Perturbation of Dirichlet Forms

Some Results and Questions

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Lecture I

General Plan

Lecture I :

* Metric and differential structures
* Perturbation of measures
* Dirichlet forms
* Orlicz spaces basics

Lecture II :

* Coercive bounds
* Poincare and log-Sobolev
* Hypercontractivity in Orlicz spaces // Time
Non-hmogeneous Diffusions
* Higher Order Coercive Inequalities
* Extended Gradient Bounds

Lecture III :

* Noncommutative $\mathbb{L}_p$ spaces
* Dirichlet forms in their perturbation
* Perturbation of Noncommutative Poincare and log-Sobolev

Lecture I :Content

* Metric and differential structures ⏩ *.1
* Perturbation of measures ⏩ *.2
* Dirichlet forms ⏩ *.3
* Orlicz spaces basics ⏩ *.3

* Space $\Omega \equiv \mathbb{R}^n$.
Natural (*abelian*) addition of vectors & scaling

$$\mathbf{x} \oplus \mathbf{y} \equiv (x_j + y_j)_{j=1,..,n}$$
$$\lambda \odot \mathbf{x} \equiv (\lambda \cdot x_j)_{j=1,..,n}$$

Metrics given by l_p norms $p \in [1, \infty]$:

$$||\mathbf{x}||_p \equiv \left(\sum_{j=1,..,n} |x_j|^p \right)^{\frac{1}{p}} \qquad if \quad p \in [1, \infty)$$
$$||\mathbf{x}||_\infty \equiv \max_{j=1,..,n} |x_j|$$

Homogeneity and triangle inequality

$$||\lambda \odot \mathbf{x}||_p = |\lambda| \cdot ||\mathbf{x}||_p$$
$$||\mathbf{x} \oplus \mathbf{y}||_p \leq ||\mathbf{x}||_p + ||\mathbf{y}||_p$$

Equivalence of metrics

$$\rho_p(\mathbf{x}, \mathbf{y}) \equiv \|\mathbf{x} - \mathbf{y}\|_p$$

$\forall p_1, p_2 \in [1, \infty] \exists C \in (0, \infty)$

$$\frac{1}{C}\rho_{p_2}(\mathbf{x}, \mathbf{y}) \leq \rho_{p_1}(\mathbf{x}, \mathbf{y}) \leq C\rho_{p_2}(\mathbf{x}, \mathbf{y})$$

* Gauge metrics :
Fix a convex set $\mathcal{O}$ and define define

$$\|\mathbf{x}\|_{\mathcal{O}} = \inf\{\lambda > 0 : \lambda^{-1} \odot \mathbf{x} \in \mathcal{O}\}$$

Non-homogeneous Space

$$\lambda \odot \mathbf{x} \equiv (\lambda^{\kappa_j} x_j)_{j=1,..,r}$$

with $1 \leq \kappa_1 \leq ... \leq \kappa_r$.

$$\lambda \odot \mathbf{x} \equiv (\lambda \cdot x_j)_{j=1,..,n}$$

Adapted metrics

$$\rho_{p,\kappa}(\mathbf{x}, \mathbf{y}) \equiv \left(\sum_{j=1,..,r} |x_j - y_j|^{\frac{p}{\kappa_j}} \right)^{\frac{1}{p}} \equiv \|\mathbf{x} \ominus \mathbf{y}\|_{p,\kappa}$$

Homogeneous norm

$$\|\lambda \odot \mathbf{x}\|_{p,\kappa} = |\lambda| \cdot \|\mathbf{x}\|_{p,\kappa}$$

E.g. $r = 3,\ p = 4,\ \kappa = (1, 2)$

$$N(x, y, z) \equiv \|(x, y, z)\|_{4,\kappa} = \left((x^2 + y^2)^2 + 16z^2\right)^{\frac{1}{4}}$$

Families of metrics

* positive linear combinations of metrics
* If $F(s,t)$ is jointly concave and vanishes at zero than for any metrics $\rho, \tilde{\rho}$, also $F(\rho, \tilde{\rho})$ is a metric.
(e.g. $F(s,t) = f(\frac{s}{h(t)})h(t)$ for perspective function f and a concave $h > 0$.)

E.g.

*Arithmetic and geometric means generate new metrics.
* A metric plus a semimetric = metric

$$\rho(\mathbf{x}, \mathbf{y}) + F(\mathbf{x}, |y_j|)$$

for some j.
* Use homogeneous norms

$$e.g.1 \qquad (d^p + N^p)^{\frac{1}{p}}$$

$$e.g.2 \qquad \left(\sum_{l=1,..,m} \rho_l^p \right)^{\frac{1}{p}}$$

Metric defined by length of a curve

$$\rho(\mathbf{x}, \mathbf{y}) \equiv \inf\left\{ l(\gamma_{\mathbf{x},\mathbf{y}}) \right\}$$

where l denotes a length function $\equiv$ an additive nonnegative function on some family of allowed paths.

FIELDS $\equiv$ DIFFERENTIAL OPERATORS

* Gradient $\nabla \equiv (\partial_j \equiv \partial_{x_j})_{j=1,..,n}$ in $\mathbb{R}^n$

**Smooth and nonsmooth metrics

$\|\cdot\|_p$, $p \in (1,\infty)$ are smooth out of origin.

*Eikonal equation

For $p, q \in (1,\infty)$, dual $\frac{1}{p} + \frac{1}{q} = 1$

$$\|\nabla \|\mathbf{x}\|_p\|_q = 1$$

for $\mathbf{x} \neq \mathbf{0}$.

Fields $\equiv$ Differential operators

* SubGradient $\nabla \equiv (X_j \equiv \partial_{x_j} + \sum_{l>j} a_l \partial_{x_j})_{j=1,..,k}$, $k < n$ in $\mathbb{R}^n$, a_l smooth.

* Assumption : Multiple commutators

$$[X_{j_1}, [X_{j_2}, ..[X_{j_{m-1}}, X_{j_m}]]]$$

of finite order $m \in \mathbb{N}$ span all tangent space.

