Analysis of Quantum Large Interacting Systems

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Before we begin

" If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is. "

John von Neumann

"Fall in love with some activity, and do it! Nobody ever figures out what life is all about, and it doesn?t matter. Explore the world. Nearly everything is really interesting if you go into it deeply enough. Work as hard and as much as you want to on the things you like to do the best. Don?t think about what you want to be, but what you want to do. Keep up some kind of a minimum with other things so that society doesn?t stop you from doing anything at all. "

Richard P. Feynman

Bibliography

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Basic Objects and Inequalities in Commutative Analysis

- Dirichlet Form

$$\mathcal{E}_{\mu,\mathbb{X}}(f) = \int |\mathbb{X}f|^2 d\mu \equiv \int \sum_j |X_j f|^2 d\mu \equiv \mu|\mathbb{X}f|^2$$

- Poincaré Inequality

$$m\mu(f - \mu f)^2 \leq \mathcal{E}_{\mu,\mathbb{X}}(f)$$

- Log-Sobolev inequality

$$Ent_{\mu,2}(f) \equiv \mu\left(f^2 \log \frac{f^2}{\mu(f^2)}\right) \leq c\mathcal{E}_{\mu,\mathbb{X}}(f)$$

Few Classical Results in Commutative Analysis

Suppose

$$d\tilde{\mu} = \tilde{\rho} d\mu$$

If

$$0 < \varepsilon \leq \frac{d\tilde{\mu}}{d\mu} \leq \varepsilon^{-1} < \infty$$

then

- Comparison of Dirichlet forms

$$\varepsilon \, \mathcal{E}_{\mu,\mathbb{X}}(f) \leq \mathcal{E}_{\tilde{\mu},\mathbb{X}}(f) \leq \varepsilon^{-1} \, \mathcal{E}_{\mu,\mathbb{X}}(f)$$

- Perturbation of Poincaré Inequality

$$\tilde{\mu}(f - \tilde{\mu}f)^2 = \inf_{a\in\mathbb{R}} \left(\tilde{\mu}(f-a)^2\right) \leq \varepsilon^{-1} \inf_{a\in\mathbb{R}} \left(\mu(f-a)^2\right)$$
$$= \varepsilon^{-1}\mu(f-\mu f)^2 \leq \varepsilon^{-1} m^{-1} \mathcal{E}_{\mu,\mathbb{X}}(f) \leq \varepsilon^{-2} m^{-1} \mathcal{E}_{\tilde{\mu},\mathbb{X}}(f)$$

i.e.

$$\tilde{m} \geq m\varepsilon^2$$

- Perturbation of Logarithmic Sobolev Inequality

$$
\begin{aligned}
Ent_{\tilde{\mu},2}(f) \equiv \mu\left(f^2 \log \frac{f^2}{\mu(f^2)}\right) &= \inf_{t>0}\left(\tilde{\mu}(f^2 \log \frac{f^2}{t} - f^2 + t)^2\right) \\
&\leq \varepsilon^{-1} \inf_{t>0}\left(\mu(f^2 \log \frac{f^2}{t} - f^2 + t)^2\right) \\
&= Ent_{\mu,2}(f) \leq \varepsilon^{-1} c\, \mathcal{E}_{\mu,\mathbb{X}}(f) \\
&\leq \varepsilon^{-2} c\, \mathcal{E}_{\tilde{\mu},\mathbb{X}}(f)
\end{aligned}
$$

L.Gross Integration Lemma:
Hypercontractivity $\iff$ Log-Sobolev inequality.
γ_n - the standard Gaussian measure on $\mathbb{R}^n$.
Ornstein-Uhlenbeck semi-group $(P_t)_{t\geq 0}$, whose infinitesimal generator is $L := \Delta - x \cdot \nabla$

$$\|P_t f\|_{q(t)} \leq \|P_s f\|_{q(s)}, \qquad s \leq t$$

where $q(t) = 1 + (q(0) - 1)e^{2t}$, $q(0) \geq 1$, and
$\|g\|_p^p := \int |g|^p d\gamma_n$, $p \geq 1$.
Log-Sobolev inequality:

$$Ent_{\gamma_n}(f^2) := \int f^2 \log f^2 d\gamma_n - \int f^2 d\gamma_n \log \int f^2 d\gamma_n \leq 2 \int |\nabla f|^2 d\gamma_n.$$

By Bounded perturbation Lemma for $d\tilde{\mu} = e^{-U} d\gamma_n$ with U bounded one gets

$$Ent_{\tilde{\mu}}(f^2) \leq 2e^{osc(U)} \mathcal{E}_{\tilde{\mu},\nabla}(f)$$

Discrete Systems

Space $\Omega = M^{\mathbb{Z}^d}$, M - finite or compact set.
Local Specification $\{E^{\cdot}_{\Lambda}\}_{\Lambda \subset\subset \mathbb{Z}^d} \equiv$ Family of compatible probability kernels.

$$\Lambda_1 \subset \Lambda_2 \Rightarrow E^{\omega}_{\Lambda_2} E^{\cdot}_{\Lambda_1} f = E^{\omega}_{\Lambda_2} f$$

Let μ be a Gibbs measure i.e.

