb{D})$ then

$$d(g(0)) \geq \frac{1}{4}|g'(0)|.$$

Let $f : \mathbb{C}_+ \to \Omega$ be a conformal map.

Claim: for any $z = x + iy \in \mathbb{C}_+$ there holds

$$d(f(z)) \geq \frac{y}{2}|f'(z)|.$$

Proof of Claim. Write $z = x + iy$. The map

$$h_z(w) = \frac{\overline{z}\, w - z}{w - 1}$$

maps conformally $\mathbb{D}$ onto $\mathbb{C}_+$ and satisfies $h_z(0) = z$, $h_z'(0) = 2y$. Then

$$g_z = f \circ h_z$$

maps conformally $\mathbb{D}$ onto Ω. It follows that

$$d(g_z(0)) \geq \frac{1}{4}|g_z'(0)|.$$

But

$$g_z(0) = f(z) \quad , \qquad g_z'(0) = 2yf'(z)$$

so

$$d(f(z)) \geq \frac{y}{2}|f'(z)|$$

[REP] $$d(f(z)) \geq \frac{y}{2}|f'(z)|$$

Therefore, given $u \in C_c^\infty(\Omega)$,

$$\begin{aligned}
\int_\Omega |\nabla u|^2 dx' dy' &= \int_{\mathbb{C}_+} |\nabla(u \circ f)|^2 dx\, dy \\
&\geq \int_{-\infty}^{\infty} \int_0^{\infty} |\nabla(u \circ f)|^2 dy\, dx \\
&\geq \frac{1}{4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{(u \circ f)^2}{y^2} dy\, dx \\
&\geq \frac{1}{4} \cdot \frac{1}{4} \int_{-\infty}^{\infty} \int_0^{\infty} (u \circ f)^2 \frac{|f'(z)|^2}{d(f(z))^2} dy\, dx \\
&= \frac{1}{16} \int_{\mathbb{C}_+} (u \circ f)^2 \frac{|f'(z)|^2}{d(f(z))^2} \\
&= \frac{1}{16} \int_\Omega \frac{u^2}{d^2} dx' dy' .
\end{aligned}$$

A modified version of the above theorem:

Theorem. Let $\Omega \subset \mathbb{R}^2$ be simply connected and satisfy an external cone condition: each $y \in \partial\Omega$ is the vertex of an infinite angle of size θ which contains Ω. Then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{\pi^2}{4\theta^2} \int_\Omega \frac{u^2}{d^2}\, dx\,, \quad \text{for all } u \in C_c^\infty(\Omega).$$

Best case: Ω convex $\longrightarrow \theta = \pi \longrightarrow H(\Omega) = \frac{1}{4}$

Worst case: $\theta = 2\pi \longrightarrow \frac{1}{16}$

The proof makes use of a modified version of Koebe's 1/4 theorem.

Open problem.

Find the best uniform Hardy constant over all simply connected domains $\Omega \subset \mathbb{R}^2$. That is, find the largest constant H^* such that

$$\int_\Omega |\nabla u|^2 dx \geq H^* \int_\Omega \frac{u^2}{d^2}\, dx$$

for all simply connected domains $\Omega \subset \mathbb{R}^2$ and for all $u \in C_c^\infty(\Omega)$.

Moreover, determine whether there are extremal domains, that is domains Ω for which $H(\Omega) = H^*$.

It follows from the preceding discussion that

$$\frac{1}{16} \leq H^* \leq \frac{1}{4}.$$

The anguler sector.

Let Λ_β denote the infinite sector of angle β,

$$\Lambda_\beta = \{(r,\theta) : r > 0\,,\ \ 0 < \theta < \beta\}.$$

Symmetry plays an important role. Note that

$$\frac{1}{d^2} = \begin{cases} \frac{1}{r^2 \sin^2\theta}, & 0 < \theta < \frac{\pi}{2}, \\ \frac{1}{r^2}, & \frac{\pi}{2} < \theta < \beta - \frac{\pi}{2}, \\ \frac{1}{r^2 \sin^2(\beta-\theta)}, & \beta - \frac{\pi}{2} < \theta < \beta. \end{cases} =: \ \frac{1}{r^2} V_\beta(\theta)$$

Lemma. The Hardy constant $H(\Lambda_\beta)$ coincides with the best constant c_β for the Hardy-type inequality

$$\int_0^\beta g'(\theta)^2 d\theta \geq c_\beta \int_0^\beta g(\theta)^2 V_\beta(\theta) d\theta\,, \quad g \in C_c^\infty(0,\beta).$$

Proof. Let $u \in C_c^\infty(\Lambda_\beta)$. Then

$$\begin{aligned}\int_{\Lambda_\beta} |\nabla u|^2 dx &= \int_0^\beta \int_0^\infty \left(u_r^2 + \frac{1}{r^2} u_\theta^2\right) r\, dr\, d\theta \\ &\geq c_\beta \int_0^\beta \int_0^\infty \frac{V_\beta(\theta)}{r^2} u^2 r\, dr\, d\theta = c_\beta \int_{\Lambda_\beta} \frac{u^2}{d^2} dx\end{aligned}$$

hence $H(\Lambda_\beta) \geq c_\beta$. For the reverse inequality, let $g \in C_c^\infty(0,\beta)$. For $\epsilon > 0$ set

$$h_\epsilon(r) = \begin{cases} r^\epsilon, & 0 < r < 1, \\ r^{-\epsilon}, & r > 1. \end{cases}$$