E.g. $\mathbb{R}^3$ $\nabla_{\mathbb{H}} \equiv (X, Y)$

$$X = \partial_x + 2y\partial_z \qquad Y = \partial_y - 2x\partial_z$$

Then

$$[X, Y] = 4\partial_z$$

Kaplan homogeneous norm

$$N(x,y,z) \equiv \left((x^2+y^2)^2 + 16z^2\right)^{\frac{1}{4}}$$

is smooth and we have

$$||\nabla_{\mathbb{H}} N||_2 = \frac{|(x,y)|}{N}$$
$$\Delta_{\mathbb{H}} N^{-2} \equiv (X^2 + Y^2)N^{-2} = 0, \qquad a.e.$$

http://www-sop.inria.fr/members/James.Inglis/Site/Publications_files/Inglis_THESIS_FINAL.pdf

Control distance

* **Chow-Rashevski Theorem**: Every two points in $\mathbb{R}^n$ can be connected by a path which is piecewise smooth with tangent vectors in Span(∇).

* Length of a curve

$$l(\gamma_{\mathbf{x},\mathbf{y}}) = \int_0^1 \|X_{\gamma(t)}\|_2 dt$$

* Control Distance

$$d(\mathbf{x},\mathbf{y}) \equiv \inf l(\gamma_{\mathbf{x},\mathbf{y}})$$

* Eikonal equation

$$\|\nabla_{\mathbb{H}} d\|_2 = 1$$

* Control distance is not smooth

$$\Delta_{\mathbb{H}} d(0,\mathbf{x}) \to_{|(x,y)|_2 \to 0} -\infty$$

Nilpotent Lie groups

$$\mathbf{x} \circ \mathbf{y} = (x_j + y_j + P_j(\mathbf{x}, \mathbf{y}))_{j=1,..,n}$$

E.g. Heisenberg Group

$$(x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + a(x'y - xy'))$$

* Noncommutative Discrete Infinite Coxeter Group associated to Nilpotent Lie Groups.

Perturbation of measures

$$d\mu = e^{-U} d\lambda$$

*Quasi invariant measures

$$\frac{d\mu(\cdot \circ e^{tX_j})}{d\mu}$$

* Associated Unitary Group and CCRs

Start

Nice Orlicz functions

$\Phi : \mathbb{R} \to \mathbb{R}$

$$\Phi(x) = \int_0^{|x|} \phi(s) ds$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\phi(0) = 0$ and otherwise $\phi > 0$, and $\phi(s) \to_{s\to\infty} \infty$.

Complementary Function

$$\Psi(y) = \int_0^{|y|} \psi(s) ds$$

where

$$\psi(t) = \inf\{s \geq 0 : \phi(s) \geq t\}$$

Young's Inequality

$$xy \leq \Phi(x) + \Psi(y)$$

Luxemburg norm

$$\|X\|_\Phi = \inf\{\lambda > 0 : E\Phi(X/\lambda) \leq 1\}.$$

Normalisation Property

$$E\Phi\left(\frac{X}{\|X\|_\Phi}\right) = 1 \qquad (\star)$$

Michał Lemańczyk, Concentration inequalities survey.

https://www.mimuw.edu.pl/~bpolaczyk/projects/resources/orlicz.pdf

Tail Estimates I.

If Φ is N-function and $0 < \|X\|_\Phi < \infty$, then

$$P(|X| \geq t) \leq \frac{1}{\Phi(t/\|X\|_\Phi)}$$

Skip MaxEstimates

Max Estimates $\max_{1\le i\le n}|X_i|$

Let Φ be a strictly increasing, non-zero function, s.t. $\Phi(1) \le 1/2$ and

$$\exists c > 0 \; \forall x, y \ge 1, \qquad \Phi(x)\Phi(y) \le \Phi(cxy).$$

Then for any $n \in \mathbb{N}$

$$\|\max_{1\le i\le n} X_i\|_\Phi \le c\,\Phi^{-1}(2n) \max_{1\le i\le n} \|X_i\|_\Phi \qquad (*)$$

If Φ is a strictly increasing, convex, non-zero function, then for any $n \in \mathbb{N}$

$$E\left(\max_{1\le i\le n} |X_i|\right) \le \Phi^{-1}(n) \max_{1\le i\le n} \|X_i\|_\Phi \qquad (**)$$

Skip Proof

Proof: The assumption says that for $x \geq y \geq 1$ we have

$$\Phi\left(\frac{x}{y}\right) \leq \frac{\Phi(cx)}{\Phi(y)}$$

Thus, for $y \geq 1$ and any $C > 0$ using the monotonicity of Φ

$$\max_{1 \leq i \leq n} \Phi\left(\frac{|X_i|}{Cy}\right) \leq \Phi(1) + \sum_{i=1}^{n} \frac{\Phi(c|X_i|/C)}{\Phi(y)}$$

For $C = c \max_{1 \leq i \leq n} \|X_i\|_\Phi$ we get

$$\begin{aligned} E\Phi\left(\frac{\max_{1 \leq i \leq n} X_i}{Cy}\right) &\leq E \max_{1 \leq i \leq n} \Phi\left(\frac{X_i}{Cy}\right) \\ &\leq \frac{n}{\Phi(y)} + \Phi(1) \leq \frac{n}{\Phi(y)} + \frac{1}{2} \leq 1 \end{aligned}$$

where the last inequality holds iff $2n \leq \Phi(y)$.

To get the second part of this lemma where Φ convex note that due to Jensen's inequality we have (w.l.o.g. $X_i \geq 0$)

$$\frac{1}{C} E \max_{1 \leq i \leq n} X_i \leq \Phi^{-1}(E \max_{1 \leq i \leq n} \frac{\Phi(X_i)}{C}) \leq \Phi^{-1}(n)$$

$\square$

Exponential Luxemburg norms

DEFINITION :

For any $\alpha \in (0, \infty)$ we define

$$\Psi_\alpha(x) = \exp(|x|^\alpha) - 1$$

and the corresponding α-exponential Luxemburg norm as

$$\|X\|_{\Psi_\alpha} = \inf\{c > 0 : E\exp(|X|^\alpha / c^\alpha) \leq 2\}$$

Tail Estimates II

For any random variable X with

$$0 < \|X\|_{\Psi_\alpha} < \infty$$

and $t > 0$ we have

$$P(|X| \geq t) \leq 2\exp(-t^\alpha/\|X\|_{\Psi_\alpha}^\alpha)$$

LuxNorm

Observations

(1) If one replaces a Luxemburg norm by some convenient estimate from above, one still gets a tail estimates.

(2) If one replaces X by $X - EX$ we get concentration estimates.

(3) More generally, given a space $\mathcal{P}$ of R.V.s , one can study $X - w$, for $w \in \mathcal{P}$.