$$\mu(E^{\omega}_{\Lambda} f) = \mu f$$

Dirichlet forms and Jump type Markov Generators

$$\mathcal{E}_X(f) = \sum_{j \in \mathbb{Z}^d} \mu \left(E_{X+j} f - f \right)^2$$

$$Lf = \sum_{j \in \mathbb{Z}^d} \left(E_{X+j} f - f \right)$$

Equivalence of Dirichlet forms : $\forall X, Y \subset\subset \mathbb{Z}^d \ \exists \kappa \in (0, \infty)$

$$\frac{1}{\kappa}\mathcal{E}_X(f) \leq \mathcal{E}_Y(f) \leq \kappa\mathcal{E}_X(f)$$

Thm: If Complete Analyticity Condition holds, then Poincare and Log-Sobolev inequality holds for E_Λ^ω uniformly in ω and Λ.

Coercive Inequalities and Perturbation Theory in Noncommutative Spaces

Some Results and Challenges

A State

$$\omega(f) \equiv Tr\,(\rho f)$$

with $\rho > 0$, $Tr\rho = 1$.

Modular Automorphism

$$\alpha_t(f) \equiv \Delta^{it}(f) \equiv e^{-itH} f e^{itH} \equiv \rho^{it} f \rho^{-it}$$

with

$$\Delta^s(f) \equiv \rho^s f \rho^{-s}.$$

Scalar Product

For $\kappa \in [0,1]$ define

$$\langle f, g\rangle_{\omega,\kappa} \equiv Tr\left(\left(\rho^{\frac{\kappa}{2}} f \rho^{\frac{1-\kappa}{2}}\right)^* \rho^{\frac{\kappa}{2}} g \rho^{\frac{1-\kappa}{2}}\right) = \omega\left(\left(\Delta^{\frac{\kappa}{2}}(f)\right)^* \Delta^{\frac{\kappa}{2}}(g)\right)$$

Later on $\kappa = \frac{1}{2}$ and $\langle f, g\rangle \equiv \langle f, g\rangle_{\omega,\frac{1}{2}}$.

Noncommutative Dirichlet Form

$$\mathcal{E}_{X,\rho}(f) \equiv \int_{-\infty}^{\infty} \Big(\langle \delta_{X_s}(f), \delta_{X_s}(f) \rangle + \langle \delta_{X_s{}^*}(f), \delta_{X_s{}^*}(f) \rangle \Big) ds$$

where

$$\delta_Z(f) \equiv i[Z, f].$$

and

$$X_s \equiv \alpha_s(X)\xi_s$$

with an operator X
and a function ξ_s s.t. ([YMP2000])

$$\begin{aligned} &(a) \quad 0 < \xi_s^2, \\ &(b) \quad 0 < \xi_{s-i/4}^2 + \xi_{s+i/4}^2 \\ &(c) \quad \exists \beta \in (0, \infty) \qquad \sup_s (e^{\beta s} \xi_s^2) < \infty \end{aligned}$$

(a) $\Rightarrow$ positivity of $\mathcal{E}_{X,\rho}$
(b) $\Rightarrow$ dissipativity of the corresponding generator $\mathcal{L}$

$$\mathcal{L}(b^* b) - b^* \mathcal{L}(b) - \mathcal{L}(b^*) b = \int \left| \delta_{\sigma_{t-\frac{i}{4}}(X)}(b) \right|^2 \left(\xi^2_{s-i/4} + \xi^2_{s+i/4} \right) dt$$

(This is for $X = X^*$)
(c) is a technical assumption so Dirichlet form and the Markov generator is well defined.
And (c) is also used in perturbation theory.

For $\tilde{\rho} \neq \rho$, let $\tilde{\alpha}_s(f) \equiv \tilde{\rho}^{it} f \tilde{\rho}^{-it}$ and

$$\tilde{X}_s \equiv \tilde{\alpha}_s(X)\xi_s = \tilde{\alpha}_s\left(\alpha_{-s}(X_s)\right)$$

where

$$X_s \equiv \alpha_s(X)\xi_s$$

note that here α_s is associated to ρ. With this notation we have the following relation

$$\tilde{X}_s = X_s + \tilde{\rho}^{is}\rho^{-is}[X_s, \rho^{is}\tilde{\rho}^{-is}] \equiv X_s + B_s. \qquad (1)$$

Perturbation of Dirichlet Forms.
Given two density matrices ρ and $\tilde{\rho}$ we define relative modular operator by

$$\Delta_{\rho,\tilde{\rho}}(f) \equiv \rho f \tilde{\rho}^{-1}$$

Theorem

Suppose the following Poincaré Inequality holds

$$\|f-\omega_\rho(f)\|^2_{\rho,\frac{1}{2}} \leq c_0\,\mathcal{E}_{X,\rho}(f)$$

Let

$$\tilde{X}_s \equiv X_s + B_s.$$

(i) If

$$a_1 \equiv 2\| \left|\Delta^{\frac{1}{4}}_{\tilde{\rho},\rho}(\mathbb{I})\right|^2 \|^2_{\rho,\infty} < \infty$$

and

$$b_1 \equiv a_1 \int ds \left(\| \left|\Delta^{\frac{1}{4}}_{\rho}(B_s)\right|^2 \|_{\rho,\infty} + \| \left|\Delta^{\frac{1}{4}}_{\rho}(B_s^*)\right|^2 \|_{\rho,\infty} \right) < \infty,$$

