Higher Order Estimates:

Some Results and Questions

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POURQUOI FAIRE DES MATHEMATIQUES ?

Parce que les mathématiques, ça sert à faire de la physique. La physique, ça sert à faire des frigidaires. Les frigidaires, ça sert à y mettre des langoustes, et les langoustes, ça sert aux mathématiciens, qui les mangent et sont alors dans de bonnes dispositions pour faire des mathématiques, qui servent à la physique, qui sert à faire des frigidaires qui ...

Laurent Schwartz

* What is mathematics helpful for ? Mathematics is helpful for physics. Physics helps us make fridges. Fridges are made to contain spiny lobsters, and spiny lobsters help mathematicians who eat them and have hence better abilities to do mathematics, which are helpful for physics, which helps us make fridges which...*

Anne-Sandrine Paumier, Laurent Schwartz (1915-2002) et la vie collective des mathématiques.
https://tel.archives-ouvertes.fr/tel-01087201/document

* Review of Hypercontractivity/Log Sobolev - Works with Bodineau - Works with Pierre (Cyril& Ivan) - U-bounds versus Lyapunov - No go thms with different metrics - Higher Order Inequalities * Martingale problem. Systems of SDE not in Hilbert space. * Coercive inequalities on groups (some groups) U-Bounds Laplacian bounds * Time dependent. Contractivities and phase transitions * Orlicz Spaces and Entropic Switch (Post Quantum Computing Coding) * Noncommutative Stochastic Analysis * Generalised Gradient Bounds. * Random walk on Toulousiens' groups. * NonMarkovian Cellular Automata and covid * Nonperiodic 3D game * Michel Ledoux & Mario Milman. * Solar mathematics * Optimisation problems : Ecological scandal

Content

*Tail Estimates and Orlicz Norms *.1

*Poincare Inequalities *.2

*Entropy Estimates *.3

*Time Non-hmogeneous Diffusions *.4

*Extended Gradient Bounds *.5

Quick Review of Probabilistic Tail Estimates

$$P(|X| \geq t) \leq \gamma(t)$$

Start

Nice Orlicz functions

$\Phi : \mathbb{R} \to \mathbb{R}$

$$\Phi(x) = \int_0^{|x|} \phi(s)ds$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\phi(0) = 0$ and otherwise $\phi > 0$, and $\phi(s) \to_{s\to\infty} \infty$.

Complementary Function

$$\Psi(y) = \int_0^{|y|} \psi(s)ds$$

where

$$\psi(t) = \inf\{s \geq 0 : \phi(s) \geq t\}$$

Young's Inequality

$$xy \leq \Phi(x) + \Psi(y)$$

Luxemburg norm

$$\|X\|_\Phi = \inf\{\lambda > 0 : E\Phi(X/\lambda) \leq 1\}.$$

Normalisation Property

$$E\Phi\left(\frac{X}{\|X\|_\Phi}\right) = 1 \qquad (\star)$$

Michał Lemańczyk, Concentration inequalities survey.
https://www.mimuw.edu.pl/~bpolaczyk/projects/resources/orlicz.pdf

Tail Estimates I.

If Φ is N-function and $0 < \|X\|_\Phi < \infty$, then

$$P(|X| \geq t) \leq \frac{1}{\Phi(t/\|X\|_\Phi)}$$

Skip MaxEstimates

Max Estimates $\max_{1\leq i\leq n}|X_i|$

Let Φ be a strictly increasing, non-zero function, s.t. $\Phi(1)\leq 1/2$ and

$$\exists c>0\,\forall x,y\geq 1,\qquad \Phi(x)\Phi(y)\leq\Phi(cxy).$$

Then for any $n\in\mathbb{N}$

$$\|\max_{1\leq i\leq n} X_i\|_\Phi \leq c\,\Phi^{-1}(2n)\max_{1\leq i\leq n}\|X_i\|_\Phi \qquad (*)$$

If Φ is a strictly increasing, convex, non-zero function, then for any $n\in\mathbb{N}$

$$E\left(\max_{1\leq i\leq n}|X_i|\right)\leq\Phi^{-1}(n)\max_{1\leq i\leq n}\|X_i\|_\Phi \qquad (**)$$

Skip Proof

Proof: The assumption says that for $x \geq y \geq 1$ we have

$$\Phi\left(\frac{x}{y}\right) \leq \frac{\Phi(cx)}{\Phi(y)}$$

Thus, for $y \geq 1$ and any $C > 0$ using the monotonicity of Φ

$$\max_{1\leq i\leq n} \Phi\left(\frac{|X_i|}{Cy}\right) \leq \Phi(1) + \sum_{i=1}^{n} \frac{\Phi(c|X_i|/C)}{\Phi(y)}$$

For $C = c \max_{1\leq i\leq n} \|X_i\|_\Phi$ we get

$$E\Phi\left(\frac{\max_{1\leq i\leq n} X_i}{Cy}\right) \leq E \max_{1\leq i\leq n} \Phi\left(\frac{X_i}{Cy}\right)$$
$$\leq \frac{n}{\Phi(y)} + \Phi(1) \leq \frac{n}{\Phi(y)} + \frac{1}{2} \leq 1$$

where the last inequality holds iff $2n \leq \Phi(y)$.