The function $u_\epsilon(x) = h_\epsilon(r) g(\theta)$ then belongs to $H_0^1(\Lambda_\beta)$. So

$$H(\Lambda_\beta) \leq \frac{\displaystyle\int_{\Lambda_\beta} |\nabla u_\epsilon|^2 dx}{\displaystyle\int_{\Lambda_\beta} \frac{u_\epsilon^2}{d^2} dx} = \frac{\displaystyle\int_0^\beta g'(\theta)^2 d\theta}{\displaystyle\int_0^\beta g(\theta)^2 V_\beta(\theta) d\theta} + \epsilon^2 \frac{\displaystyle\int_0^\beta g(\theta)^2 d\theta}{\displaystyle\int_0^\beta g(\theta)^2 V_\beta(\theta) d\theta}$$

Letting $\epsilon \to 0+$ we obtain

$$H(\Lambda_\beta) \leq \frac{\displaystyle\int_0^\beta g'(\theta)^2 d\theta}{\displaystyle\int_0^\beta g(\theta)^2 V_\beta(\theta) d\theta}.$$

Hence $H(\Lambda_\beta) \leq c_\beta$. $\square$

Change variables to pass from $(0, \beta)$ to $(0, \pi)$,

$$c_\beta = \inf \frac{\displaystyle\int_0^\pi g'(\theta)^2 d\theta}{\displaystyle\int_0^\pi g(\theta)^2 V_\beta(\beta\theta/\pi) d\theta}$$

so

$$\frac{1}{c_\beta} = \sup \frac{\displaystyle\int_0^\pi g(\theta)^2 V_\beta(\beta\theta/\pi) d\theta}{\displaystyle\int_0^\pi g'(\theta)^2 d\theta}.$$

We can easily see the following:

- The function $\beta \mapsto V_\beta(\beta\theta/\pi)$ is convex in $(0, 2\pi)$; hence $\beta \mapsto c_\beta^{-1}$ is convex
- $c_4 = \frac{1}{4}$ (the function $\phi(\theta) = \theta^{\frac{1}{2}}(4 - \theta)^{\frac{1}{2}}$ is a positive supersolution)
- $c_{2\pi} < \frac{1}{4}$ (test with $g(\theta) = \sin(\frac{\theta}{2})$)

Hence there exists a critical angle $\beta_{cr} \in (4, 2\pi)$ such that

$$c_\beta = \begin{cases} \frac{1}{4}, & \text{for } 0 < \beta \leq \beta_{cr}, \\ \text{decreases monotonically}, & \text{for } \beta_{cr} \leq \beta \leq 2\pi\,. \end{cases}$$

Numerical computations give $\beta_{cr} \simeq 1.546\pi$, $c_{2\pi} \simeq 0.205$

More work is required in order to obtain a better understanding of the constant c_β.

Fix $\beta \in (0, 2\pi)$ and for $c > 0$ consider the boundary value problem

$$(*) \qquad \begin{cases} \psi''(\theta) + cV_\beta(\theta)\psi(\theta) = 0, & 0 \le \theta \le \beta, \\ \psi(0) = \psi(\beta) = 0, \end{cases}$$

where, we recall,

$$V_\beta(\theta) = \begin{cases} \frac{1}{\sin^2\theta}, & 0 < \theta < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < \theta < \beta - \frac{\pi}{2}, \\ \frac{1}{\sin^2(\beta-\theta)}, & \beta - \frac{\pi}{2} < \theta < \beta. \end{cases}$$

The Hardy constant c_β of Λ_β is the largest constant c for which the boundary value problem $(*)$ has a positive solution.

In the inteval $(0, \pi/2)$ the general solution of the ODE can be expressed in terms of hypergeometric functions. In case $0 < c < \frac{1}{4}$ we have

$$
\begin{aligned}
y(\theta) &= c_1 \sin^{\alpha}(\frac{\theta}{2}) \cos^{1-\alpha}(\frac{\theta}{2}) F(\frac{1}{2}, \frac{1}{2}, \alpha + \frac{1}{2}; \sin^2(\frac{\theta}{2})) \\
&\quad + c_2 \sin^{1-\alpha}(\frac{\theta}{2}) \cos^{1-\alpha}(\frac{\theta}{2}) F(1-\alpha, 1-\alpha, \frac{3}{2} - \alpha; \sin^2(\frac{\theta}{2})).
\end{aligned}
$$

while for $c = \frac{1}{4}$ we have

$$
\begin{aligned}
y(\theta) &= c_1 \sin^{1/2}(\frac{\theta}{2}) \cos^{1/2}(\frac{\theta}{2}) F(\frac{1}{2}, \frac{1}{2}, 1; \sin^2(\frac{\theta}{2})) \\
&\quad + c_2 \sin^{1/2}(\frac{\theta}{2}) \cos^{1/2}(\frac{\theta}{2}) F(\frac{1}{2}, \frac{1}{2}, 1; \sin^2(\frac{\theta}{2})) \int_{\sin^2(\theta/2)}^{1/2} \frac{dt}{t(1-t)F^2(\frac{1}{2}, \frac{1}{2}, 1; t)}.
\end{aligned}
$$

From the above one can analyze the boundary value problem $(*)$.