then

$$\mathcal{E}_{\tilde{X},\tilde{\rho}}(f) \leq C \cdot \mathcal{E}_{X,\rho}(f)$$

with a constant

$$C \equiv a_1 + b_1 c_0.$$

(ii) *If*

$$a_2 \equiv \| \left| \Delta_{\tilde{\rho},\rho}^{\frac{1}{4}}(\mathbb{I}) \right|^2 \|_{\rho,\infty}^2 < \infty$$

and

$$b_2 c_0 \equiv 4c_0 \int ds \left(\| \left| \Delta_{\rho}^{\frac{1}{4}}(B_s) \right|^2 \|_{\rho,\infty} + \| \left| \Delta_{\rho}^{\frac{1}{4}}(B_s^*) \right|^2 \|_{\rho,\infty} \right) < 1,$$

then

$$\mathcal{E}_{X,\rho}(f) \leq \tilde{C}\, \mathcal{E}_{\tilde{X},\tilde{\rho}}(f)$$

with a constant

$$\tilde{C} \equiv \frac{2\| \left| \Delta_{\rho,\tilde{\rho}}^{\frac{1}{4}}(\mathbb{I}) \right|^2 \|_{\tilde{\rho},\infty}^2}{1 - b_2 c_0}.$$

Proof.

(i) We start with noticing that using 1, we have

$$\|\delta_{\tilde{X}_s}(f)\|^2_{\tilde{\rho},\frac{1}{2}} = \|\delta_{X_s+B_s}(f)\|^2_{\tilde{\rho},\frac{1}{2}} \leq 2\|\delta_{X_s}(f)\|^2_{\tilde{\rho},\frac{1}{2}} + 2\|\delta_{B_s}(f)\|^2_{\tilde{\rho},\frac{1}{2}}$$

For the norm square in the first term on the right hand side, we have

$$\begin{aligned}\|\delta_{X_s}(f)\|^2_{\tilde{\rho},\frac{1}{2}} &= \langle(\delta_{X_s}(f))^*, (\delta_{X_s}(f))\rangle_{\tilde{\rho},\frac{1}{2}} \\ &\leq \|\left|\Delta^{\frac{1}{4}}_{\tilde{\rho},\rho}(\mathbb{I})\right|^2\|^2_{\rho,\infty} \cdot \|\delta_{X_s}(f)\|^2_{\rho,\frac{1}{2}}\end{aligned}$$

For the norm square in the second term on the right hand side, we have

$$\begin{aligned}\|\delta_{B_s}(f)\|^2_{\tilde{\rho},\frac{1}{2}} &= \langle(\delta_{B_s}(f))^*, (\delta_{B_s}(f))\rangle_{\tilde{\rho},\frac{1}{2}} \\ &\leq 2\|B_s(f-\omega_\rho(f))\|^2_{\tilde{\rho},\frac{1}{2}} + 2\|(f-\omega_\rho(f))B_s\|^2_{\tilde{\rho},\frac{1}{2}}\end{aligned}$$

Since

$$\|B_s g\|^2_{\tilde\rho,\frac{1}{2}} \le \| \left|\Delta^{\frac{1}{4}}_{\tilde\rho,\rho}(\mathbb{I})\right|^2 \|^2_{\rho,\infty} \cdot \|B_s g\|^2_{\rho,\frac{1}{2}}$$
$$\le \| \left|\Delta^{\frac{1}{4}}_{\tilde\rho,\rho}(\mathbb{I})\right|^2 \|^2_{\rho,\infty} \cdot \| \left|\Delta^{\frac{1}{4}}_{\rho}(B_s)\right|^2 \|_{\rho,\infty} \cdot \|g\|^2_{\rho,\frac{1}{2}}$$

and

$$\|g B_s\|^2_{\tilde\rho,\frac{1}{2}} \le \| \left|\Delta^{\frac{1}{4}}_{\tilde\rho,\rho}(\mathbb{I})\right|^2 \|^2_{\rho,\infty} \cdot \|g B_s\|^2_{\rho,\frac{1}{2}}$$
$$\le \| \left|\Delta^{\frac{1}{4}}_{\tilde\rho,\rho}(\mathbb{I})\right|^2 \|^2_{\rho,\infty} \cdot \| \left|\Delta^{\frac{1}{4}}_{\rho}(B_s^*)\right|^2 \|_{\rho,\infty} \cdot \|g\|^2_{\rho,\frac{1}{2}},$$

we obtain

$$\|\delta_{B_s}(f)\|^2_{\tilde\rho,\frac{1}{2}} \le \frac{b_1}{2} \cdot \|(f - \omega_\rho(f))\|^2_{\rho,\frac{1}{2}}$$

with

$$b_1 \equiv 2\| \left|\Delta^{\frac{1}{4}}_{\tilde\rho,\rho}(\mathbb{I})\right|^2 \|^2_{\rho,\infty} \cdot \int ds \left(\| \left|\Delta^{\frac{1}{4}}_{\rho}(B_s)\right|^2 \|_{\rho,\infty} + \| \left|\Delta^{\frac{1}{4}}_{\rho}(B_s^*)\right|^2 \|_{\rho,\infty} \right).$$