How to get bounds on Luxemburg Norms

Coercive Inequalities

Start

• Poincaré Inequality

$$m\,E(f - Ef)^2 \leq E|\nabla f|^2, \qquad (\mathrm{PI}_2)$$

with

$$E(f - Ef)^2 = \inf_{a\in\mathbb{R}} E(f-a)^2$$

H. Poincaré (Eqn (11) p. 253), « Sur les Equations aux Dérivées Partielles de la Physique Mathématique »,

Amer. J. of Math. vol. 12, no 3, 1890, pp. 211–294, doi10.2307/2369620

([BE'1985].., [Antoniuk Antoniuk'1993] ,..[V'06],[BCG'08],[HZ'09],..)

Start

Facts

∘ [HZ'09]

Let $dE = e^{-U} d\lambda$ with U locally bounded. Let $q \in [1,2]$.

If $\exists\, 0 \leq \eta(x) \xrightarrow[dist(0,x)\to\infty]{} \infty$ (no matter how slowly)

and $\exists\, C, D \in (0,\infty)$

$$E\,(\eta|f|^q) \leq C\,E|\nabla f|^q + D\,E|f|^q, \qquad (\star)$$

then $\exists\, m \in (0,\infty)$ s.t.

$$m \inf_{a\in\mathbb{R}} E|f-a|^q \leq E|\nabla f|^q \qquad (\mathrm{IP}_q)$$

∘ If (IP_q) holds, then for every Lip function f with finite first moment there exists $\varepsilon > 0$ s.t.

$$E\,e^{\varepsilon|f|} < \infty$$

i.e. Lip functions (with finite 1^{st} moment) belong to Orlicz space with exp norm.

- Poincaré Inequality with a Weight

([BL'1976],..,[Antoniuk Antoniuk'1993],..,[HZ'09],..)

$$m\, E|f - Ef|^q \leq E\frac{|\nabla f|^q}{W}, \qquad q \geq 1 \qquad (\mathrm{IP}_{q,W})$$

Facts ○ [HZ'09]

Let $dE = e^{-U} d\lambda$ with U locally bounded. Let $q \in [1,2]$.

If $\quad \exists\, 0 \leq \eta(x), \frac{|\nabla \eta|^q}{\eta^{q+\delta}} \xrightarrow[dist(0,x)\to\infty]{} \infty$ (with some $0 < \delta < 1$)

and $\exists\, C, D \in (0,\infty)$

$$E\,(\eta |f|^q) \leq C\, E|\nabla f|^q + D\, E|f|^q, \qquad (\star)$$

then $\exists\, m \in (0,\infty)$ s.t.

$$m \inf_{a\in\mathbb{R}} E|f-a|^q \leq E\frac{|\nabla f|^q}{W} \qquad (\mathrm{IP}_{q,W})$$

○ If $(\mathrm{IP}_{q,W})$ holds, then for every Lip_W function f

i.e. $\|\frac{|\nabla f|^q}{W}\|_\infty < \infty$, (with $E|f| < \infty$), there exists $\varepsilon > 0$ s.t.

$$E\, e^{\varepsilon |f|} < \infty$$

i.e. Lip_W functions (with $E|f| < \infty$) belong to

Orlicz space with exp norm.

- Higher order Poincaré inequalities

$\exists\, m \in (0,\infty)$ s.t.

$$m_k \inf_{w \in \mathcal{P}_k} E|f-w|^q \leq E|\boldsymbol{\nabla}^k f| \qquad (\mathrm{IP}_{q,k})$$

where $k \in \mathbb{N}$ and $\mathcal{P}_k$ denotes space of polynomials of order k, and

$$|\boldsymbol{\nabla}^k f| \equiv \sum_{|\alpha|=k} |\nabla^\alpha f|^q$$

Facts

- [WZ'21] **Downhill Induction $\mathbf{L}_q$-Case.**

$$\forall k \in \mathbb{N} \qquad (\mathrm{IP}_q) \implies (\mathrm{IP}_{q,k})$$

Theorem

For $q \in (1, \infty)$, there exists $C_{k,q} \in (0, \infty)$ and a polynomial $m_{k,q}(f)$ of order $k-1$, such that

$$\mu \left| f - m_{k,q}(f) \right|^q \leq C_{k,q} \sum_{|\alpha|=k} \mu |\nabla^\alpha f|^q. \qquad (PI_{k,q})$$

Y.Wang & BZ, Higher Order Coercive Inequalities, Potential Analysis (2021) https://link.springer.com/article/10.1007/s11118-021-09940-1

Assume that $(PI_{q,1})$ holds. Then

$$\begin{aligned}\sum_{|\alpha|=k} \mu|\nabla^\alpha f|^q &= \sum_{|\beta|=k-1} \sum_j \mu|\nabla_j \nabla^\beta f|^q \\ &\geq C_{1,q}^{-1} \sum_{|\beta|=k-1} \mu|\nabla^\beta f - M_{1,q}(\nabla^\beta f)|^q \\ &= C_{1,q}^{-1} \sum_{|\beta|=k-1} \mu|\nabla^\beta \left(f - B_{1,q,k-1}(f)\right)|^q\end{aligned}$$

with

$$B_{1,q,k-1}(f) \equiv \sum_{|\beta|=k-1} \frac{x^\beta}{\beta!} M_{1,q}(\nabla^\beta f)$$

Applied inductively, yields

$$\sum_{|\alpha|=k} \mu|\nabla^\alpha f|^q \geq C_{1,q}^{-(k-j)} \sum_{|\alpha|=k-j} \mu|\nabla^\alpha \left(f - B_{1,q,j-1}(f)\right)|^q$$

with

$$B_{1,q,j-1}(f) \equiv \sum_{|\alpha|=k-j} \frac{x^\alpha}{\alpha!} M_{1,q}(\nabla^\alpha (f - B_{1,q,j}(f))).$$

Hence ($\mathrm{PI}_{k,q}$) follows $m_{k,q} \equiv B_{1,q,1}(f)$ and $C_{k,q} \leq C_{1,q}^k$. □

Remark The constant in the above is exponential in k and is generally not optimal and different then the optimal constant in the $(PI_{k,q})$ with the minimizing polynomial $M_{k,q}(f)$.