Theorem. The critical angle β_{cr} is the unique solution in the interval $(\pi, 2\pi)$ of the equation

$$\tan\left(\frac{\beta_{cr}-\pi}{4}\right) = 4\left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right)^2.$$

Moreover for any $\beta \in (0, 2\pi)$ the Hardy constant of the sector Λ_β is given by

$$H(\Lambda_\beta) = \begin{cases} \frac{1}{4}, & \text{if } \pi < \beta < \beta_{cr}, \\ \text{the unique solution of } (**), & \text{if } \beta_{cr} \leq \beta \leq 2\pi. \end{cases}$$

$$(**) \qquad \sqrt{c_\beta}\tan\left(\sqrt{c_\beta}\left(\frac{\beta-\pi}{2}\right)\right) = 2\left(\frac{\Gamma\left(\frac{3+\sqrt{1-4c_\beta}}{4}\right)}{\Gamma\left(\frac{1+\sqrt{1-4c_\beta}}{4}\right)}\right)^2$$

Note. Slightly modifying the above argument we can see that the bounded sector $\Lambda_\beta \cap D(1)$ also has Hardy constant c_β.

Q: What about other bounded domains with Hardy constant smaller than $1/4$? What about domains where there is no symmetry ?

A comment on the positive supersolution method.

"If the Hardy quotient admits a minimizer u_0, then to apply the positive supersolution method we need to now exactly the minimizer."

Proposition. If the Hardy quotient admits a minimizer u_0 then any positive supersolution is a scalar multiple of u_0.

Proof. Let ϕ be a positive supersolution and $(u_n) \subset C_c^\infty(\Omega)$ be such that $u_n \to u_0$ in $H_0^1(\Omega)$. Then

$$0 \leq \int_\Omega |\nabla u_n - \frac{\nabla\phi}{\phi} u_n|^2 dx \leq \int_\Omega |\nabla u_n|^2 dx - H(\Omega) \int_\Omega \frac{u_n^2}{d^2} dx.$$

Hence

$$\begin{aligned} 0 &\leq \int_\Omega |\nabla u_0 - \frac{\nabla\phi}{\phi} u_0|^2 dx = \int_\Omega \lim |\nabla u_n - \frac{\nabla\phi}{\phi} u_n|^2 dx \\ &\leq \liminf \int_\Omega |\nabla u_n - \frac{\nabla\phi}{\phi} u_n|^2 dx \leq \liminf \Big(\int_\Omega |\nabla u_n|^2 dx - H(\Omega) \int_\Omega \frac{u_n^2}{d^2} dx \Big) \\ &= \int_\Omega |\nabla u_0|^2 dx - H(\Omega) \int_\Omega \frac{u_0^2}{d^2} dx \\ &= 0. \end{aligned}$$

It follows that $\phi = cu_0$ for some $c \in \mathbb{R}$. □

The Hardy constant of a quadrilateral.

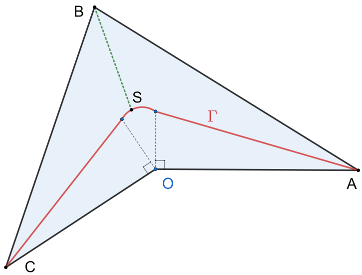
Theorem. Let Q be a non-convex quadrilateral with non- convex angle $\beta \in (\pi, 2\pi)$. Then $H(Q) = c_\beta$.

Proof. Set-up:

- Denote by α the largest solution of the equation $\alpha(1-\alpha) = c_\beta$
- Denote by $\psi(\theta)$ the positive solution of the boundary value problem

$$\begin{cases} \psi''(\theta) + c_\beta V_\beta(\theta)\psi(\theta) = 0, \quad 0 \le \theta \le \beta, \\ \psi(0) = \psi(\beta) = 0. \end{cases}$$

- Assume that the non-convex vertex lies at the origin. Let A, B, C denote the other three vertices, with A lying on the positive x-semiaxis.



Divide Q into two parts Ω_- and Ω_+ by means of the equidistance curve

$$\Gamma = \{(x, y) \in Q : \mathrm{dist}((x, y), OA \cup O\Gamma) = \mathrm{dist}((x, y), AB \cup B\Gamma)\}$$

Denote by $\vec{n}$ the unit normal vector along Γ which is exterior with respect to Ω_-.

Apply the positive supersolution method separately on each subdomain $\Omega_\pm$.