By a similar consideration we get an analogous bound for the adjoint term in the Dirichlet form. Adding them together and integrating with respect to s, we get

$$\mathcal{E}_{\tilde{X},\tilde{\rho}}(f) \leq a_1 \cdot \mathcal{E}_{X,\rho}(f) + b_1 \cdot \|(f - \omega_\rho(f))\|^2_{\rho,\frac{1}{2}}$$

with

$$a_1 \equiv 2\| \left| \Delta^{\frac{1}{4}}_{\tilde{\rho},\rho}(\mathbb{I}) \right|^2 \|^2_{\rho,\infty}$$

Hence, if the following Poincaré inequality holds

$$\|(f - \omega_\rho(f))\|^2_{\rho,\frac{1}{2}} \leq c_0 \cdot \mathcal{E}_{X,\rho}(f)$$

with some constant $c_0 \in (0, \infty)$ independent of f, we get

$$\mathcal{E}_{\tilde{X},\tilde{\rho}}(f) \leq C \cdot \mathcal{E}_{X,\rho}(f)$$

with

$$C \equiv a_1 + b_1 c_0$$

$\square$ $QED(i)$

(ii) The proof of this part of theorem is similar to the previous one. Using now $X_s = \tilde{X}_s - B_s$, we have

$$\|\delta_{X_s}(f)\|^2_{\rho,\frac{1}{2}} \le 2\|\delta_{\tilde{X}_s}(f)\|^2_{\rho,\frac{1}{2}} + 2\|\delta_{B_s}(f)\|^2_{\rho,\frac{1}{2}}$$

For the first term on the right hand side we have

$$2\|\delta_{\tilde{X}_s}(f)\|^2_{\rho,\frac{1}{2}} \le 2\| \left|\Delta^{\frac{1}{4}}_{\rho,\tilde{\rho}}(\mathbb{I})\right|^2 \|^2_{\tilde{\rho},\infty} \cdot \|\delta_{\tilde{X}_s}(f)\|^2_{\tilde{\rho},\frac{1}{2}}$$

On the other hand

$$\|\delta_{B_s}(f)\|^2_{\rho,\frac{1}{2}} \le 2\|B_s(f-\omega_\rho(f))\|^2_{\rho,\frac{1}{2}} + 2\|(f-\omega_\rho(f))B_s\|^2_{\rho,\frac{1}{2}}$$
$$\le 2\left(\| \left|\Delta^{\frac{1}{4}}_{\rho}(B_s)\right|^2 \|_{\rho,\infty} + \| \left|\Delta^{\frac{1}{4}}_{\rho}(B_s^*)\right|^2 \|_{\rho,\infty}\right) \cdot \|(f-\omega_\rho(f))\|^2_{\rho,\frac{1}{2}}$$

Combining the above with similar relations for terms involving adjoint operators and integrating over s, we arrive at

$$\mathcal{E}_{X,\rho}(f) \leq a_2 \mathcal{E}_{\tilde{X},\tilde{\rho}}(f) + b_2 \|(f - \omega_\rho(f))\|^2_{\rho,\frac{1}{2}}$$

with

$$a_2 \equiv 2 \| \left| \Delta^{\frac{1}{4}}_{\rho,\tilde{\rho}}(\mathbb{I}) \right|^2 \|_{\tilde{\rho},\infty}$$

$$b_2 \equiv 4 \int ds \left(\| \left| \Delta^{\frac{1}{4}}_{\rho}(B_s) \right|^2 \|_{\rho,\infty} + \| \left| \Delta^{\frac{1}{4}}_{\rho}(B_s^*) \right|^2 \|_{\rho,\infty} \right)$$

Hence, if the Poincare Inequality associated to the state ω_ρ holds, we obtain

$$\mathcal{E}_{X,\rho}(f) \leq a_2 \mathcal{E}_{\tilde{X},\tilde{\rho}}(f) + b_2 c_0 \mathcal{E}_{X,\rho}(f).$$

Thus if

$$b_2 c_0 < 1,$$

we arrive at

$$\mathcal{E}_{X,\rho}(f) \leq \frac{a_2}{1 - b_2 c_0} \mathcal{E}_{\tilde{X},\tilde{\rho}}(f).$$

Example.1G

Let $\rho \equiv e^{-U}$ be a density matrix with respect to a trace. Let

$$\Delta^s(f) \equiv \Delta^s_\rho(f) \equiv \rho^s f \rho^{-s}.$$

For a bounded operator a and a (YMP) function ξ_s, consider

$$X_s \equiv \Delta^{is}(\mathrm{a})\xi_s,$$

Let $\tilde{\rho} \equiv e^{-\tilde{U}}$ be another density matrix s.t.

$$2\| \left|\Delta^{\frac{1}{4}}_{\tilde{\rho},\rho}(\mathbb{I})\right|^2 \|^2_{\rho,\infty} < \infty$$

Define

$$\tilde{X}_s \equiv \tilde{\Delta}^{is}(\mathrm{a})\xi_s,$$

with $\tilde{\Delta}^{is} \equiv \Delta^{is}_{\tilde{\rho}}$. Then

$$\tilde{X}_s = \tilde{\Delta}^{is}\Delta^{-is}(X_s) \equiv X_s + B_s$$

with

$$B_s \equiv \tilde{\rho}^{is}\rho^{-is}[X_s, \rho^{is}\tilde{\rho}^{-is}].$$

Estimates of norms $\| \left|\Delta_\rho^{\frac{1}{4}}(B_s)\right|^2 \|_{\rho,\infty}, \| \left|\Delta_\rho^{\frac{1}{4}}(B_s^*)\right|^2 \|_{\rho,\infty}$.