To get the second part of this lemma where Φ convex note that due to Jensen's inequality we have (w.l.o.g. $X_i \geq 0$)

$$\frac{1}{C} E \max_{1 \leq i \leq n} X_i \leq \Phi^{-1}(E \max_{1 \leq i \leq n} \frac{\Phi(X_i)}{C}) \leq \Phi^{-1}(n)$$

□

Exponential Luxemburg norms

DEFINITION :

For any $\alpha \in (0, \infty)$ we define

$$\Psi_\alpha(x) = \exp(|x|^\alpha) - 1$$

and the corresponding α-exponential Luxemburg norm as

$$\|X\|_{\Psi_\alpha} = \inf\{c > 0 : E \exp(|X|^\alpha / c^\alpha) \leq 2\}$$

Tail Estimates II

For any random variable X with

$$0 < \|X\|_{\Psi_\alpha} < \infty$$

and $t > 0$ we have

$$P(|X| \geq t) \leq 2\exp(-t^\alpha / \|X\|_{\Psi_\alpha}^\alpha)$$

LuxNorm

Observations

(1) If one replaces a Luxemburg norm by some convenient estimate from above, one still gets a tail estimates.

(2) If one replaces X by $X - EX$ we get concentration estimates.

(3) More generally, given a space $\mathcal{P}$ of R.V.s , one can study $X - w$, for $w \in \mathcal{P}$.

How to get bounds on Luxemburg Norms

Coercive Inequalities

Start

- Poincaré Inequality

$$mE(f - Ef)^2 \leq E|\nabla f|^2, \qquad (\mathrm{PI}_2)$$

with

$$E(f - Ef)^2 = \inf_{a \in \mathbb{R}} E(f - a)^2$$

H. Poincaré (Eqn (11) p. 253), « Sur les Equations aux Dérivées Partielles de la Physique Mathématique »,

Amer. J. of Math. vol. 12, no 3, 1890, pp. 211–294, doi10.2307/2369620

([BE'1985].., [Antoniuk Antoniuk'1993] ,..[V'06],[BCG'08],[HZ'09],..)

Start

Facts

◦ [HZ'09]

Let $dE = e^{-U} d\lambda$ with U locally bounded. Let $q \in [1,2]$.

If $\quad \exists 0 \leq \eta(x) \xrightarrow[dist(0,x)\to\infty]{} \infty$ (no matter how slowly)

and $\exists C, D \in (0,\infty)$

$$E(\eta|f|^q) \leq C\,E|\nabla f|^q + D\,E|f|^q, \qquad (\star)$$

then $\exists m \in (0,\infty)$ s.t.

$$m \inf_{a\in\mathbb{R}} E|f-a|^q \leq E|\nabla f|^q \qquad (\mathrm{IP}_q)$$

◦ If (IP_q) holds, then for every Lip function f with finite first moment there exists $\varepsilon > 0$ s.t.

$$E\,e^{\varepsilon|f|} < \infty$$

i.e. Lip functions (with finite 1^{st} moment) belong to Orlicz space with exp norm.

- Poincaré Inequality with a Weight

([BL'1976],..,[Antoniuk Antoniuk'1993],..,[HZ'09]...)

$$m\,E|f - Ef|^q \leq E\frac{|\nabla f|^q}{W}, \qquad q \geq 1 \qquad\qquad (\mathrm{IP}_{q,W})$$

Facts ∘ [HZ'09]

Let $dE = e^{-U} d\lambda$ with U locally bounded. Let $q \in [1,2]$.

If $\quad \exists 0 \leq \eta(x), \frac{|\nabla\eta|^q}{\eta^{q+\delta}} \xrightarrow[dist(0,x)\to\infty]{} \infty \qquad$ (with some $0 < \delta < 1$)

and $\exists C, D \in (0,\infty)$

$$E(\eta|f|^q) \leq C\,E|\nabla f|^q + D\,E|f|^q, \qquad (\star)$$

then $\exists m \in (0,\infty)$ s.t.

$$m \inf_{a\in\mathbb{R}} E|f-a|^q \leq E\frac{|\nabla f|^q}{W} \qquad (\mathrm{IP}_{q,W})$$

∘ If ($\mathrm{IP}_{q,W}$) holds, then for every Lip_W function f

i.e. $\|\frac{|\nabla f|^q}{W}\|_\infty < \infty$, (with $E|f| < \infty$), there exists $\varepsilon > 0$ s.t.

$$E\,e^{\varepsilon|f|} < \infty$$

i.e. Lip_W functions (with $E|f| < \infty$) belong to

Orlicz space with exp norm.

- Higher order Poincaré inequalities

$\exists m \in (0, \infty)$ s.t.

$$m_k \inf_{w \in \mathcal{P}_k} E|f - w|^q \leq E|\boldsymbol{\nabla}^k f| \qquad (\mathrm{IP}_{q,k})$$

where $k \in \mathbb{N}$ and $\mathcal{P}_k$ denotes space of polynomials of order k, and

$$|\boldsymbol{\nabla}^k f| \equiv \sum_{|\alpha|=k} |\nabla^\alpha f|^q$$

Facts

- [WZ'21] **Downhill Induction L_q-Case.**

$$\forall k \in \mathbb{N} \qquad (\mathrm{IP}_q) \implies (\mathrm{IP}_{q,k})$$

Theorem

For $q \in (1, \infty)$, there exists $C_{k,q} \in (0, \infty)$ and a polynomial $m_{k,q}(f)$ of order $k-1$, such that

$$\mu \left| f - m_{k,q}(f) \right|^q \leq C_{k,q} \sum_{|\alpha|=k} \mu |\nabla^\alpha f|^q. \qquad (PI_{k,q})$$

Y.Wang & BZ, Higher Order Coercive Inequalities, Potential Analysis (2021) https://link.springer.com/article/10.1007/s11118-021-09940-1