E. Bou Dagher, Y.Wang & BZ, Coercive inequalities on Carnot groups and applications, , Advances in Analysis and Geometry, Vol. 6 (2022)
https://doi.org/10.1515/9783110741711

Statistical Polynomials $m_{k,q}(f)$

Questions :

What is the structure & possible applications of $m_{k,q}(f)$?

Are they orthogonal for different values of k if $q = 2$?

Facts CND

- Similar techniques based on $\star$-bounds for $(\mathrm{IP}_{q,k})$ and

$$m \inf_{\mathcal{P}\in\mathcal{P}_{k-1}} E|f-_{\mathcal{P}}|^q \le E\left(\frac{|\nabla^k f|^q}{W}\right) \qquad (\mathrm{IP}_{q,k,W})$$

Facts CND

- $$\forall k \in \mathbb{N} \qquad (\mathrm{IP}_{q,k}) \implies \mathit{ExpBounds}$$

Suppose $\|\frac{|\nabla^k f|^q}{W}\|_\infty < \infty$

$$\exists\ \varepsilon > 0 \qquad E e^{\varepsilon |f - m_{k,q}(f)|} < \infty$$

S. G. Bobkov, F. Gotze, H. Sambale, Higher Order Concentration of Measure, Commun. Contemporary Math., Vol. 21, No. 03 (2018)
https://doi.org/10.1142/S0219199718500438
Friedrich Götze and Holger Sambale, Higher Order Concentration in Presence of Poincaré-Type Inequalities, Ch. 6, in "High Dimensional Probability VIII, The Oaxaca Volume", Birkhäuser 2019, Eds. N.Gozlan, R.Latała, K.Lounici, M.Madiman,
https://doi.org/10.1007/978-3-030-26391-1

Multiparticle decay

Higher order estimates for Gaussian semigroup Let

$$L = \Delta - x \cdot \nabla \quad \text{and} \quad P_t = e^{tL}.$$

Then we have

Theorem

$$|\nabla^\alpha P_t f|^2 \leq e^{-2|\alpha|} P_t |\nabla^\alpha f|^2$$

Hence with the corresponding invariant Gaussian measure γ, we have

$$\int |\nabla^\alpha P_t f|^2 d\gamma \leq e^{-2|\alpha|} \int |\nabla^\alpha f|^2 d\gamma$$

Skip Proof/Jump to Optima

Proof In the case of interest to us it is well known that the kernel is smooth.
Following an idea of Bakry-Emery, we have

$$\partial_s P_{t-s}|\nabla^\alpha P_s f|^2 = P_{t-s}\Big(-L|\nabla^\alpha P_s f|^2 + 2\nabla^\alpha L P_s f \cdot \nabla^\alpha P_s f\Big)$$

and so

$$\partial_s P_{t-s}|\nabla^\alpha P_s f|^2$$
$$= P_{t-s}\Big(\Big(-L|\nabla^\alpha P_s f|^2 + 2\nabla^\alpha P_s f \cdot L\nabla^\alpha P_s f\Big) + 2[\nabla^\alpha, L]P_s f \cdot \nabla^\alpha P_s f\Big).$$

Since

$$[\nabla, L] = -\nabla,$$

using the following inductive formula

$$[\nabla^n, L] \equiv \nabla[\nabla^{n-1}, L] + [\nabla, L]\nabla^{n-1},$$

we have

$$[\nabla^n, L] = -n\nabla^n.$$

Hence for a component ∇^α, we get

$$[\nabla^\alpha, L] = -|\alpha|\nabla^\alpha.$$

Using the fact that the Markovian form of L satisfies

$$L(g^2) - 2gLg = 2|\nabla g|^2 \geq 0,$$

we obtain the following differential inequality

$$\partial_s P_{t-s} |\nabla^\alpha P_s f|^2 \leq -2|\alpha| P_{t-s} \left(|\nabla^\alpha P_s f|^2 \right),$$

thereby concluding

$$|\nabla^\alpha P_t f|^2 \leq e^{-2|\alpha|} P_t |\nabla^\alpha f|^2$$

Accordingly, with the corresponding invariant Gaussian measure γ, we have

$$\int |\nabla^\alpha P_t f|^2 d\gamma \leq e^{-2|\alpha|} \int |\nabla^\alpha f|^2 d\gamma$$

$\square$

Optimal higher order Poincaré constants for O-U

Note that for Hermite polynomials H_k, $k \in \mathbb{N}$, in one dimensions, we have

$$\nabla H_k(t) = \sqrt{k} H_{k-1}(t).$$

and hence

$$\nabla^n H_k(t) = \sqrt{k \cdot .. \cdot (k-n)} H_{k-n}(t) \quad \text{for all} \quad k \geq n.$$

Thus, using representation

$$f = \sum_{k \in \mathbb{N}} f_k H_k$$

with respect to the O-N basis of Hermite polynomials, we compute

$$\int |\nabla^n f|^2 d\gamma = \sum_{k \geq n+1} k..(k-n)|f_{k-n}|^2 \geq n! \int |f - M_{2,n,\gamma}(f)|^2 d\gamma$$

where in $\mathbb{L}_2(\gamma)$, we have

$$M_{2,n,\gamma}(f) = \sum_{k \leq n} f_k H_h.$$

The optimal k-th order Poincare Inequality for O-U

Theorem

$$n! \int |f - M_{2,n,\gamma}(f)|^2 d\gamma \leq \int |\nabla^n f|^2 d\gamma.$$

Remark: This is better than Downhill Induction gives.

Entropy Estimates
and Unrestricted Exp Bounds

Start

Log-Sobolev Inequality

[Stamm'59], [Federbush'69][Gross'75],.., [JRosen'76], [Adams'79],..,[B-E'85],..

$$E\left(f^2 \log \frac{f^2}{Ef^2}\right) \leq c\, E|\nabla f|^2$$

$$\Updownarrow$$

$$\|(f - Ef)^2\|_\Phi \leq c' E|\nabla f|^2$$

with $\Phi(x) = |x| \log(1 + |x|)$, [BG'JFA99].