Note that

$$\zeta_s \equiv \rho^{\frac{1}{4}} \tilde{\rho}^{is} \rho^{-is} \rho^{-\frac{1}{4}},$$

with $V \equiv \tilde{U} - U$, satisfies

$$\frac{d}{ds}\zeta_s = i\zeta_s \cdot \Delta^{is+\frac{1}{4}}(V)$$

If

$$\|\Delta^{\pm\frac{1}{4}}(V)\| < \varepsilon < \infty,$$

then the solution is given by

$$\zeta_s = \mathbb{I} + \sum_{n\in\mathbb{N}} (-1)^n \int_0^s ds_1 .. \int_0^{s_{n-1}} ds_n \, \Delta^{\frac{1}{4}-is_1}(V) ... \Delta^{\frac{1}{4}-is_n}(V)$$

and satisfies

$$\|\zeta_s\| \leq e^{s\|\Delta^{\frac{1}{4}}(V)\|}$$

$$\|\zeta_s^*\| \leq e^{s\|\Delta^{-\frac{1}{4}}(V)\|}$$

Thus if

$$\|\Delta^{\pm\frac{1}{4}}(\mathrm{a})\| < \infty$$

and ξ_s goes to zero at infinity exponentially sufficiently fast (or $\varepsilon \in (0, \infty)$ is sufficiently small), then

$$\int ds \left(\| \left| \Delta_\rho^{\frac{1}{4}}(B_s) \right|^2 \|_{\rho,\infty} + \| \left| \Delta_\rho^{\frac{1}{4}}(B_s^*) \right|^2 \|_{\rho,\infty} \right) < \infty.$$

Then conditions of our Dirichlet forms perturbation theory are satisfied.

Perturbation of Log-Sobolev Inequality

Define

$$Ent_\rho(f) \equiv Tr|\rho^{\frac{1}{4}} f \rho^{\frac{1}{4}}|^2 \left(\log\left(\frac{|\rho^{\frac{1}{4}} f \rho^{\frac{1}{4}}|^2}{Tr|\rho^{\frac{1}{4}} f \rho^{\frac{1}{4}}|^2} \right) - \log\rho \right)$$

and

$$\mathcal{E}_{\rho,X}(f) \equiv \int \left(\|\delta_{X_s}(f)\|^2_{2,\rho} + \|\delta_{X_s{}^*}(f)\|^2_{2,\rho} \right) ds$$

with

$$X_s \equiv \alpha_s(X)\xi(s).$$

where $\alpha_s(\cdot)$ is the modular unitary group and $\xi(s)$ is a (YMP) function.

Assume Log-Sobolev inequality :

$$Ent_\rho(f) \leq c_0\, \mathcal{E}_{\rho,X}(f) \qquad (LS_{\rho,X})$$

Theorem: (Log-Sobolev Perturbation)
Suppose the conditions of the DFPL are satisfied and

$$\|\log \tilde{\rho} - \log \rho\| < \infty$$

Then $\exists \tilde{C} \in (0, \infty)$

$$Ent_{\tilde{\rho}}(g) \leq \tilde{C}\mathcal{E}_{\tilde{\rho},\tilde{X}}(g).$$

If $(LS_{\rho,X})$ holds, then in particular for

$$f \equiv \rho^{-\frac{1}{4}} \tilde{\rho}^{\frac{1}{4}} g \tilde{\rho}^{\frac{1}{4}} \rho^{-\frac{1}{4}} \equiv \gamma^* g \gamma$$

we have

$$Ent_{\tilde{\rho}}(g) \leq c_0 \mathcal{E}_{\rho,X}(\gamma^* g \gamma) - Tr\left(|\tilde{\rho}^{\frac{1}{4}} g \tilde{\rho}^{\frac{1}{4}}|^2 (\log \tilde{\rho} - \log \rho) \right) \quad (2)$$

Proposition D :
Suppose the conditions of the DFPL are satisfied.
Then $\exists C_2, D_2 \in (0, \infty)$

$$\mathcal{E}_{\tilde{\rho},\tilde{X}}(\gamma^* g \gamma) \leq C_2\, \mathcal{E}_{\tilde{\rho},\tilde{X}}(g) + D_2\, \|g\|^2_{2,\tilde{\rho}} \qquad (3)$$

Proof of Proposition D:
If the conditions of the DFPL are satisfied, then there is a constant $C_1 \in (0, \infty)$ such that

$$\mathcal{E}_{\rho,X}(f) \leq C_1 \mathcal{E}_{\tilde{\rho},\tilde{X}}(f) \tag{4}$$

and in particular, we have

$$\mathcal{E}_{\rho,X}(\gamma^* g \gamma) \leq C_1 \mathcal{E}_{\tilde{\rho},\tilde{X}}(\gamma^* g \gamma)$$