Assume that $(PI_{q,1})$ holds. Then

$$\sum_{|\alpha|=k} \mu|\nabla^\alpha f|^q = \sum_{|\beta|=k-1} \sum_j \mu|\nabla_j \nabla^\beta f|^q$$
$$\geq C_{1,q}^{-1} \sum_{|\beta|=k-1} \mu|\nabla^\beta f - M_{1,q}(\nabla^\beta f)|^q$$
$$= C_{1,q}^{-1} \sum_{|\beta|=k-1} \mu|\nabla^\beta \left(f - B_{1,q,k-1}(f)\right)|^q$$

with

$$B_{1,q,k-1}(f) \equiv \sum_{|\beta|=k-1} \frac{x^\beta}{\beta!} M_{1,q}(\nabla^\beta f)$$

Applied inductively, yields

$$\sum_{|\alpha|=k} \mu|\nabla^\alpha f|^q \geq C_{1,q}^{-(k-j)} \sum_{|\alpha|=k-j} \mu|\nabla^\alpha (f - B_{1,q,j-1}(f))|^q$$

with

$$B_{1,q,j-1}(f) \equiv \sum_{|\alpha|=k-j} \frac{x^\alpha}{\alpha!} M_{1,q}(\nabla^\alpha (f - B_{1,q,j}(f))).$$

Hence $(\mathrm{PI}_{k,q})$ follows $m_{k,q} \equiv B_{1,q,1}(f)$ and $C_{k,q} \leq C_{1,q}^k$. □

Remark The constant in the above is exponential in k and is generally not optimal and different then the optimal constant in the $(PI_{k,q})$ with the minimizing polynomial $M_{k,q}(f)$.

E. Bou Dagher, Y.Wang & BZ, Coercive inequalities on Carnot groups and applications, , Advances in Analysis and Geometry, Vol. 6 (2022)
https://doi.org/10.1515/9783110741711

Statistical Polynomials $m_{k,q}(f)$

Questions :

What is the structure & possible applications of $m_{k,q}(f)$?

Are they orthogonal for different values of k if $q = 2$?

Facts CND

- Similar techniques based on $\star$-bounds for $(\mathrm{IP}_{q,k})$ and

$$m \inf_{\mathcal{P}\in\mathcal{P}_{k-1}} E|f-\mathcal{P}|^q \leq E\left(\frac{|\nabla^k f|^q}{W}\right) \qquad (\mathrm{IP}_{q,k,W})$$

Facts CND

- $$\forall k \in \mathbb{N} \qquad (\mathrm{IP}_{q,k}) \implies \textit{ExpBounds}$$

Suppose $\|\frac{|\nabla^k f|^q}{W}\|_\infty < \infty$

$$\exists\, \varepsilon > 0 \qquad E e^{\varepsilon |f - m_{k,q}(f)|} < \infty$$

S. G. Bobkov, F. Gotze, H. Sambale, Higher Order Concentration of Measure, Commun. Contemporary Math., Vol. 21, No. 03 (2018)
https://doi.org/10.1142/S0219199718500438
Friedrich Götze and Holger Sambale, Higher Order Concentration in Presence of Poincaré-Type Inequalities, Ch. 6, in "High Dimensional Probability VIII, The Oaxaca Volume", Birkhäuser 2019, Eds. N.Gozlan, R.Latała, K.Lounici, M.Madiman,
https://doi.org/10.1007/978-3-030-26391-1

Multiparticle decay

Higher order estimates for Gaussian semigroup Let

$$L = \Delta - x \cdot \nabla \quad \text{and} \quad P_t = e^{tL}.$$

Then we have

Theorem

$$|\nabla^\alpha P_t f|^2 \leq e^{-2|\alpha|} P_t |\nabla^\alpha f|^2$$

Hence with the corresponding invariant Gaussian measure γ, we have

$$\int |\nabla^\alpha P_t f|^2 d\gamma \leq e^{-2|\alpha|} \int |\nabla^\alpha f|^2 d\gamma$$

Skip Proof/Jump to Optima

Proof In the case of interest to us it is well known that the kernel is smooth.
Following an idea of Bakry-Emery, we have

$$\partial_s P_{t-s}|\nabla^\alpha P_s f|^2 = P_{t-s}\left(-L|\nabla^\alpha P_s f|^2 + 2\nabla^\alpha L P_s f \cdot \nabla^\alpha P_s f\right)$$

and so

$$\partial_s P_{t-s}|\nabla^\alpha P_s f|^2$$
$$= P_{t-s}\left(\left(-L|\nabla^\alpha P_s f|^2 + 2\nabla^\alpha P_s f \cdot L\nabla^\alpha P_s f\right) + 2[\nabla^\alpha, L]P_s f \cdot \nabla^\alpha P_s f\right.$$

Since

$$[\nabla, L] = -\nabla,$$

using the following inductive formula

$$[\nabla^n, L] \equiv \nabla[\nabla^{n-1}, L] + [\nabla, L]\nabla^{n-1},$$

we have

$$[\nabla^n, L] = -n\nabla^n.$$

Hence for a component ∇^α, we get

$$[\nabla^\alpha, L] = -|\alpha|\nabla^\alpha.$$

Using the fact that the Markovian form of L satisfies

$$L(g^2) - 2gLg = 2|\nabla g|^2 \geq 0,$$

we obtain the following differential inequality

$$\partial_s P_{t-s}|\nabla^\alpha P_s f|^2 \leq -2|\alpha| P_{t-s}\left(|\nabla^\alpha P_s f|^2\right),$$

thereby concluding

$$|\nabla^\alpha P_t f|^2 \leq e^{-2|\alpha|} P_t |\nabla^\alpha f|^2$$

Accordingly, with the corresponding invariant Gaussian measure γ, we have

$$\int |\nabla^\alpha P_t f|^2 d\gamma \leq e^{-2|\alpha|} \int |\nabla^\alpha f|^2 d\gamma$$