M. Ledoux, Logarithmic Sobolev Inequalities, *what they are, some history analytic, geometric, optimal transportation proofs, last decade developments, at the interface between analysis, probability, geometry*
https://www.math.univ-toulouse.fr/~ledoux/Logsobwpause.pdf

Entropy Bounds & Exponential Moments

If , for $q \in (1, \infty)$,

$$E\left(|f|^q \log \frac{|f|^q}{E|f|^q}\right) \leq c\, E|\nabla f|^q$$

then

$$Ee^{tf} \leq \exp\left\{const \cdot t^q \, \|\nabla f\|_\infty^q + Ef\right\}$$

Proof Idea

$$E\left(e^{tf}\log\frac{e^{tf}}{Ee^{tf}}\right)\leq\frac{c}{q^q}\,t^q\,E\left(|\nabla f|^q e^{tf}\right)$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\frac{1}{t}\log Ee^{tf}\right)\leq\frac{c}{q^q}\,t^{q-2}\,\|\nabla f\|_\infty^q$$

$$\Downarrow$$

$$Ee^{tf}\leq\exp\left\{\frac{c}{q^q(q-1)}\,t^q\,\|\nabla f\|_\infty^q+Ef\right\}$$

M. Ledoux, Remarks on logarithmic Sobolev constants, exponential integrability and bounds on the diameter, J. Math. Kyoto Univ. (JMKYAZ) 35-2 (1995) 211-220 https://perso.math.univ-toulouse.fr/ledoux/files/2019/10/Kyoto.pdf

S.G.Bobkov and B.Zegarlinski, Entropy bounds and isoperimetry, Mem.Amer.Math. Soc. 2005; Vol.176, Nr 829 http://www.ams.org/books/memo/0829

Generalised Entropy Bounds

Let

$$d\mu \equiv e^{-U} d\lambda$$

Adams Regularity Conditions: *[Adams'JFA1979]*

$\exists \varepsilon, C \in (0, \infty)$

$$\sum_{|\alpha|=2} |\nabla^\alpha U| \leq C(1 + |\nabla U|)^{2-\varepsilon}. \qquad \text{(ARC)}$$

Remark :

- Note that in particular we have

$$LU \leq C(1 + |\nabla U|)^{2-\varepsilon} - |\nabla U|^2.$$

where

$$Lf = \Delta f - \nabla U \cdot \nabla f$$

Smooth Orlicz functions

Let

$$\Phi_{A,p}(t) = |t|^p \prod_{j=1}^{n} (\log_j^*(|t|))^{p_j} \equiv |t|^p A(\log^* t),$$

and

$$\Phi(t) = t \prod_{j=1}^{n} (\log_j(\gamma_j + |t|))^{p_j} \equiv t\boldsymbol{\Theta}(t).$$

Then there exists $C \in (0, \infty)$ s.t.

$$\Phi(|x|^p) \leq C\Phi_{A,p}(|x|).$$

Lemma

Suppose the following Adams Inequality holds

$$\mu(\Phi_{A,p}(f)) \leq \tilde{C}_A \, \mu \left| \nabla^k f \right|^p + \tilde{D}_A \, \Phi_{A,p}(\|f\|_p^p) \qquad \text{(AI)}$$

with some $\tilde{C}_A, \tilde{D}_A \in (0, \infty)$ *independent of* f. *Then*

$$\| \, |f|^p \, \|_\Phi \leq C' \mu \left| \nabla^k f \right|^p + D' \|f\|_p^p. \qquad \text{(AOI)}$$

with some $C', D' \in (0, \infty)$ *independent of* f.

Theorem

If the following (p,k)-Poincaré inequality holds

$$\|(f - M_{p,k}(f))\|_p^p \leq c_{p,k}\, \mu \left|\nabla^k f\right|^p \qquad (\mathrm{PI}_{p,k})$$

with some $c_{p,k} \in (0,\infty)$ for all f for which the r.h.s. is well defined, then the following tight (Φ,p,k)-Inequality holds

$$\| \left|f - M_{p,k}(f)\right|^p \|_\Phi \leq C\, \mu \left|\nabla^k f\right|^p . \qquad (\mathrm{OSI})$$

Class $\mathfrak{N}$

If we have with $\alpha \in (1, \infty)$

$$p.p.N(u) = \frac{|u|^\alpha}{\alpha} \prod_{k=1}^{n} (\log_k |u|)^{\gamma_k}$$

then with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$p.p.N^*(u) = \frac{|u|^\beta}{\beta} \left(\prod_{k=1}^{n} (\log_k |u|)^{-\gamma_k} \right)^{\beta - 1}$$

Krasnosielski, M., Ruticki, B.: Convex functions and Orlicz spaces. Groningen: Noordhoff 1961. Chapter 1 §7

Questions:

- For $q = 2$, what are the optimal estimates

$$\|P_t(f - M_n(f))\|_2^2 \leq e^{-m_n t}\|f - M_n(f)\|_2^2$$

-

$$\|P_t(f - M_n(f))\|_\Phi \leq e^{-\varepsilon_n t}\|f - M_n(f)\|_\Phi$$

- Description of Invariant subspaces and Martingales.
- Analysis on Nilpotent Lie Groups:

Kaplan versus Carnot-Caratheodory.

No Adams regularity for $U(d)$ with C-C distance d, $(|\nabla d| = 1)$.
No Log-Sobolev with Kaplan Distance$_{[HZ'09]}$:(and any smooth homogeneous norm),
Some $(\mathrm{Log})^\beta$-Sobolev $_{[EBZ'21–22]}$ and Poincaré Inequality are still OK:)$_{[In'10],[CFZ'21]}$

○ E.Bou Dagher and B.Zegarlinski, Coercive Inequalities and U-Bounds on Step-Two Carnot Groups, Potential Analysis 2021 https://doi.org/10.1007/s11118-021-09979-0

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Time Non Homogeneous Processes with Variable Smoothing Properties

Start

+Contractivity along Orlicz spaces

Let $(\mathbb{L}_{\Phi_t})_{t\in\mathbb{R}^+}$ be a nested family of Orlicz spaces.