Using Leibnitz rule for the derivation and convexity inequality for the norm, we have

$$\begin{aligned} \|\delta_{\tilde{X}_s}(\gamma^* g \gamma)\|_{2,\tilde{\rho}}^2 \leq \frac{1}{3}\|\delta_{\tilde{X}_s}(\gamma^*) g \gamma\|_{2,\tilde{\rho}}^2 + \frac{1}{3}\|\gamma^* \delta_{\tilde{X}_s}(g) \gamma\|_{2,\tilde{\rho}}^2 \\ + \frac{1}{3}\|\gamma^* g \delta_{\tilde{X}_s}(\gamma)\|_{2,\tilde{\rho}}^2. \end{aligned}$$

Next we note the following

Lemma

$$\|agb\|_{2,\tilde\rho}^2 \le \| \left|\tilde\Delta^{\frac14}(a)\right|^2 \| \cdot \| \left|\tilde\Delta^{\frac14}(b^*)\right|^2 \| \cdot \|g\|_{2,\tilde\rho}^2.$$

Proof of Lemma: One has

$$\|agb\|_{2,\tilde\rho}^2 = Tr\left(\tilde\rho^{\frac12} b^* g^* a^* \tilde\rho^{\frac12} agb\right) = Tr\left(\tilde\rho^{\frac12} b^* g^* \tilde\rho^{\frac14} \left|\tilde\rho^{\frac14} a \tilde\rho^{-\frac14}\right|^2 \tilde\rho^{\frac14} gb\right)$$

$$= Tr\left(\tilde\rho^{\frac14} g^* \tilde\rho^{\frac14} \left|\tilde\rho^{\frac14} a \tilde\rho^{-\frac14}\right|^2 \tilde\rho^{\frac14} g \tilde\rho^{\frac14} \left|\tilde\rho^{\frac14} b^* \tilde\rho^{-\frac14}\right|^2\right)$$

$$\le \left(\tilde\rho^{\frac14} g^* \tilde\rho^{\frac14} \left|\tilde\rho^{\frac14} a \tilde\rho^{-\frac14}\right|^4 \tilde\rho^{\frac14} g \tilde\rho^{\frac14}\right)^{\frac12} \cdot$$

$$\left(\tilde\rho^{\frac14} g \tilde\rho^{\frac14} \left|\tilde\rho^{\frac14} b^* \tilde\rho^{-\frac14}\right|^4 \tilde\rho^{\frac14} g^* \tilde\rho^{\frac14}\right)^{\frac12}.$$

from which the lemma follows.

Applying the lemma, we get

$$\begin{aligned}\|\delta_{\tilde{X}_s}(\gamma^* g\gamma)\|^2_{2,\tilde{\rho}} \leq & \frac{1}{3}\|\delta_{\tilde{X}_s}(\gamma^*)g\gamma\|^2_{2,\tilde{\rho}} + \frac{1}{3}\|\gamma^*\delta_{\tilde{X}_s}(g)\gamma\|^2_{2,\tilde{\rho}} + \frac{1}{3}\|\gamma^* g\delta_{\tilde{X}_s}(\\ \leq & \frac{1}{3}\|\tilde{\Delta}^{\frac{1}{4}}(\gamma^*)\|^4 \cdot \|\delta_{\tilde{X}_s}(g)\|^2_{2,\tilde{\rho}} \\ + & \frac{1}{3}\left(\|\tilde{\Delta}^{\frac{1}{4}}(\delta_{\tilde{X}_s}(\gamma^*))\|^2 + \|\tilde{\Delta}^{\frac{1}{4}}(\delta_{\tilde{X}_s{}^*}(\gamma^*))\|^2\right) \cdot \|\tilde{\Delta}^{\frac{1}{4}}(\gamma^*)\end{aligned}$$

This together with similar result with $\tilde{X}_s{}^*$ replacing $\tilde{X}_s$, imply the following bound

$$\mathcal{E}_{\tilde{\rho},\tilde{X}}(\gamma^* g\gamma) \leq C_2\, \mathcal{E}_{\tilde{\rho},\tilde{X}}(g) + D_2\, \|g\|^2_{2,\tilde{\rho}} \qquad (5)$$

with

$$C_2 \equiv \frac{1}{3}\|\tilde{\Delta}^{\frac{1}{4}}(\gamma^*)\|^4$$

$$D_2 \equiv \frac{2}{3}\int ds \left(\|\tilde{\Delta}^{\frac{1}{4}}(\delta_{\tilde{X}_s}(\gamma^*))\|^2 + \|\tilde{\Delta}^{\frac{1}{4}}(\delta_{\tilde{X}_s{}^*}(\gamma^*))\|^2\right) \cdot \|\tilde{\Delta}^{\frac{1}{4}}(\gamma^*)\|^2$$

Using (2) together with (4) and (5), we obtain

$$Ent_{\tilde{\rho}}(g) \leq C\,\mathcal{E}_{\tilde{\rho},\tilde{X}}(g) + D\,\|g\|_{\tilde{\rho}}^2$$

with $C \equiv c_0 C_1 C_2$ and $D \equiv c_0 C_1 D_2 + \|\log\tilde{\rho} - \log\rho\|$.