$\square$

Optimal higher order Poincaré constants for O-U

Note that for Hermite polynomials H_k, $k \in \mathbb{N}$, in one dimensions, we have

$$\nabla H_k(t) = \sqrt{k} H_{k-1}(t).$$

and hence

$$\nabla^n H_k(t) = \sqrt{k \cdot .. \cdot (k-n)} H_{k-n}(t) \quad \text{for all} \quad k \geq n.$$

Thus, using representation

$$f = \sum_{k \in \mathbb{N}} f_k H_k$$

with respect to the O-N basis of Hermite polynomials, we compute

$$\int |\nabla^n f|^2 d\gamma = \sum_{k \geq n+1} k..(k-n)|f_{k-n}|^2 \geq n! \int |f - M_{2,n,\gamma}(f)|^2 d\gamma$$

where in $\mathbb{L}_2(\gamma)$, we have

$$M_{2,n,\gamma}(f) = \sum_{k \leq n} f_k H_h.$$

The optimal k-th order Poincare Inequality for O-U

Theorem

$$n! \int |f - M_{2,n,\gamma}(f)|^2 d\gamma \leq \int |\nabla^n f|^2 d\gamma.$$

Remark: This is better than Downhill Induction gives.

Entropy Estimates
and Unrestricted Exp Bounds

Start

Log-Sobolev Inequality

[Stamm'59], [Federbush'69][Gross'75],.., [JRosen'76], [Adams'79],..,[B-E'85]...

$$E\left(f^2 \log \frac{f^2}{Ef^2}\right) \leq c\, E|\nabla f|^2$$

$$\Updownarrow$$

$$\|(f - Ef)^2\|_\Phi \leq c' E|\nabla f|^2$$

with $\Phi(x) = |x| \log(1 + |x|)$, [BG'JFA99].

M. Ledoux, Logarithmic Sobolev Inequalities, *what they are, some history analytic, geometric, optimal transportation proofs, last decade developments, at the interface between analysis, probability, geometry*
https://www.math.univ-toulouse.fr/~ledoux/Logsobwpause.pdf

Entropy Bounds & Exponential Moments

If , for $q \in (1, \infty)$,

$$E\left(|f|^q \log \frac{|f|^q}{E|f|^q}\right) \leq c\, E|\nabla f|^q$$

then

$$Ee^{tf} \leq \exp\left\{const \cdot t^q \|\nabla f\|_\infty^q + Ef\right\}$$

Proof Idea

$$E\left(e^{tf}\log\frac{e^{tf}}{Ee^{tf}}\right) \leq \frac{c}{q^q}\,t^q\,E\left(|\nabla f|^q e^{tf}\right)$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\frac{1}{t}\log Ee^{tf}\right) \leq \frac{c}{q^q}\,t^{q-2}\,\|\nabla f\|_\infty^q$$

$$\Downarrow$$

$$Ee^{tf} \leq \exp\left\{\frac{c}{q^q(q-1)}\,t^q\,\|\nabla f\|_\infty^q + Ef\right\}$$

M. Ledoux, Remarks on logarithmic Sobolev constants, exponential integrability and bounds on the diameter, J. Math. Kyoto Univ. (JMKYAZ) 35-2 (1995) 211-220 https://perso.math.univ-toulouse.fr/ledoux/files/2019/10/Kyoto.pdf

S.G.Bobkov and B.Zegarlinski, Entropy bounds and isoperimetry, Mem.Amer.Math. Soc. 2005; Vol.176, Nr 829 http://www.ams.org/books/memo/0829

Generalised Entropy Bounds

Let

$$d\mu \equiv e^{-U} d\lambda$$

Adams Regularity Conditions:*[Adams'JFA1979]*

$\exists \varepsilon, C \in (0,\infty)$

$$\sum_{|\alpha|=2} |\nabla^\alpha U| \leq C(1+|\nabla U|)^{2-\varepsilon}. \qquad \text{(ARC)}$$

Remark :

- Note that in particular we have

$$LU \leq C(1+|\nabla U|)^{2-\varepsilon} - |\nabla U|^2.$$

where

$$Lf = \Delta f - \nabla U \cdot \nabla f$$

Smooth Orlicz functions

Let

$$\Phi_{A,p}(t) = |t|^p \prod_{j=1}^{n} (\log_j^*(|t|))^{p_j} \equiv |t|^p A(\log^* t),$$

and

$$\Phi(t) = t \prod_{j=1}^{n} (\log_j(\gamma_j + |t|))^{p_j} \equiv t\mathbf{\Theta}(t).$$

Then there exists $C \in (0, \infty)$ s.t.

$$\Phi(|x|^p) \leq C\Phi_{A,p}(|x|).$$

Lemma

Suppose the following Adams Inequality holds

$$\mu(\Phi_{A,p}(f)) \leq \tilde{C}_A \, \mu \left| \nabla^k f \right|^p + \tilde{D}_A \, \Phi_{A,p}(\|f\|_p^p) \qquad \text{(AI)}$$

with some $\tilde{C}_A, \tilde{D}_A \in (0, \infty)$ *independent of* f*. Then*

$$\| \, |f|^p \, \|_\Phi \leq C' \mu \left| \nabla^k f \right|^p + D' \|f\|_p^p. \qquad \text{(AOI)}$$

with some $C', D' \in (0, \infty)$ *independent of* f*.*

Theorem

If the following (p,k)-Poincaré inequality holds

$$\|(f - M_{p,k}(f))\|_p^p \leq c_{p,k}\, \mu \left|\nabla^k f\right|^p \qquad (\mathrm{PI}_{p,k})$$

with some $c_{p,k} \in (0,\infty)$ for all f for which the r.h.s. is well defined, then the following tight (Φ, p, k)-Inequality holds

$$\| \left|f - M_{p,k}(f)\right|^p \|_{\Phi} \leq C\, \mu \left|\nabla^k f\right|^p . \qquad (\mathrm{OSI})$$