- On Ω_- choose the function $\phi = \psi(\theta)$ (it is a solution)
- On Ω_+ choose $\phi = d^\alpha$ Note that, since $\alpha(1-\alpha) = c_\beta$, we have

$$\Delta d^\alpha + \frac{c_\beta}{d^2} d^\alpha = \mathrm{div}(\alpha d^{\alpha-1} \nabla d) + c_\beta d^{\alpha-2} = \alpha d^{\alpha-1} \Delta d \leq 0, \quad \text{in } \Omega_+$$

We obtain that for any $u \in C_c^\infty(\Omega)$,

$$\int_{\Omega_-} |\nabla u|^2 dx \geq c_\beta \int_{\Omega_-} \frac{u^2}{d^2} dx + \int_\Gamma \frac{\nabla \psi}{\psi} \cdot \vec{n}\, u^2 dS$$

$$\int_{\Omega_+} |\nabla u|^2 dx \geq c_\beta \int_{\Omega_+} \frac{u^2}{d^2} dx - \alpha \int_\Gamma \frac{\nabla d}{d} \cdot \vec{n}\, u^2 dS$$

Adding we obtain

$$\int_\Omega |\nabla u|^2 dx \geq c_\beta \int_\Omega \frac{u^2}{d^2} dx + \int_\Gamma \Big(\frac{\nabla \psi}{\psi} - \alpha \frac{\nabla d}{d}\Big) \cdot \vec{n}\, u^2 dS$$

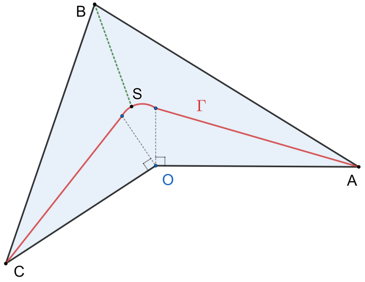
Hence what is required is to prove that

$$\Big(\frac{\nabla \psi}{\psi} - \alpha \frac{\nabla d}{d}\Big) \cdot \vec{n} \geq 0\,, \qquad \text{along } \Gamma.$$

$\longrightarrow$ one-dimensional inequalities, parametrized by $\theta \in (0, \beta)$

Everything can be written explicitly except for $\psi(\theta)$. Also, different types of quadrilaterals must be distinguished.

Consider this quadrilateral:



Starting from the point A (i.e. $\theta = 0$) the curve Γ has the form L-P-P-L. These four segment correspond to angles θ as follows:

Γ_1	:	$0 \leq \theta \leq \frac{\pi}{2}$	(line segment)
Γ_2	:	$\frac{\pi}{2} \leq \theta \leq \theta_0$	(parabola segment)
Γ_3	:	$\theta_0 \leq \theta \leq \beta - \frac{\pi}{2}$	(parabola segment)
Γ_4	:	$\beta - \frac{\pi}{2} \leq \theta \leq \beta$	(line segment)

For the line segment Γ_1 we need

$$\sin\theta\,\cos(\theta+\frac{\gamma}{2})\frac{\psi'(\theta)}{\psi(\theta)}+\alpha\,\cos\frac{\gamma}{2}\ \geq 0\,, \qquad 0\leq\theta\leq\frac{\pi}{2}$$

Indeed, letting $f(\theta)=\frac{\psi'(\theta)}{\psi(\theta)}$ we have

Lemma. Let $0\leq\omega\leq\pi/4$. Then

$$f(\theta)\,\sin\theta\,\cos(\theta+\omega)+\alpha\cos\omega\geq 0\,, \qquad 0\leq\theta\leq\frac{\pi}{2}.$$

This is applied for $\omega=\frac{\gamma}{2}$, where γ is the angle at the vertex A.

For the parabola segment Γ_2 we need

Lemma. Let $3\pi/2-\beta\leq\omega\leq 2\pi-\beta$. Then

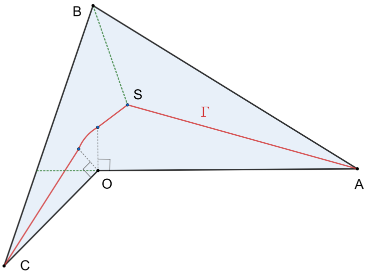
$$f(\theta)\cos(\theta+\omega)+\alpha\big(1+\sin(\theta+\omega)\big)\geq 0\,, \quad \frac{\pi}{2}\leq\theta\leq\beta-\frac{\pi}{2}$$

This is applied for $\omega=\gamma$.

Both proofs use the fact that the function $f(\theta)$ solves

$$f'+f^2+c_\beta V_\beta=0\,, \qquad f(0)=\alpha\,.$$

Consider this quadrilateral:

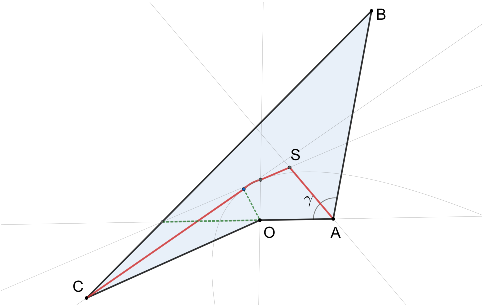


Now the curve Γ has the form L-L-P-L. These four segments correspond to angles θ as follows:

$$\begin{array}{lll} \Gamma_1 : & 0 \leq \theta \leq \theta_0 & \text{(line segment)} \\ \Gamma_2 : & \theta_0 \leq \theta \leq \frac{\pi}{2} & \text{(line segment)} \\ \Gamma_3 : & \frac{\pi}{2} \leq \theta \leq \beta - \frac{\pi}{2} & \text{(parabola segment)} \\ \Gamma_4 : & \beta - \frac{\pi}{2} \leq \theta \leq \beta & \text{(line segment)} \end{array}$$

The segment Γ_2 is different from before.