Now suppose the following Poincaré inequality holds

$$\tilde{m} \cdot \|g - \omega_{\tilde{\rho}}(g)\|_{\tilde{\rho},2} \leq \mathcal{E}_{\tilde{\rho},\tilde{X}}(g) \qquad (PI_{\tilde{\rho},\tilde{X}})$$

with some constant $\tilde{m} \in (0, \infty)$ independent of g.
Then using noncommutative Rothaus Lemma
(p.276 in R.Olkiewicz, B. Zegarlinski, JFA 161 (1999) 246-285)

$$Ent_{\tilde{\rho}}(g) \leq Ent_{\tilde{\rho}}(g - \omega_{\tilde{\rho}}(g)) + 2\|g - \omega_{\tilde{\rho}}(g)\|_{\tilde{\rho},2}$$

we arrive at

$$Ent_{\tilde{\rho}}(g) \leq \left(C + \frac{D+2}{\tilde{m}} \right) \mathcal{E}_{\tilde{\rho},\tilde{X}}(g).$$

□

REM : Note that under $(LS_{\rho,X})$ the corresponding Poincaré inequality $(PI_{\rho,X})$ holds. Then under the assumptions of the DFPL the Poincaré inequality $(PI_{\tilde{\rho},\tilde{X}})$ holds.

Log-Sobolev on CCR Algebra

R. Carbone, E. Sasso, Hypercontractivity for a quantum Ornstein-Uhlenbeck semigroup. PTRF 140 (2008), no. 3-4, 505-522

CCR-algebra $[A, A^*] = \mathbb{I}$. Particle Number Operator $N = A^*A$

State $\omega(f) \equiv \frac{1}{Z} Tr(e^{-\beta N} f)$

Dirichlet form - Quatum OU semigroup

$$\mathcal{E} = \kappa \langle \delta_A(f), \delta_A(f) \rangle_{\omega, \frac{1}{2}} + \nu \langle \delta_{A^*}(f), \delta_{A^*}(f) \rangle_{\omega, \frac{1}{2}}$$

Thm[R. Carbone, E. Sasso] Log-Sobolev holds.

Our Perturbation Theory provides infinitely many new examples.

More sophisticated examples in :

- *Fabio E.G. Cipriani, Boguslaw Zegarlinski, Perturbation of Dirichlet forms and coercive inequalities, in preparation.*

Poincaré Inequality in Noncommutative Setup

Assume ω is constructed with classical potential of finite range R

$$\Phi \equiv \{(\Phi_Y)_{Y \subset\subset \mathcal{R}} : \forall Y, X\ [\Phi_Y, \Phi_X] = 0;\ \&\ \Phi_Y = 0 \ if\ diam(Y) \geq R\}$$

Let

$$E_X(f) \equiv Tr_X\left(\gamma_X^* f \gamma_X\right)$$

with

$$\gamma_X \equiv \gamma_X(\Phi) \equiv e^{-\frac{1}{2}U_X}\left(Tr_X(e^{-U_X})\right)^{-\frac{1}{2}}$$

where

$$U_X \equiv \sum_{Y \ni X} \Phi_Y$$

and we use a partial trace

$$Tr_X f \equiv \int_{\mathcal{U}_X} d\nu_X(v) v^* f v$$

were we have integration with the Haar measure $d\nu_X$ on the unitary group $\mathcal{U}_X$.

For $\omega(f) \equiv Tr\,(\rho f)$, define

$$\langle f, g\rangle_\omega \equiv Tr\left(\rho^{\frac{1}{2}} f^* \rho^{\frac{1}{2}} g\right)$$

$$\|f\|_\omega^2 \equiv \langle f, f\rangle_\omega$$

Basic Properties of $\mathbb{L}_2$-Generalised Conditional Expectation

- $$E_X(\mathbb{I}) = \mathbb{I}$$
- $$E_X(f^* f) \geq E_X(f)^* E_X(f)$$
- $$\|E_X(f)\|^2_{\omega,2} \leq \|f\|^2_{\omega,2}$$
- $$\|E_X(f)\|^2_{\omega\circ Tr_X,2} \leq \|f\|^2_{\omega,2}$$
- $$\langle f, E_X(g)\rangle_\omega = \langle E_X(f), g\rangle_\omega$$
- $$\langle f, E_X(g)\rangle_\omega = \langle E_X(f), E_X(g)\rangle_{\omega\circ Tr_X}$$

∘

Markovian Forms

With E a completely positive unit preserving map, let

$$L = E - I.$$

Then

$$L(f^* f) - f^* L(f) - L(f^*)f = -E(f^*)f - f^* E(f) + E(f^* f) + f^* f$$

$$\geq (E(f^*) - f^*)(E(f) - f) \geq 0$$

Dirichlet Forms and Their Properties

Definition:

$$\mathcal{E}_X(f) \equiv \mathcal{E}_{X,\omega}(f) \equiv -\langle L_X(f), f\rangle_\omega \equiv \langle f - E_X(f), f\rangle_\omega$$