∘ For a Markov semigroup $(P_t)_{t\in\mathbb{R}^+}$ defined in the given family of Orlicz spaces spaces, when do we have the following strong contractivity property ?

$$||P_t f||_{\Phi_t} \leq ||f||_{\Phi_0}.$$

∘ Can we also consider variable family $(P_t^{(t)})_{t\in\mathbb{R}^+}$ semigroups

$$||P_t^{(t)} f||_{\Phi_t} \leq ||f||_{\Phi_0}.$$

∘ C.Roberto and B.Zegarlinski, Hypercontractivity for Markov Semigroups, J. Funct. Analysis (2022) https://doi.org/10.1016/j.jfa.2022.109439

Theorem Let $(\Phi_t)_{t\geq 0}$ be a family of $\mathcal{C}^2$ Young functions. Assume that for some $t,s\geq 0$ there exist two positive constants $C(t,s)$ and $\widetilde{C}(t,s)$ such that

(i)

$$\dot{\Phi}_t(\Phi_t^{-1}) \leq C(t,s)\dot{\Phi}_s(\Phi_s^{-1}),$$

(ii)

$$\frac{\Phi_t''}{\Phi_t'^2}\circ\Phi_t^{-1} \geq \widetilde{C}(t,s)\frac{\Phi_s''}{\Phi_s'^2}\circ\Phi_s^{-1}.$$

Assume that for some $c\in(0,\infty)$ and for any f (smooth enough), it holds

$$\|f\|_{\Phi_s}^2\int\dot{\Phi}_s\left(\frac{f}{\|f\|_{\Phi_s}}\right)d\mu \leq c\int\Phi_s''\left(\frac{f}{\|f\|_{\Phi_s}}\right)|\nabla f|^2 d\mu. \qquad (1)$$

Then, for any f smooth enough it holds

$$\|f\|_{\Phi_t}^2\int\dot{\Phi}_t\left(\frac{f}{\|f\|_{\Phi_t}}\right)d\mu \leq c\frac{C(t,s)}{\widetilde{C}(t,s)}\int\Phi_t''\left(\frac{f}{\|f\|_{\Phi_t}}\right)|\nabla f|^2 d\mu.$$

Proof

Let

$$\begin{aligned} I_{s,t} : \mathbb{L}_{\Phi_t} &\rightarrow \mathbb{L}_{\Phi_s} \\ f &\mapsto \|f\|_{\Phi_t} \Phi_s^{-1} \circ \Phi_t\left(\frac{f}{\|f\|_{\Phi_t}}\right). \end{aligned}$$

For any $f \in \mathbb{L}_{\Phi_t}$, by the very definition of the Luxembourg norm, it holds $\|I_{s,t}(f)\|_{\Phi_s} = \|f\|_{\Phi_t}$. Therefore, $I_{s,t}(f)$ is an isometry between the two Orlicz spaces $\mathbb{L}_{\Phi_t}$ and $\mathbb{L}_{\Phi_s}$.
Applying (1) to $I_{s,t}(f)$ leads to

$$\|f\|_{\Phi_t}^2 \int \dot{\Phi}_s\left(\Phi_s^{-1} \circ \Phi_t\left(\frac{f}{\|f\|_{\Phi_t}}\right)\right) d\mu$$

$$\leq c \int \Phi_s'' \circ \Phi_s^{-1} \circ \Phi_t\left(\frac{f}{\|f\|_{\Phi_t}}\right) \frac{\Phi_t'\left(\frac{f}{\|f\|_{\Phi_t}}\right)^2 |\nabla f|^2}{\Phi_s' \circ \Phi_s^{-1} \circ \Phi_t\left(\frac{f}{\|f\|_{\Phi_t}}\right)^2} d\mu.$$

The result follows by (i) and (ii).

The Standard Orlicz Family

Let $F:(0,\infty)\to\mathbb{R}$ be a $\mathcal{C}^2$ increasing function s.t. $F(1)=0$. Assume that $(0,\infty)\ni x\mapsto xF(x)$ is convex and that $1/xF(x)$ is not integrable at $x=0$, $x=1$ and $x=+\infty$. Let $\mathcal{F}_1:(0,1)\to\mathbb{R}$ and $\mathcal{F}_2:(1,+\infty)\to\mathbb{R}$ be two primitives of $x\mapsto 1/(xF(x))$. Let Φ_0 be a N-Young function of class $\mathcal{C}^2$ on $(0,\infty)$, and x_o the unique positive point s.t. $\Phi_0(x_o)=1$.

Assume $-\left(\frac{\Phi_0}{\Phi_0'}\right)' F(\Phi_0)-\Phi_0 F'(\Phi_0)$ is non-increasing on $\mathbb{R}^+$.

Let $\lambda\colon\mathbb{R}^+\to\mathbb{R}^+$ be an increasing function, s.t. $\lambda(0)=0$.

Definition The *standard Orlicz family* $(\Phi_t)_{t\geq 0}$ *built from* F*,* Φ_0 *and* λ is defined by

$$\Phi_t(x):=\begin{cases}0 & \text{for } x=0\\ \mathcal{F}_1^{-1}\left(\mathcal{F}_1(\Phi_0(x))+\lambda(t)\right) & \text{for } x\in(0,x_o)\\ 1 & \text{for } x=x_o\\ \mathcal{F}_2^{-1}\left(\mathcal{F}_2(\Phi_0(x))+\lambda(t)\right) & \text{for } x\in(x_o,+\infty).\end{cases}\qquad \forall t>0.$$

Examples

Example [G'75] For $F(x) = \log(x)$ and any N-function Φ_0 ,
$\mathcal{F}_1(x) = \log(\log(1/x))$, $x \in (0,1)$ and $\mathcal{F}_2(x) = \log(\log(x))$, $x > 1$
so that $\mathcal{F}_1^{-1}(x) = e^{-e^x}$ and $\mathcal{F}_2^{-1}(x) = e^{e^x}$, $x \in \mathbb{R}$.
Hence, $\Phi_t(x) = \Phi_0^{e^{\lambda(t)}}$.
If $q(t) = 1 + e^{(4/\rho)t}$ and $\lambda(t) = \log(q(t)/2)$, with $\Phi_0(x) = x^2$, we have $\Phi_t(x) = |x|^{q(t)}$, we get $\mathbb{L}_p$ scale of *Gross*' setting.

Example [BCR'] For $F(x) = \log(1+x)^\beta - \log(2)^\beta$, $\beta \in (0,1)$,
one gets non explicit $\mathcal{F}_1$ and $\mathcal{F}_2$
with an asymptotic of the corresponding $\Phi_t(x)$, when x tends to ∞ or $+\infty$ equivalent to $\Phi_0 e^{a_\beta \lambda (\log \phi_0)^\beta}$,
with a numerical constant a_β depend only on β.
This is the family of Young functions $x^2 e^{ctF(x)}$ considered in *Barthe-Cattiaux-Roberto*.