Consider the following quadrilateral: now $\gamma > \pi/2$. The curve Γ has the form L-L-P-L.



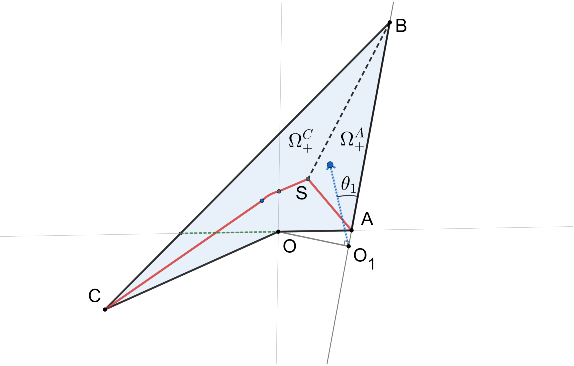
Note that in this case $\beta < 3\pi/2 < \beta_{cr}$ and hence $c_\beta = \frac{1}{4}$.

The previous argmument does not work: the required inequality is not valid on Γ_1 (the segment AS).

An alternative approach is required.

Use the positive supersolution method as follows:

Consider a second coordinate system (x_1, y_1) with origin $O_1 = \mathrm{proj}_{AB}(O)$ and the positive x_1 semiaxis containing the side AB. Denote by θ_1 the polar angle in this new system. Divide Ω_+ in two parts, Ω_+^A and Ω_+^C with the bisector at B.



- On Ω_- use the solution $\phi = \psi(\theta)$ (as before)
- On Ω_+^A use the solution $\phi = \psi(\theta_1)$
- On Ω_+^C use the supersolution $\phi = d^{\frac{1}{2}}$

For the segment AS we need the following: define $g(\theta) = \frac{\psi'(\theta)}{\psi(\theta)} \sin\theta$.

Lemma. Let $\frac{\pi}{2} \leq \gamma \leq \pi$. For $\theta \in (0, \frac{\pi}{2})$ let $\theta_1 = \theta_1(\theta)$ be the angle determined by

$$\cot\theta_1 = -\cos\gamma \cot\theta + \sin\gamma$$

Then there holds

$$g(\theta)\cos(\theta + \frac{\gamma}{2}) + g(\theta_1)\cos(\theta_1 - \frac{\gamma}{2}) \geq 0\,, \qquad 0 \leq \theta \leq \frac{\pi}{2}.$$

But we may also have $\gamma > \frac{\pi}{2}$ and Γ be of the form L-P-P-L. In this case the construction is the same and for the (first) parabola we need

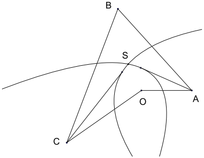
Lemma. Let $\pi \leq \beta \leq 2\pi$ and $\frac{\pi}{2} \leq \gamma \leq \pi$. For $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} - \gamma$ denote by $\theta_1 = \theta_1(\theta)$ the angle in $[0, \pi/2]$ uniquely determined by

$$\cot\theta_1 = -\cos(\theta + \gamma)$$

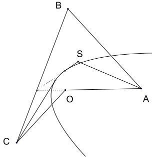
Then

$$f(\theta_1) \leq f(\theta)\frac{1 + \cos^2(\theta + \gamma)}{2 + \sin(\theta + \gamma)}\,, \qquad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} - \gamma\,.$$

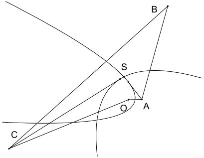
In all, five types of quadrilaterals must be considered:



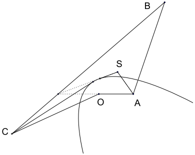
Type A1 (L-P-P-L)



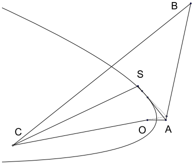
Type A2 (L-L-P-L)



Type B1 (L-P-P-L)



Type B2 (L-L-P-L)



Type B3 (L-P-L-L)

Lemma. Let $0 \le \omega \le \pi/4$. Then

$$f(\theta)\,\sin\theta\,\cos(\theta+\omega) + \alpha\cos\omega \ge 0\,, \qquad 0 \le \theta \le \frac{\pi}{2}.$$