∘

Proposition

:

$$\mathcal{E}_X(f) = \mathcal{E}_X(f - Tr_{\tilde{X}} f) \leq C \sum_{j \in \tilde{X}} \|f - Tr_j f\|_\omega^2$$

$$\leq 4C(1+C)\,\frac{|\tilde{X}|}{|Y|} \sum_{i:Y+i \ni j} \mathcal{E}_{Y+i}(f)$$

and so

$$\mathcal{E}_X(f) \leq CA \sum_{j \in \tilde{X}} \|\delta_j(f)\|_\omega^2 \leq 4A^2 C(1+C)\,\frac{|\tilde{X}|}{|Y|} \sum_{i:Y+i \ni j} \mathcal{E}_{Y+i}(f)$$

○

Equivalence of Dirichlet Forms for Infinite Systems

For a finite set $X \subset\subset \mathcal{R}$, let

$$\boldsymbol{\mathcal{E}}_{X,\omega}(f) = \sum_j \mathcal{E}_{X+j,\omega}(f)$$

Theorem

Then $\forall\ Y \subset\subset \mathcal{R} \qquad \exists\ D \in (0, \infty)$

$$D^{-1}\, \boldsymbol{\mathcal{E}}_{Y,\omega}(f) \leq \boldsymbol{\mathcal{E}}_{X,\omega}(f) \leq D\, \boldsymbol{\mathcal{E}}_{Y,\omega}(f)$$

$\circ$

Telescopic Expansion of the $\mathbb{L}_2$-Variance

Let $(j_k)_{k\in\mathbb{N}}$ be a lexicographic order in the lattice $\mathcal{R}$. Set $f_0 \equiv f$ and

$$f_k \equiv E_{j_k} f_{k-1} = E_{X_{j_k}} f_{k-1},\ k \in \mathbb{N}$$

Then we have the following Telescopic Expansion

$$\langle f, f\rangle_\omega - \langle f, \mathbb{I}\rangle_\omega^2 = \sum_{k=1}^{n} \Big(\langle f_{k-1}, f_{k-1}\rangle_{\omega_{k-1}} - \langle f_k, f_k\rangle_{\omega_k} \Big) + \langle f_n, f_n\rangle_{\omega_n} - \langle f, \mathbb{I}\rangle_\omega^2$$

where

$$\omega_k(f) \equiv \omega_{k-1}(Tr_{X_{j_k}} f).$$

We have

$$\begin{aligned}\|f_{k-1}\|^2_{\omega_{k-1}} - \|f_k\|^2_{\omega_k} &= \mathcal{E}_{\omega_{k-1}}(f_{k-1}) \\ &= \mathcal{E}_{\omega_{k-1}}(f_{k-1} - Tr_{\tilde{X}_{j_k}} f_{k-1}) \\ &\leq C\,|\tilde{X}_{j_k}| \sum_{i\in\tilde{X}_{j_k}} \|f_{k-1} - Tr_i f_{k-1}\|^2_{\omega_{k-1}}\end{aligned}$$

Lem.E1:

For $i \in X_k$

$$\|\boldsymbol{\delta}_i E_k f\|^2_{\omega_k} = 0$$

For $i \in \mathcal{R} \setminus \tilde{X}_k$

$$\|\boldsymbol{\delta}_i E_k f\|^2_{\omega_k} \leq \|\boldsymbol{\delta}_i(f)\|^2_{\omega_{k-1}} \qquad fori \in \mathcal{R} \setminus \tilde{X}.$$

For $i \in \mathcal{R} \setminus X_k$, we have

$$\|\boldsymbol{\delta}_i E_k f\|^2_{\omega_k} \leq 3\|\left(\boldsymbol{\delta}_i f\right)\|^2_{\omega_{k-1}} + \sum_{i \in \tilde{X}_k} a_{ik} \|\boldsymbol{\delta}_i(f)\|^2_{\omega_{k-1}}$$

with

$$a_{ik} \equiv 3C\,A\,|\tilde{X}_k| \left(\|\rho^{\frac{1}{4}} \left(\boldsymbol{\delta}_i(\gamma^*_{X_k})(\gamma^*_{X_k})^{-1}\right) \rho^{-\frac{1}{4}}\|^2 + \|\rho^{-\frac{1}{4}} \left(\boldsymbol{\delta}_i(\gamma_{X_k})\gamma^{-1}_{X_k}\right)\right.$$

if $0 < dist(i, X_k) \leq R$, and zero otherwise

Smallness of Matrix $\{\mathbf{a}\}$

Thm.F1:

$$\|\delta_{I,j}(\gamma_X)\| \leq 2\|D_{I,j}\| \cdot \|U_X\| \cdot e^{\|U_X\|}$$

$\circ$

Conclusion: Poincare Inequality for Infinite System with Finite Range Classical Interaction

Theorem: Let ω_Φ be the Gibbs state for a classical potential Φ. Let

$$L_X \equiv \sum_{j \in \mathcal{R}} E_{X+j} - \mathbb{I}$$

be a symmetric jump type generator.
If the potential Φ is sufficiently small then the Poincare Inequality holds

$$\|f - \omega_\Phi(f)\|_{2,\omega} \leq c\, \mathcal{E}_X(f)$$

with a constant $c \in (0, \infty)$ independent of the operator f.

Reference: *B. Zegarlinski, Poincaré Inequality For Infinite Quantum System with Classical Potential, 2018 manuscript in progress.*