Integration Lemma for standard Orlicz family

Theorem

Let $(\Phi_t)_{t\geq 0}$ be a standard Orlicz family built from F, Φ_0 and λ. Let $c>0$. Then the following are equivalent

(i)

$$\|f\|^2_{\Phi_0}\int \Phi_0\Big(\frac{f}{\|f\|_{\Phi_0}}\Big)F\left(\Phi_0\Big(\frac{f}{\|f\|_{\Phi_0}}\Big)\right)d\mu \leq c\int \Phi_0''\left(\frac{f}{\|f\|_{\Phi_0}}\right)|\nabla f|^2 d\mu \qquad ;$$

for any function f for which the r.h.s. is well defined;

(ii) $\forall\ t\geq s\geq 0$, it holds

$$\|P_t f\|_{\Phi_t}\leq \|P_s f\|_{\Phi_s}.$$

for any function $f\in \mathbb{L}_{\Phi_s}$.

Moreover $(i)\Rightarrow(ii)$ with any (increasing) λ such that $\frac{\Phi_t''}{\Phi_t'^2}\circ\Phi_t^{-1}\geq c\lambda'(t)\frac{\Phi_0''}{\Phi_0'^2}\circ\Phi_0^{-1}$ for any $t\geq 0$ (in particular, any λ satisfying $\lambda'(t)\leq 1/c$ would do); and $(ii)\Rightarrow(i)$ with $c=1/\lambda'(0)$.

Hyperboundedness in $\mathbb{L}_p$-scales

Let

$$L_t := \Delta - \nabla V_t \cdot \nabla$$

$t \geq 0$, on $\mathbb{R}^n$, with V_t smooth (tbs) and s.t. $\int e^{-V_t} = 1$.

The associated semi-group $(P_s^{(t)})_{s \geq 0}$ is reversible with respect to the probability measure

$$\mu_t(dx) := e^{-V_t(x)} dx$$

Define

$$a_t := \|(\dot{V}_t)_-\|_\infty, b_t := \||\nabla \dot{V}_t|\|_\infty, c_t := \|(\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t)_-\|_\infty$$

Theorem

Consider the inhomogeneous diffusion operator L_t *as above.*
Assume $\forall t \geq 0 \;\; a_t, b_t, c_t < \infty$.
Suppose $\forall t \geq 0 \; \exists \rho_t \in \mathbb{R}$ *s.t.* $\mathrm{Hess}(V_t) \geq \rho_t$.
Assume $\exists \bar{c}_t \in (0, \infty)$

$$\int f^2 \log(f^2) d\mu_t \leq \bar{c}_t \int |\nabla f|^2 d\mu_t,$$

for all f *with* $\int f^2 d\mu_t = 1$ *for which the r.h.s is well defined.*
Then, for any $p > 1$ *and* $0 \leq s \leq t < \infty$,

$$\|P_t^{(t)} f\|_{\Phi_t, \mu_t} \leq m(s,t) \|P_s^{(s)} f\|_{\Phi_s, \mu_s}$$

where $\Phi_t(x) = |x|^{q(t)}$, $q(t) = 1 + (p-1)\exp\{\int_0^t (2/\bar{c}_s) ds\}$, *and*

$$m(s,t) := \exp\left\{ \int_s^t \frac{a_u}{q(u)} + u c_u + b_u^2 \frac{1 - e^{-\rho_u u}}{2\rho_u} (q(u) - 1) du \right\}.$$

Example

$$V_t(x) = U(x) + \alpha(t)V(x) + \gamma(t)$$

with V unbounded and $\gamma(t) := \log \int e^{-U-\alpha V}dx$ so that μ_t is a probability measure.
Let $U(x) = \frac{|x|^2}{2}$, $\alpha\colon \mathbb{R}^+ \to \mathbb{R}^+$ non-decreasing and $V(x) = (1+|x|^2)^{\frac{\beta}{2}}$, with $\beta \in (0,1]$.
Then,

$$\dot{V}_t = \alpha'(t)(1+|x|^2)^{\frac{\beta}{2}}$$

so that $a_t = 0$; $\nabla \dot{V}_t = \alpha'(t)\beta(1+|x|^2)^{\frac{\beta}{2}-1}x$, whence $b_t = \alpha'(t)\beta\sqrt{\frac{(1-\beta)^{1-\beta}}{(2-\beta)^{2-\beta}}}$ (and when $\beta \to 1$, we get $b_t = \alpha'(t)$.
One can prove that $c_t \leq n^2\alpha'(t)(\alpha(t)+2)$ and $\mathrm{Hess}(V_t) \geq 1$ so that $\rho_t = 1$ and $\bar{c}_t = 2$. Hence $(P_t^{(t)})_{t\geq 0}$ is hyper-bounded in the $\mathbb{L}_{q(t)}$ scale, with $q(t) = 1+(p-1)e^t$.

General result

Let $V_t(x) = U(x) + \alpha(t)V(x) + \gamma(t)$.
Let $L_t = \Delta - \nabla V_t \cdot \nabla$.
Theorem Assume $\forall t \geq 0$, $b_t := |||\nabla \dot{V}_t|||_\infty < \infty$ and $\exists \rho_t \in \mathbb{R}$ s.t. $\mathrm{Hess}(V_t) \geq \rho_t I$.
Let $(\Phi_t)_{t\geq 0}$ be a family of N-functions s.t. $\Phi_t(x) \leq x\Phi_t'(x) \leq B_t\Phi_t(x)$, $\Phi_t'^2 \leq C_t\Phi_t\Phi_t''$ and $x^2\Phi_t''(x) \leq D_t\Phi_t(x) + E_t$ for all $x \geq 0$ and some constants B_t, C_t, D_t, E_t.
Assume $\forall t \geq 0$ $\exists \delta_t \in [0,1)$ and $F_t \in \mathbb{R}$ such that $(\dot{V}_t)_- \leq \frac{\delta_t}{4C_t}\left(|\nabla V_t|^2 - 2\Delta V_t\right) + F_t$.