Proof. The inequality is trivially true for $0 \le \theta \le \frac{\pi}{2} - \omega$, so we restrict our attention to the interval $\frac{\pi}{2} - \omega \le \theta \le \frac{\pi}{2}$. We must prove that

$$f(\theta) \le q(\theta)\,, \qquad \frac{\pi}{2} - \omega \le \theta \le \frac{\pi}{2},$$

where

$$q(\theta) = -\alpha\frac{\cos\omega}{\sin\theta\cos(\theta+\omega)}.$$

We have $\sqrt{c_\beta} \le \alpha$, hence

$$\begin{aligned}
q(\frac{\pi}{2}) - f(\frac{\pi}{2}) &= \alpha\cot\omega - \sqrt{c_\beta}\tan[\sqrt{c_\beta}(\frac{\beta}{2} - \frac{\pi}{2})] \\
&\ge \alpha\Big\{\cot\omega - \tan[\sqrt{c_\beta}(\frac{\beta}{2} - \frac{\pi}{2})]\Big\} \\
&= \frac{\alpha}{\sin\omega\cos[\sqrt{c_\beta}(\frac{\beta}{2} - \frac{\pi}{2})]}\cos\Big(\sqrt{c_\beta}(\frac{\beta}{2} - \frac{\pi}{2}) + \omega\Big) \\
&\ge 0,
\end{aligned}$$

since $0 < \sqrt{c_\beta}(\frac{\beta}{2} - \frac{\pi}{2}) + \omega \le \frac{\beta}{4} - \frac{\pi}{4} + \omega \le \pi/2$.

In view of the above, it is enough to establish that

$$q'(\theta) + q(\theta)^2 + \frac{c_\beta}{\sin^2\theta} \leq 0\,, \qquad \text{for } \frac{\pi}{2} - \omega \leq \theta \leq \frac{\pi}{2}.$$

Indeed, for $\theta \in [\pi/2 - \omega, \pi/2]$ we find

$$\begin{aligned} & q'(\theta) + q^2(\theta) + \frac{c_\beta}{\sin^2\theta} \\ = \quad & \alpha\frac{2\cos\omega\cos\theta\cos(\theta+\omega) - (1-\alpha)\sin\theta\sin(\theta+2\omega)}{\sin^2\theta\cos^2(\theta+\omega)} \\ \leq \quad & 0, \end{aligned}$$

since

$$\frac{\pi}{2} \leq \theta + \omega \leq \frac{\pi}{2} + \omega \leq \frac{3\pi}{4}$$

and

$$\frac{\pi}{2} \leq \theta + 2\omega \leq \frac{\pi}{2} + 2\frac{\pi}{2} = \pi\,.$$

□

Exercise. Consider the initial value problem

$$\begin{cases} g' = -\frac{1}{\sin\theta}\Big(g^2 - \cos\theta\, g + \alpha(1-\alpha)\Big) \\ g(0) = \alpha \end{cases}$$

Prove that

(i) For $\frac{1}{2} < \alpha < 1$ there there exists a unique solution

(ii) For $\alpha = \frac{1}{2}$ there exists a continuum of solutions

Moreover, any solution is strictly decreasing in $(0, \pi/2)$.

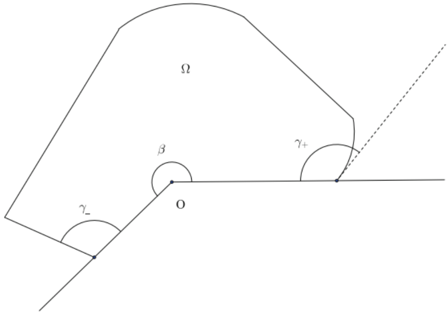
Beyond the quadrilateral...

Are there other planar domains for which these ideas can be applied ?

- More general polygons with one non-convex angle
- Polygons with two or more non-convex angles
- Domains that are not polygons

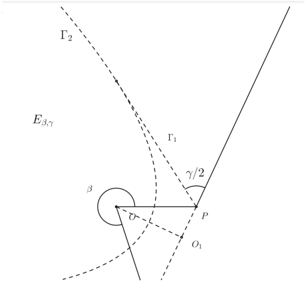
Note. If a planar domain Ω has an angle of size $\beta > \pi$ then $H(\Omega) \leq c_\beta$.

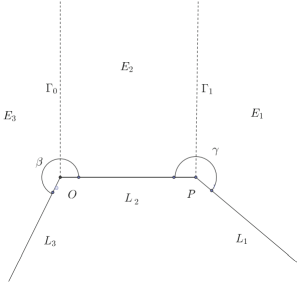
Theorem. Let $\Omega = K \cap \Lambda_\beta$, $\beta \in (\pi, 2\pi]$, where K is a bounded convex planar domain containing the origin. Let γ_+ and γ_- denote the interior angles of intersection of K with Λ_β.



There exists an angle $\gamma_\beta \in (\pi/2, \pi)$ such that if $\gamma_+, \gamma_- \leq \gamma_\beta$, then the Hardy constant of Ω is c_β.

Theorem. Consider the domain $E_{\beta,\gamma}$ (diagram) where $\pi < \beta < 2\pi$, $0 < \gamma < 2\pi$ and $\beta + \gamma \leq 3\pi$.





(i) If $\gamma \leq \pi$ then the Hardy constant of $E_{\beta,\gamma}$ is c_β.

(ii) Assume that $\gamma > \pi$. If in addition

$$|\beta - \gamma| \leq \frac{2}{\sqrt{c_{\beta+\gamma-\pi}}} \arccos(2\sqrt{c_{\beta+\gamma-\pi}}).$$

then the Hardy constant of $E_{\beta,\gamma}$ is $c_{\beta+\gamma-\pi}$.

