$S(m, g)$ calculus and applications to the analysis of PDEs

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In this talk we present some applications to the analysis of PDEs by introducing suitable metrics on the phase-space.

PROBLEM 1: WELL-POSEDNESS FOR A DEGENERATE SCHRÖDINGER EQUATION

We consider a class of degenerate Schrödinger equations corresponding to a Hamiltonian: $\mathcal{H}_V = a_2(x, D) + V(x)$, where $a_2(x, D)$ is a second order degenerate elliptic operator on $\mathbb{R}^n$ and the potential V is a real-valued measurable function of quadratic order at ∞.

$$\begin{cases} i\partial_t u &= \mathcal{H}_V u, \\ u(0) &= f. \end{cases} \tag{1}$$

But in which setting??

PROBLEM 2: ANHARMONIC OSCILLATORS

We consider a class of anharmonic oscillators on $\mathbb{R}^n$ of the form

$$\mathcal{H}_V = (-\Delta)^\ell + |x|^{2k}$$

for k, ℓ integers ≥ 1.

The $\ell = 1$ context

In the study of the Schrödinger equation

$$i\partial_t \psi = -\Delta\psi + V(x)\psi,$$

the analysis of energy levels E_j is reduced to the corresponding eigenvalue problem for the operator $-\Delta + V(x)$.

Herein we will be interested in the rate of growth of eigenvalues for our anharmonic oscillators.

More generally, our prototype includes:

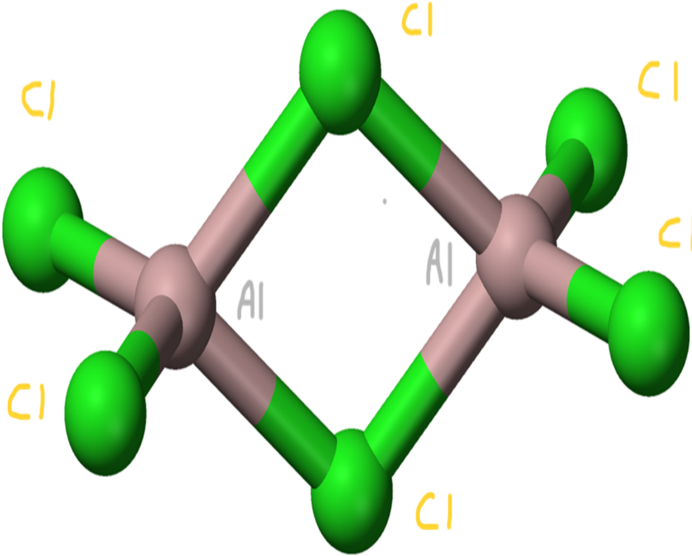
$$A_{2k} = -\frac{d^2}{dx^2} + x^{2k}$$

In particular, the quartic oscillator

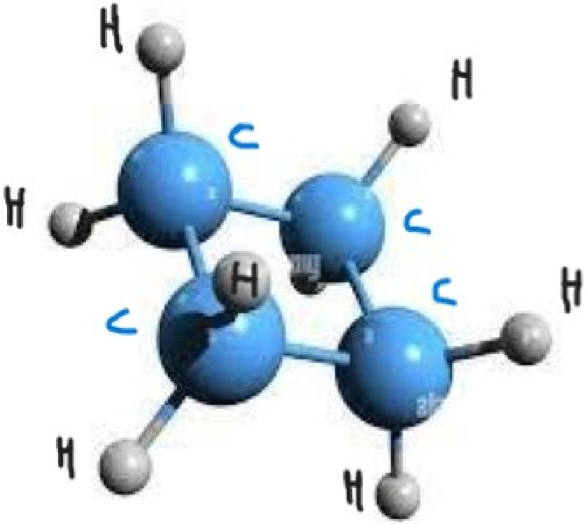
$$A_4 = -\frac{d^2}{dx^2} + x^4$$

Despite the intensive research on the quartic oscillator in the last 50 years, the exact solution for the corresponding eigenvalue problem is unknown. This fact is a further motivation for the research in approximative and qualitative methods around this problem.