Set $W_t := (\nabla V_t \cdot \nabla \dot{V}_t - \Delta \dot{V}_t)_-$ and denote by $\bar{\rho}_t \in (0,\infty]$ the best constant such that for all f with $\|f\|_{\Phi_0} = 1$ it holds

$$\int \dot{\Phi}_t(f) d\mu_t \leq \bar{\rho}_t \int \Phi_t''(f) |\nabla f|^2 d\mu_t. \qquad (2)$$

Finally, assume either that (i) $c_t := \|W_t\|_\infty < \infty$ and $\bar{\rho}_t < 1 - \delta_t$;or (ii) $c_t' := \max\left(2|||\nabla W_t|||_\infty / b_t, \sup_{x: W_t(x) \neq 0} \left(\frac{L_t W_t}{W_t} - \rho_t\right)_-\right) < \infty$ and that for all $t \geq 0$ there exists $\delta_t' \in [0,1)$ and $F_t' \in [0,\infty)$ such that $\delta_t' \int_0^t e^{(c_s' - \rho_s)s} ds < 1$ and $W_t \leq \frac{\delta_t'}{4 B_t C_t}\left(|\nabla V_t|^2 - 2\Delta V_t\right) + F_t'$. Then, for any $f : \mathbb{R}^n \to \mathbb{R}_+$ smooth enough, it holds

$$\|P_t^{(t)} f\|_{\Phi_t, \mu_t} \leq m(s,t) \|P_s^{(s)} f\|_{\Phi_s, \mu_s}$$

where under assumption (i),

$$m(s,t) = \exp\left\{ \int_s^t F_u + \left(b_u \frac{1 - e^{-\rho_u u}}{\rho_u}\right)^2 \frac{D_u + E_u}{2(1 - \delta_u - \bar{\rho}_u)} + c_u B_u u du \right\},$$

and under assumption (ii),

$$m(s,t) =$$

$$\exp\left\{\int_s^t \left(\int_0^u e^{(c_v'-\rho_v)v}dv\right)^2 \frac{b_u(D_u+E_u)}{2(1-\delta_u-\bar{\rho}_u-\delta_u'\int_0^u e^{(c_v'-\rho_v)v}dv)} + \int_0^u e^{(c_v'-\rho_v)v}dv\,(B_uF_u'+F_u)\,du\right\}.$$

endtheorem

Open Problems

- An interesting example not yet covered : with $\alpha \in (1,2)$,

$$V_t(x) = (1-t)_+^2 |x|^2 + |x|^\alpha$$

we have a critical point $t = 1$ in which hypercontractivity in $\mathbb{L}_p$ spaces is replaced by a weaker property. Such an example requires $b_t = \infty$ and is not covered yet by the theorem.

- Time Dependent Diffusion Equation

For

$$L_t = L + V(t) \cdot \nabla,$$

$$\partial_t P_t^{(t)} f = L_{(t)} f + \int_0^t \left(e^{\tau L_{(t)}} \dot{V}(t) e^{(t-\tau) L_{(t)}} f \right) d\tau.$$

- What are nonhommogeneous stochastic processes with nonconstant smoothing properties ?

Extended Gradient Bounds

For $W\colon \mathbb{R}^n \to \mathbb{R}_+$, $W^2 \in \mathcal{C}^2$, define

$$\Gamma^W(f,g) := \Gamma(f,g) + W^2 fg$$

with

$$\Gamma(f) \equiv \frac{1}{2}\left(Lf^2 - 2fLf\right)$$

and define the **iterated operator**

$$\Gamma_2^W(f,g) := \frac{1}{2}\left(L\Gamma^W(f,g) - \Gamma^W(Lf,g) - \Gamma^W(f,Lg)\right)$$

$$= \Gamma_2(f,g) + \frac{1}{2} fgL(W^2) + W^2\Gamma(f,g) + 2W\nabla W\nabla(fg).$$

Start

∘ C. Roberto, B. Zegarlinski, Bakry-Emery Calculus For Diffusion With Additional Multiplicative Term, (2021) arXiv:2102.10633

Theorem [Extended Gradient Bound]

Assume that for some $\rho \in \mathbb{R}$

$$\Gamma_2 \geq \rho\Gamma$$

and

$$\gamma := \inf_{x\in\mathbb{R}^n : W(x)\neq 0}\left(\frac{LW}{W} - 3\frac{|\nabla W|^2}{W^2}\right) > -\infty.$$

Then, for t ≥ 0,

$$\Gamma^W(P_t f) \leq e^{-2\min(\rho,\gamma)t} P_t(\Gamma^W(f)) \qquad .$$

for all $f \in \mathcal{C}^2$.

EquivThm

Examples.Skip EquivThm

Remark

For $\rho = 0$, the ratio $\frac{1-e^{-2\rho t}}{\rho}$ is understood as its limit (i.e. $2t$). Notice that it is always non-negative.

Observe that, applying (ii) to constant functions $f \equiv C$, $C \neq 0$, leads to

$$W^2 \leq e^{-2\rho t} P_t(W^2).$$

Therefore, if $\int W^2 d\mu < \infty$ and $\rho > 0$,
taking the limit $t \to \infty$ and by ergodicity, we would conclude that $W \equiv 0$.
Therefore, for

$$\Gamma_2^W(f) \geq \rho \Gamma^W(f)$$

to hold for a non trivial W, either $\rho \leq 0$ or $\int W^2 d\mu = \infty$.
But, we have no such restriction removing mean value $\mu(f)$ of the function f.

Example

For $p, q \geq 1$, consider

$$U(x) = c + \frac{(1+|x|^2)^{p/2}}{p} \quad \textit{and} \quad W(x) = \frac{(1+|x|^2)^{q/2}}{q},$$

with c s.t. $\int e^{-U(x)}dx = 1$ and $|x| = (\sum x_i^2)^{1/2}$. Then

$$\frac{LW}{W} - 3\frac{|\nabla W|^2}{W^2}$$
$$= \frac{qn}{1+|x|^2} - \frac{q|x|^2}{1+|x|^2}\left(\frac{2(q+1)}{1+|x|^2} + (1+|x|^2)^{(p-2)/2}\right)$$

is bounded from below iff $1 \leq p \leq 2$.

Extended Gradient Bounds

Complete Characterisation

$$\langle \nabla P_t f, \nabla P_t f\rangle \leq e^{\alpha t} P_t \langle \nabla f, \nabla f\rangle$$

The description of all allowed $\langle \cdot, \cdot \rangle$ comes from
Theory of Markov Semigroups in Noncommutative Spaces

see $\Downarrow$

• F.Cipriani, B.Zegarlinski, KMS Dirichlet forms, coercivity and super-bounded Markovian semigroups, arXiv:2105.06000

•• F.Cipriani, B.Zegarlinski, Noncommutative Perturbation Theory. https://it.overleaf.com/project/6214cf6524f31131f50ea0ad

Thank you for your attention

Start

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