Some open problems.

Problem 1. Prove more results about the Hardy constant of domains in $\mathbb{R}^2$.

For example:

- Find the Hardy constant of various simply connected domains (polygons, Koch snowflake...)
- Find a simply connected domain with Hardy constant smaller than $c_{2\pi}$
- ...
- ...

Find the best uniform Hardy constant valid over all simply connected domains $\Omega \subset \mathbb{R}^2$, that is the largest constant H^* such that

$$\int_\Omega |\nabla u|^2 dx \geq H^* \int_\Omega \frac{u^2}{d^2}\, dx$$

for all simply connected domains $\Omega \subset \mathbb{R}^2$ and for all $u \in C_c^\infty(\Omega)$.

We know that

$$c_{2\pi} \leq H^* \leq \frac{1}{4} \qquad (c_{2\pi} \simeq 0.205)$$

Problem 2. Davies' conjecture on the *weak Hardy constant*

Let $\Omega \subset \mathbb{R}^n$ be bounded. Assume that there exist $\alpha, \beta > 0$ such that

$$\int_\Omega |\nabla u|^2 dx \geq \alpha \int_\Omega \frac{u^2}{d^2}\, dx - \beta \int_\Omega u^2 dx\,, \quad u \in C_c^\infty(\Omega). \tag{1}$$

We define the *weak Hardy constant* of Ω as

$$H_w(\Omega) = \sup \left\{ \alpha > 0 \ : \ \text{there exists } \beta > 0 \text{ such that (1) is valid} \right\}$$

The weak Hardy constant depends only on the boundary regularity. To make this precise, for $y \in \partial\Omega$ and $r > 0$ define

$$\begin{aligned} H_w(y, r) \quad = \quad & \sup \left\{ \alpha > 0 \ : \ \text{there exists } \beta > 0 \text{ such that} \right. \\ & \left. \qquad\qquad \text{(1) is valid for all } u \in C_c^\infty(B_r(y)) \right\} \end{aligned}$$

and

$$H_w(y) = \sup_{r>0} H_w(y, r) = \lim_{r \to 0} H_w(y, r).$$

Theorem. Let $\Omega \subset \mathbb{R}^n$ be bounded. The function $y \mapsto H_w(y)$ is lower semicontinuous and

$$H_w(\Omega) = \min_{y \in \partial\Omega} H_w(y)$$

Conjecture. For any domain $\Omega \subset \mathbb{R}^n$ and any $y \in \partial\Omega$ there holds $H_w(y) \leq 1/4$.

Problem 3. Improved Hardy inequalities.

Let $\Omega \subset \mathbb{R}^n$ be a weakly mean convex domain. So

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2}\, dx\,, \qquad u \in C_c^\infty(\Omega)\,.$$

Q: Can this be improved?

Define

$$X(t) = \frac{1}{1 - \log t}, \qquad t \in (0,1).$$

Theorem. If Ω is weakly mean convex and $D := \sup_\Omega d < +\infty$, then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2}\, dx + \frac{1}{4} \int_\Omega \frac{u^2}{d^2} X^2(d/D)\, dx\,, \qquad u \in C_c^\infty(\Omega)\,.$$

The inequality is sharp.

Can this be additionally improved?

Theorem. If Ω is weakly mean convex and $D := \sup_\Omega d < +\infty$, then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,dx + \frac{1}{4}\int_\Omega \frac{u^2}{d^2} X^2(d/D)\,dx$$
$$+\frac{1}{4}\int_\Omega \frac{u^2}{d^2} X^2(d/D) X^2(X(d/D))\,dx\ , \qquad u \in C_c^\infty(\Omega)\,.$$

The inequality is sharp.

More generally, define

$$X_1(t) = X(t)\ , \quad X_{k+1}(t) = X_1(X_k(t))\ , \qquad t \in (0,1).$$

Theorem. Let Ω be weakly mean convex and $D := \sup_\Omega d < +\infty$, then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,dx + \frac{1}{4}\sum_{k=1}^{\infty}\int_\Omega \frac{u^2}{d^2} X_1^2(d/D) X_2^2(d/D)\ldots X_k^2(d/D)\,dx$$

for all $u \in C_c^\infty(\Omega)$. The inequality is sharp at each step.

Q: Can this inequality be improved?

Problem 4. Improved Hardy-Sobolev inequalities.

Hardy-Sobolev inequalities are inequalities of the form

$$\int_\Omega |\nabla u|^2 dx \geq H(\Omega) \int_\Omega \frac{u^2}{d^2}\, dx + c\Big(\int_\Omega |u|^q dx \Big)^{\frac{2}{q}}, \quad u \in C_c^\infty(\Omega),$$

or weighted variations of this.

Theorem. For any $n \geq 3$ there exists $c_n > 0$ such that for any convex domain $\Omega \subset \mathbb{R}^n$ there holds

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2}\, dx + c_n\Big(\int_\Omega |u|^{\frac{2n}{n-2}} dx \Big)^{\frac{n-2}{n}}, \quad u \in C_c^\infty(\Omega).$$

Q: Does this remain true for weakly mean convex domains?