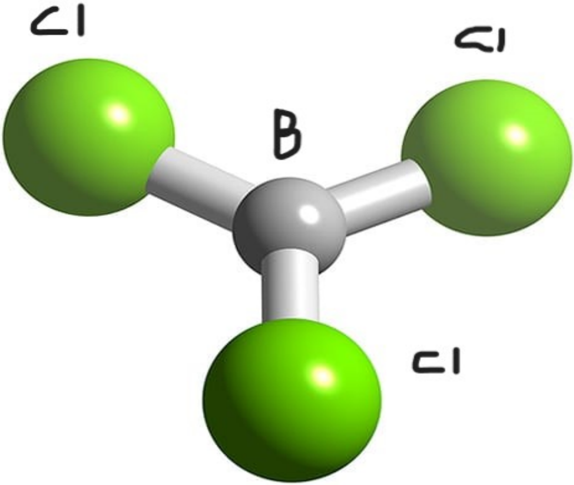
ALUMINIUM TRICHLORIDE: Al_2Cl_6 $\quad -\frac{d^2}{dx^2} + x^4$



CYCLOBUTANE: C_4H_8 $\quad -\frac{d^2}{dx^2} + x^4$



BORON TRICHLORIDE: BCl_3 $-\frac{d^2}{dx^2} + x^4$



General idea

From the Hamiltonian $\mathcal{H}_V$ we can introduce a suitable pseudo-differential calculus on $\mathbb{R}^n$ by mean of a Hörmander metric on the phase-space, and then make use of this powerful machinery. In particular, one can define Sobolev spaces adapted to the metric and therefore to the Hamiltonian.

Weyl-Hörmander Calculus, Lars Hörmander 1979

Let $X \in \mathbb{R}^{2n}$, g_X positive definite quadratic form on $\mathbb{R}^{2n}$. We say that g is an Hörmander metric if:

1. **Continuity**:

$$\exists C > 0, g_X(X - Y) \leq C^{-1} \implies \left(\frac{g_X(\cdot)}{g_Y(\cdot)}\right)^{\pm 1} \leq C$$

.

2. **Uncertainty principle:** We define $g^\sigma_X(T) = \sup_{W \neq 0} \frac{[T,W]^2}{g_X(W)}$,

$$g \leq g^\sigma.$$

3. **Temperateness**: We say that g is temperate if $\exists \overline{C} > 0, J \in \mathbb{N}$ such that

$$\left(\frac{g_X(\cdot)}{g_Y(\cdot)}\right)^{\pm 1} \leq \overline{C}(1 + g^\sigma_Y(X - Y))^J.$$

Some definitions

$g-$weight

We say that a strictly positive function M is a $g-$admissible weight if:

1. M **is** $g-$**continuous**: $\exists C > 0$ such that and $N \in \mathbb{N}$

$$g_X(X - Y) \leq \frac{1}{\tilde{C}} \implies \left(\frac{M(X)}{M(Y)}\right)^{\pm 1} \leq \tilde{C}.$$

2. M **is temperate**: $\exists \tilde{C} > 0$ and $\exists N \in \mathbb{N}$ such that

$$\left(\frac{M(Y)}{M(X)}\right)^{\pm 1} \leq \tilde{C}(1 + g_Y^\sigma(X - Y))^N.$$

Classes $S(M, g)$

For a metric g and a weight M, we shall denoted by $S(M, g)$ the set of smooth functions a on $\mathbb{R}^{2n}$ such that for every integer k there exists $C_k > 0$, such that if $X, T_1, ..., T_k \in \mathbb{R}^{2n}$ then

$$|a^{(k)}(X)(T_1, ..., T_k)| \leq C_k M(X) \prod_{i=1}^{k} g_X^{1/2}(T_i).$$

Examples of Classes $S(M, g)$

$$g_X^{\rho,\delta} = <\xi>^{2\delta} dx^2 + <\xi>^{-2\rho} d\xi^2, \; X = (x, \xi) \in \mathbb{R}^{2n},$$

is a Hörmander's metric. The function

$$X = (x, \xi) \mapsto <\xi>^m,$$

is a weight for the Hörmander metric $g^{\rho,\delta}$ and

$$S^m_{\rho,\delta} = S(<\xi>^m, g^{\rho,\delta}).$$

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Metrics adapted to the Hamiltonians: PROBLEM 1

Theorem

Let $a_2 \in S^2(\mathbb{R}^n \times \mathbb{R}^n)$ *be non-negative* (≥ 0)*. We define the following Riemannian metric on* $\mathbb{R}^n \times \mathbb{R}^n$*,*

$$g_X(dx, d\xi) := m^{-1}(x,\xi)(\langle\xi\rangle^2 + |x|^2)dx^2 + m^{-1}(x,\xi)d\xi^2\,, \qquad (2)$$

where

$$m(x,\xi) = a(x,\xi) + \langle X\rangle, \qquad (3)$$

and the symbol a is given by

$$a(x,\xi) = a_2(x,\xi) + |x|^2\,. \qquad (4)$$

Then g is a Hörmander metric.