•

Class $\mathfrak{N}$

If we have with $\alpha \in (1, \infty)$

$$p.p.N(u) = \frac{|u|^\alpha}{\alpha} \prod_{k=1}^{n} (\log_k |u|)^{\gamma_k}$$

then with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$p.p.N^*(u) = \frac{|u|^\beta}{\beta} \left(\prod_{k=1}^{n} (\log_k |u|)^{-\gamma_k} \right)^{\beta - 1}$$

Krasnosielski, M., Ruticki, B.: Convex functions and Orlicz spaces. Groningen: Noordhoff 1961. Chapter 1 §7

Questions:

- For $q = 2$, what are the optimal estimates

$$\|P_t(f - M_n(f))\|_2^2 \le e^{-m_n t}\|f - M_n(f)\|_2^2$$

- $$\|P_t(f - M_n(f))\|_\Phi \le e^{-\varepsilon_n t}\|f - M_n(f)\|_\Phi$$

- Description of Invariant subspaces and Martingales.
- Analysis on Nilpotent Lie Groups:

Kaplan versus Carnot-Caratheodory.

No Adams regularity for $U(d)$ with C-C distance d, ($|\nabla d| = 1$). No Log-Sobolev with Kaplan Distance$_{[HZ'09]}$:(and any smooth homogeneous norm),
Some (Log)$^\beta$-Sobolev $_{[EBZ'21-22]}$ and Poincaré Inequality are still OK:)$_{[In'10],[CFZ'21]}$

- E.Bou Dagher and B.Zegarlinski, Coercive Inequalities and U-Bounds on Step-Two Carnot Groups, Potential Analysis 2021 https://doi.org/10.1007/s11118-021-09979-0
- E.Bou Dagher, B.Zegarliński, Coercive inequalities in higher-dimensional anisotropic heisenberg group. Anal.Math.Phys. 12, 3 (2022). https://doi.org/10.1007/s13324-021-00609-x
- E. Bou Dagher and B. Zegarlinski,Coercive Inequalities on Carnot Groups: Taming Singularities, arXiv:2105.03922
- M. Chatzakou, Serena Federico and Boguslaw Zegarlinski, q-Poincaré inequalities on Carnot Groups, (2020) arXiv:2007.04689
- B. Zegarlinski, Crystallographic Groups of Hörmander Fields, Special Issue in Honour of Alexander Grigor'yan, Math. Phys. & Computer Simulations Vol 20 Nr.3 (2017) 43-64, https://doi.org/10.15688/mpcm.jvolsu.2017.3.4 ; (hal-01160736).
- James Devear Inglis, Coercive Inequalities for Generators of Hörmander Type, Thesis IC2010 http://www-sop.inria.fr/members/James.Inglis/Site/Publications_files/Inglis_THESIS_FINAL.pdf

Time Non Homogeneous Processes with Variable Smoothing Properties

Start

+CONTRACTIVITY ALONG ORLICZ SPACES

Let $(\mathbb{L}_{\Phi_t})_{t\in\mathbb{R}^+}$ be a nested family of Orlicz spaces.

∘ For a Markov semigroup $(P_t)_{t\in\mathbb{R}^+}$ defined in the given family of Orlicz spaces spaces, when do we have the following strong contractivity property ?

$$\|P_t f\|_{\Phi_t} \leq \|f\|_{\Phi_0}.$$

∘ Can we also consider variable family $(P_t^{(t)})_{t\in\mathbb{R}^+}$ semigroups

$$\|P_t^{(t)} f\|_{\Phi_t} \leq \|f\|_{\Phi_0}.$$

∘ C.Roberto and B.Zegarlinski, Hypercontractivity for Markov Semigroups, J. Funct. Analysis (2022) https://doi.org/10.1016/j.jfa.2022.109439

Theorem Let $(\Phi_t)_{t\geq 0}$ be a family of $\mathcal{C}^2$ Young functions. Assume that for some $t, s \geq 0$ there exist two positive constants $C(t,s)$ and $\widetilde{C}(t,s)$ such that
(i)

$$\dot{\Phi}_t(\Phi_t^{-1}) \leq C(t,s)\dot{\Phi}_s(\Phi_s^{-1}),$$

(ii)

$$\frac{\Phi_t''}{\Phi_t'^2} \circ \Phi_t^{-1} \geq \widetilde{C}(t,s)\frac{\Phi_s''}{\Phi_s'^2} \circ \Phi_s^{-1}.$$

Assume that for some $c \in (0,\infty)$ and for any f (smooth enough), it holds

$$\|f\|_{\Phi_s}^2 \int \dot{\Phi}_s\left(\frac{f}{\|f\|_{\Phi_s}}\right) d\mu \leq c \int \Phi_s''\left(\frac{f}{\|f\|_{\Phi_s}}\right) |\nabla f|^2 d\mu. \quad (1)$$

Then, for any f smooth enough it holds

$$\|f\|_{\Phi_t}^2 \int \dot{\Phi}_t\left(\frac{f}{\|f\|_{\Phi_t}}\right) d\mu \leq c\frac{C(t,s)}{\widetilde{C}(t,s)} \int \Phi_t''\left(\frac{f}{\|f\|_{\Phi_t}}\right) |\nabla f|^2 d\mu.$$

Proof

Let

$$\begin{aligned} I_{s,t} : \mathbb{L}_{\Phi_t} &\to \mathbb{L}_{\Phi_s} \\ f &\mapsto \|f\|_{\Phi_t} \Phi_s^{-1} \circ \Phi_t \left(\frac{f}{\|f\|_{\Phi_t}} \right). \end{aligned}$$