Note. In case $\Omega = \mathbb{R}^3_+$ there holds

$$\int_{\mathbb{R}^3_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^3_+} \frac{u^2}{d^2}\, dx + S_3\Big(\int_{\mathbb{R}^3_+} |u|^6 dx \Big)^{\frac{1}{3}}, \quad u \in C_c^\infty(\mathbb{R}^3_+)$$

where S_3 is the Sobolev constant in three dimensions!

Problem 5. The best L^p Rellich constant.

Recall that the L^p Hardy inequality in one dimension reads

$$\int_0^1 |u'(t)|^p dt \geq \left(\frac{p-1}{p}\right)^p \int_0^1 \frac{|u|^p}{t^p} dt \ , \quad \text{for all } u \in C_c^\infty(0,1)$$

What about higher dimensions?

Theorem. If Ω is weakly mean convex then

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \ , \qquad u \in C_c^\infty(\Omega)$$

The computation of the L^p Hardy constant for other domains is more difficult.

The Rellich inequality.

The L^2 Rellich inequality in one dimension reads

$$\int_0^1 (u^{(m)})^2 dt \geq \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot \ldots \cdot (2m-1)^2}{4^m} \int_0^1 \frac{u^2}{t^{2m}} dt \ , \quad u \in C_c^\infty(0,1).$$

What about higher dimensions?

Theorem. If $\Omega \subset \mathbb{R}^n$ is convex then

$$\int_\Omega (\Delta^{m/2} u)^2 dx \geq \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot \ldots \cdot (2m-1)^2}{4^m} \int_\Omega \frac{u^2}{d^{2m}} dx \ , \qquad u \in C_c^\infty(\Omega)$$

Proof. The proof uses the mean distance function method. Given a direction $\omega \in S^{n-1}$ and applying the Rellich inequality in one dimension we obtain

$$\int_\Omega (\partial_\omega^m u)^2 dx \ge A(m) \int_\Omega \frac{u^2}{d_\omega^{2m}} dx \,.$$

Applying the Fourier transform this gives

$$\int_{\mathbb{R}^n} (\xi \cdot \omega)^{2m} |\hat{u}(\xi)|^2 d\xi \ge A(m) \int_\Omega \frac{u^2}{d_\omega^{2m}} dx \,.$$

Now average over all directions:

$$⨍_{S^{n-1}} \int_{\mathbb{R}^n} (\xi \cdot \omega)^{2m} |\hat{u}(\xi)|^2 dx \ge A(m) ⨍_{S^{n-1}} \int_\Omega \frac{u^2}{d_\omega^{2m}} dx \,.$$

But for any $p \in \mathbb{R}^n$, $⨍_{S^{n-1}} (p \cdot \omega)^{2m} dS(\omega) = c_{n,m} |p|^{2m}$, hence

$$c_{n,m} \int_{\mathbb{R}^n} |\xi|^{2m} |\hat{u}(\xi)|^2 d\xi \ge A(m) \int_\Omega \frac{u^2}{d_{av}^{2m}} dx$$

where

$$\frac{1}{d_{av}^{2m}(x)} = ⨍_{S^{n-1}} \frac{u^2}{d_\omega^{2m}(x)} dS(\omega) \,, \quad x \in \Omega \,.$$

By the convexity of Ω we obtain $d_{av}^{-2m}(x) \ge c_{n,m} d^{-2m}(x)$, in Ω. □

Q1: Is the above true when Ω is weakly mean convex?

Q2: What about the corrsponding L^p inequality?

In the case of an interior point singularity, the situation is very well understood.

Theorem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin. Assume that $mp < n$. There exists $D \geq \sup_\Omega |x|$ such that for any $u \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega |\Delta^{m/2} u|^p dx \geq A(m,p) \int_\Omega \frac{|u|^p}{|x|^{mp}} dx + B(m,p) \sum_{k=1}^{\infty} \int_\Omega \frac{|u|^p}{|x|^{mp}} X_1^2 X_2^2 \dots X_k^2 dx .$$

Here $X_i = X_i(|x|/D)$ and

$$A(m,p) = \prod_{i=0}^{[(m-1)/2]} \Big(\frac{n-(m-2i)p}{p}\Big)^p \times \prod_{j=1}^{[m/2]} \Big(\frac{np-n+(m-2j)p}{p}\Big)^p$$

and

$$B(m,p) = \frac{p-1}{2p} A(m,p) \left(\sum_{i=0}^{[(m-1)/2]} \Big(\frac{n-(m-2i)p}{p}\Big)^{-2} + \sum_{j=1}^{[m/2]} \Big(\frac{np-n+(m-2j)p}{p}\Big)^{-2} \right)$$

The inequality is sharp at each step.

But the L^p Rellich inequality is little understood in case $p \neq 2$ when we take the distance to the boundary.

For example, the best constant for the inequality

$$\int_{\mathbb{R}^n_+} |\Delta u|^p dx \geq C \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{2p}} dx \; , \qquad u \in C_c^\infty(\mathbb{R}^n_+)$$

is not known.

By local considerations we have

$$C \leq \frac{(p-1)^p(2p-1)^p}{p^{2p}}$$

but we do not have any nice lower bound.

THE END