For any $f \in \mathbb{L}_{\Phi_t}$, by the very definition of the Luxembourg norm, it holds $\|I_{s,t}(f)\|_{\Phi_s} = \|f\|_{\Phi_t}$. Therefore, $I_{s,t}(f)$ is an isometry between the two Orlicz spaces $\mathbb{L}_{\Phi_t}$ and $\mathbb{L}_{\Phi_s}$. Applying (1) to $I_{s,t}(f)$ leads to

$$\begin{aligned} &\|f\|_{\Phi_t}^2 \int \dot{\Phi}_s \left(\Phi_s^{-1} \circ \Phi_t \left(\frac{f}{\|f\|_{\Phi_t}} \right) \right) d\mu \\ &\qquad \leq c \int \Phi_s'' \circ \Phi_s^{-1} \circ \Phi_t \left(\frac{f}{\|f\|_{\Phi_t}} \right) \frac{\Phi_t' \left(\frac{f}{\|f\|_{\Phi_t}} \right)^2 |\nabla f|^2}{\Phi_s' \circ \Phi_s^{-1} \circ \Phi_t \left(\frac{f}{\|f\|_{\Phi_t}} \right)^2} d\mu. \end{aligned}$$

The result follows by (i) and (ii).

The Standard Orlicz Family

Let $F:(0,\infty)\to\mathbb{R}$ be a $\mathcal{C}^2$ increasing function s.t. $F(1)=0$. Assume that $(0,\infty)\ni x\mapsto xF(x)$ is convex and that $1/xF(x)$ is not integrable at $x=0$, $x=1$ and $x=+\infty$. Let $\mathcal{F}_1:(0,1)\to\mathbb{R}$ and $\mathcal{F}_2:(1,+\infty)\to\mathbb{R}$ be two primitives of $x\mapsto 1/(xF(x))$. Let Φ_0 be a N-Young function of class $\mathcal{C}^2$ on $(0,\infty)$, and x_o the unique positive point s.t. $\Phi_0(x_o)=1$.

Assume $-\left(\frac{\Phi_0}{\Phi_0'}\right)' F(\Phi_0)-\Phi_0F'(\Phi_0)$ is non-increasing on $\mathbb{R}^+$.

Let λ: $\mathbb{R}^+\to\mathbb{R}^+$ be an increasing function, s.t. $\lambda(0)=0$.

Definition The *standard Orlicz family* $(\Phi_t)_{t\geq 0}$ *built from* F*,* Φ_0 *and* λ is defined by

$$\Phi_t(x):=\begin{cases}0 & \text{for } x=0\\ \mathcal{F}_1^{-1}\left(\mathcal{F}_1(\Phi_0(x))+\lambda(t)\right) & \text{for } x\in(0,x_o)\\ 1 & \text{for } x=x_o\\ \mathcal{F}_2^{-1}\left(\mathcal{F}_2(\Phi_0(x))+\lambda(t)\right) & \text{for } x\in(x_o,+\infty).\end{cases}\qquad \forall t>0$$

Examples

Example [G'75] For $F(x) = \log(x)$ and any N-function Φ_0 ,
$\mathcal{F}_1(x) = \log(\log(1/x))$, $x \in (0,1)$ and $\mathcal{F}_2(x) = \log(\log(x))$,
$x > 1$
so that $\mathcal{F}_1^{-1}(x) = e^{-e^x}$ and $\mathcal{F}_2^{-1}(x) = e^{e^x}$, $x \in \mathbb{R}$.
Hence, $\Phi_t(x) = \Phi_0^{e^{\lambda(t)}}$.
If $q(t) = 1 + e^{(4/\rho)t}$ and $\lambda(t) = \log(q(t)/2)$, with $\Phi_0(x) = x^2$,
we have $\Phi_t(x) = |x|^{q(t)}$, we get $\mathbb{L}_p$ scale of *Gross'* setting.

Example [BCR'] For $F(x) = \log(1+x)^\beta - \log(2)^\beta$, $\beta \in (0,1)$,
one gets non explicit $\mathcal{F}_1$ and $\mathcal{F}_2$
with an asymptotic of the corresponding $\Phi_t(x)$, when x tends
to ∞ or $+\infty$ equivalent to $\Phi_0 e^{a_\beta \lambda (\log \phi_0)^\beta}$,
with a numerical constant a_β depend only on β.
This is the family of Young functions $x^2 e^{ctF(x)}$ considered in
Barthe-Cattiaux-Roberto.

Integration Lemma for standard Orlicz family

Theorem

Let $(\Phi_t)_{t\geq 0}$ be a standard Orlicz family built from F, Φ_0 and λ. Let $c>0$. Then the following are equivalent

(i)

$$\|f\|_{\Phi_0}^2 \int \Phi_0\Big(\frac{f}{\|f\|_{\Phi_0}}\Big) F\left(\Phi_0\Big(\frac{f}{\|f\|_{\Phi_0}}\Big)\right) d\mu \leq c \int \Phi_0''\left(\frac{f}{\|f\|_{\Phi_0}}\right) |\nabla f|^2 d\mu$$

for any function f for which the r.h.s. is well defined;

(ii) $\forall\, t \geq s \geq 0$, it holds

$$\|P_t f\|_{\Phi_t} \leq \|P_s f\|_{\Phi_s}.$$

for any function $f \in \mathbb{L}_{\Phi_s}$.

Moreover $(i) \Rightarrow (ii)$ with any (increasing) λ such that $\frac{\Phi_t''}{\Phi_t'^2} \circ \Phi_t^{-1} \geq c\lambda'(t) \frac{\Phi_0''}{\Phi_0'^2} \circ \Phi_0^{-1}$ for any $t \geq 0$ (in particular, any λ satisfying $\lambda'(t) \leq 1/c$ would do); and $(ii) \Rightarrow (i)$ with $c = 1/\lambda'(0)$.

Hyperboundedness in $\mathbb{L}_p$-scales

Let

$$L_t := \Delta - \nabla V_t \cdot \nabla$$

$t \geq 0$, on $\mathbb{R}^n$, with V_t smooth (tbs) and s.t. $\int e^{-V_t} = 1$.
The associated semi-group $(P_s^{(t)})_{s\geq 0}$ is reversible with respect to the probability measure

$$\mu_t(dx) := e^{-V_t(x)} dx$$

Define

$$a_t := \|(\dot{V}_t)_-\|_\infty, b_t := \||\nabla \dot{V}_t|\|_\infty, c_t := \|(\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t)_-\|_\infty$$

Theorem

Consider the inhomogeneous diffusion operator L_t *as above.*
Assume $\forall t \geq 0\ a_t, b_t, c_t < \infty$.
Suppose $\forall t \geq 0\ \exists \rho_t \in \mathbb{R}$ *s.t.* $\mathrm{Hess}(V_t) \geq \rho_t$.
Assume $\exists \bar{c}_t \in (0, \infty)$

$$\int f^2 \log(f^2) d\mu_t \leq \bar{c}_t \int |\nabla f|^2 d\mu_t,$$

for all f *with* $\int f^2 d\mu_t = 1$ *for which the r.h.s is well defined.*
Then, for any $p > 1$ *and* $0 \leq s \leq t < \infty$,

$$\|P_t^{(t)} f\|_{\Phi_t, \mu_t} \leq m(s,t) \|P_s^{(s)} f\|_{\Phi_s, \mu_s}$$

where $\Phi_t(x) = |x|^{q(t)}$, $q(t) = 1 + (p-1)\exp\{\int_0^t (2/\bar{c}_s) ds\}$,
and

$$m(s,t) := \exp\left\{ \int_s^t \frac{a_u}{q(u)} + u c_u + b_u^2 \frac{1 - e^{-\rho_u u}}{2\rho_u} (q(u) - 1) du \right\}.$$

Example

$$V_t(x) = U(x) + \alpha(t)V(x) + \gamma(t)$$

with V unbounded and $\gamma(t) := \log \int e^{-U-\alpha V} dx$ so that μ_t is a probability measure.
Let $U(x) = \frac{|x|^2}{2}$, $\alpha: \mathbb{R}^+ \to \mathbb{R}^+$ non-decreasing and $V(x) = (1+|x|^2)^{\frac{\beta}{2}}$, with $\beta \in (0,1]$.
Then,

$$\dot{V}_t = \alpha'(t)(1+|x|^2)^{\frac{\beta}{2}}$$

so that $a_t = 0$; $\nabla \dot{V}_t = \alpha'(t)\beta(1+|x|^2)^{\frac{\beta}{2}-1}x$, whence $b_t = \alpha'(t)\beta\sqrt{\frac{(1-\beta)^{1-\beta}}{(2-\beta)^{2-\beta}}}$ (and when $\beta \to 1$, we get $b_t = \alpha'(t)$.
One can prove that $c_t \leq n^2\alpha'(t)(\alpha(t)+2)$ and $\mathrm{Hess}(V_t) \geq 1$ so that $\rho_t = 1$ and $\bar{c}_t = 2$. Hence $(P_t^{(t)})_{t\geq 0}$ is hyper-bounded in the $\mathbb{L}_{q(t)}$ scale, with $q(t) = 1+(p-1)e^t$.

General result

Let $V_t(x) = U(x) + \alpha(t)V(x) + \gamma(t)$.
Let $L_t = \Delta - \nabla V_t \cdot \nabla$.
Theorem Assume $\forall t \geq 0$, $b_t := |||\nabla \dot{V}_t|||_\infty < \infty$ and $\exists \rho_t \in \mathbb{R}$ s.t. $\mathrm{Hess}(V_t) \geq \rho_t I$.
Let $(\Phi_t)_{t \geq 0}$ be a family of N-functions s.t.
$\Phi_t(x) \leq x\Phi_t'(x) \leq B_t\Phi_t(x)$, $\Phi_t'^2 \leq C_t\Phi_t\Phi_t''$ and $x^2\Phi_t''(x) \leq D_t\Phi_t(x) + E_t$ for all $x \geq 0$ and some constants B_t, C_t, D_t, E_t.
Assume $\forall t \geq 0$ $\exists \delta_t \in [0,1)$ and $F_t \in \mathbb{R}$ such that
$(\dot{V}_t)_- \leq \frac{\delta_t}{4C_t}\left(|\nabla V_t|^2 - 2\Delta V_t\right) + F_t$.

Set $W_t := (\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t)_-$ and denote by $\bar{\rho}_t \in (0,\infty]$ the best constant such that for all f with $\|f\|_{\Phi_0} = 1$ it holds

$$\int \dot{\Phi}_t(f) d\mu_t \leq \bar{\rho}_t \int \Phi_t''(f) |\nabla f|^2 d\mu_t. \tag{2}$$

Finally, assume either that (i) $c_t := \|W_t\|_\infty < \infty$ and $\bar{\rho}_t < 1 - \delta_t$;or (ii)
$c_t' := \max\left(2\||\nabla W_t|\|_\infty / b_t, \sup_{x:W_t(x)\neq 0}\left(\frac{L_t W_t}{W_t} - \rho_t\right)_-\right) < \infty$
and that for all $t \geq 0$ there exists $\delta_t' \in [0,1)$ and $F_t' \in [0,\infty)$ such that $\delta_t' \int_0^t e^{(c_s' - \rho_s)s} ds < 1$ and
$W_t \leq \frac{\delta_t'}{4B_t C_t}\left(|\nabla V_t|^2 - 2\Delta V_t\right) + F_t'$.
Then, for any $f: \mathbb{R}^n \to \mathbb{R}_+$ smooth enough, it holds

$$\|P_t^{(t)} f\|_{\Phi_t,\mu_t} \leq m(s,t) \|P_s^{(s)} f\|_{\Phi_s,\mu_s}$$

where under assumption (i),

$$m(s,t) = \exp\left\{\int_s^t F_u + \left(b_u \frac{1 - e^{-\rho_u u}}{\rho_u}\right)^2 \frac{D_u + E_u}{2(1 - \delta_u - \bar{\rho}_u)} + c_u B_u u du\right\}$$

and under assumption (*ii*),

$$m(s,t) = \exp\left\{\int_s^t \left(\int_0^u e^{(c'_v-\rho_v)v}\,dv\right)^2 \frac{b_u(D_u+E_u)}{2(1-\delta_u-\bar{\rho}_u-\delta'_u\int_0^u e^{(c'_v-\rho_v)v}\,dv)} + \int_0^u e^{(c'_v-\rho_v)v}\,dv\left(B_uF'_u+F_u\right)du\right\}.$$

▸▸ endtheorem

Open Problems

- An interesting example not yet covered : with $\alpha \in (1,2)$,

$$V_t(x) = (1-t)_+^2 |x|^2 + |x|^\alpha$$

we have a critical point $t = 1$ in which hypercontractivity in $\mathbb{L}_p$ spaces is replaced by a weaker property. Such an example requires $b_t = \infty$ and is not covered yet by the theorem.

- Time Dependent Diffusion Equation

For

$$L_t = L + V(t) \cdot \nabla,$$

$$\partial_t P_t^{(t)} f = L_{(t)} f + \int_0^t \left(e^{\tau L_{(t)}} \dot{V}(t) e^{(t-\tau) L_{(t)}} f \right) d\tau.$$

- What are nonhommogeneous stochastic processes with nonconstant smoothing properties ?

Extended Gradient Bounds

For $W : \mathbb{R}^n \to \mathbb{R}_+$, $W^2 \in \mathcal{C}^2$, define

$$\Gamma^W(f,g) := \Gamma(f,g) + W^2 fg$$

with

$$\Gamma(f) \equiv \frac{1}{2}\left(Lf^2 - 2fLf\right)$$

and define the **iterated operator**

$$\Gamma_2^W(f,g) := \frac{1}{2}\left(L\Gamma^W(f,g) - \Gamma^W(Lf,g) - \Gamma^W(f,Lg)\right)$$

$$= \Gamma_2(f,g) + \frac{1}{2} fgL(W^2) + W^2\Gamma(f,g) + 2W\nabla W\nabla(fg).$$

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○ C. Roberto, B. Zegarlinski, Bakry-Emery Calculus For Diffusion With Additional Multiplicative Term, (2021) arXiv:2102.10633

THEOREM [Extended Gradient Bound]

Assume that for some $\rho \in \mathbb{R}$

$$\Gamma_2 \geq \rho\Gamma$$

and

$$\gamma := \inf_{x\in\mathbb{R}^n : W(x)\neq 0} \left(\frac{LW}{W} - 3\frac{|\nabla W|^2}{W^2} \right) > -\infty.$$

Then, for t ≥ 0,

$$\Gamma^W(P_t f) \leq e^{-2\min(\rho,\gamma)t} P_t(\Gamma^W(f)) \qquad .$$

for all $f \in \mathcal{C}^2$.

EquivThm

Examples.Skip EquivThm

Remark

For $\rho = 0$, the ratio $\frac{1-e^{-2\rho t}}{\rho}$ is understood as its limit (i.e. $2t$). Notice that it is always non-negative.

Observe that, applying (ii) to constant functions $f \equiv C$, $C \neq 0$, leads to

$$W^2 \leq e^{-2\rho t} P_t(W^2).$$

Therefore, if $\int W^2 d\mu < \infty$ and $\rho > 0$,
taking the limit $t \to \infty$ and by ergodicity, we would conclude that $W \equiv 0$.
Therefore, for

$$\Gamma_2^W(f) \geq \rho \Gamma^W(f)$$

to hold for a non trivial W, either $\rho \leq 0$ or $\int W^2 d\mu = \infty$.
But, we have no such restriction removing mean value $\mu(f)$ of the function f.

Example

For $p, q \geq 1$, consider

$$U(x) = c + \frac{(1+|x|^2)^{p/2}}{p} \quad \textit{and} \quad W(x) = \frac{(1+|x|^2)^{q/2}}{q},$$

with c s.t. $\int e^{-U(x)} dx = 1$ and $|x| = (\sum x_i^2)^{1/2}$. Then

$$\frac{LW}{W} - 3\frac{|\nabla W|^2}{W^2} = \frac{qn}{1+|x|^2} - \frac{q|x|^2}{1+|x|^2}\left(\frac{2(q+1)}{1+|x|^2} + (1+|x|^2)^{(p-2)/2}\right)$$

is bounded from below iff $1 \leq p \leq 2$.

Extended Gradient Bounds

COMPLETE CHARACTERISATION

$$\langle \nabla P_t f, \nabla P_t f \rangle \leq e^{\alpha t} P_t \langle \nabla f, \nabla f \rangle$$

The description of all allowed $\langle \cdot, \cdot \rangle$ comes from
Theory of Markov Semigroups in Noncommutative Spaces

see ⇓

• F.Cipriani, B.Zegarlinski, KMS Dirichlet forms, coercivity and super-bounded Markovian semigroups, arXiv:2105.06000

•• F.Cipriani, B.Zegarlinski, Noncommutative Perturbation Theory. https://it.overleaf.com/project/6214cf6524f31131f50ea0ad

Thank you for your attention

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