Definition

Let $C_b^\infty(\mathbb{R}^n)$ be the class of C^∞ functions over $\mathbb{R}^n$ with bounded derivatives of any order. We denote by $\mathrm{Diff}\,^2_+(\mathbb{R}^n)$ the class of differential operators of order 2 on $\mathbb{R}^n$ with $C_b^\infty(\mathbb{R}^n)$ coefficients and non-negative symbol.

Lemma B Let $a_2(x, D) \in \mathrm{Diff}\,^2_+(\mathbb{R}^n)$. We consider $a(x, \xi) = a_2(x, \xi) + |x|^2$ and the corresponding Hörmander metric g and the weight m as in (2) and (3), respectively. Then a, $m \in S(m, g)$.

Some applications to well-posedness for degenerate Schrödinger equations

We are now going to establish some implications in the L^2 and Sobolev spaces $H(M, g)$ context. Indeed, the construction of the metric g and the class $S(m, g)$ adapted to our degenerate harmonic oscillators has other consequences, regarding the well-posedness for degenerate Schrödinger equations in the L^2-setting. We introduce the following appropriate class of potentials on $\mathbb{R}^n$.

Definition

We will denote by $\mathcal{P}_2(\mathbb{R}^n)$, the class of Borel functions $V : \mathbb{R}^n \to \mathbb{R}$ satisying the following conditions:

(V1) There exist $C, C_1 > 0$ such that $|V(x)| \leq C|x|^2$, for a.e. $|x| \geq C_1$.

(V2) There exists $C_2 > 0$ such that $V(x) \geq -C_2$, for a.e. $x \in \mathbb{R}^n$.

Theorem

Let $a_2(x, D) \in \mathit{Diff}\,^2_+(\mathbb{R}^n)$ *be formally self-adjoint,* $V \in \mathcal{P}_2(\mathbb{R}^n)$*,* g *the Hörmander metric and* m *the* g*-weight associated to* $a_2(x, D)$ *as in* (2)*. Then,* $-i\mathcal{H}_V$ *is the infinitesimal generator of a* C_0-*group of unitary operators. Consequently, the following Cauchy problem for the corresponding degenerate Schrödinger equation is well-posed on* L^2*:*

$$\begin{cases} i\partial_t u &= \mathcal{H}_V u, \\ u(0) &= f. \end{cases} \tag{5}$$

Problem 2: The metric associated to our anharmonic oscillators

We associate to the Hamiltonian $\mathcal{H}_V$, the following metric

$$g = \frac{dx^2}{(1 + |x|^{2k} + |\xi|^{2\ell})^{\frac{1}{k}}} + \frac{d\xi^2}{(1 + |x|^{2k} + |\xi|^{2\ell})^{\frac{1}{\ell}}} \,. \tag{6}$$

Corollary

Let $1 \leq r < \infty$, $k, \ell \geq 1$, where k, ℓ are integers ≥ 1.

1. *If $\mu > \frac{(k+\ell)n}{2k\ell r}$, then*

$$((-\Delta)^{\ell} + |x|^{2k} + 1)^{-\mu} \in S_r(L^2(\mathbb{R}^n)).$$

2. *Let $\nu > n$ and $a \in S(\lambda_g^{-\nu}, g)$. Then $a(x, D)$ is trace class and*

$$\mathrm{Tr}(a(x, D)) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) dx d\xi. \tag{7}$$

RATE OF GROWTH OF EIGENVALUES

We now derive an immediate consequence on the rate of growth of the eigenvalues of the anharmonic oscillators.
Summarising we have obtained the following: Let k, ℓ be integers ≥ 1 and $r > \frac{(k+\ell)n}{2k\ell}$. Then, for every $L \in \mathbb{N}$ there exists $L_0 \in \mathbb{N}$ such that

$$Lj^{\frac{1}{r}} \leq \lambda_j((-\Delta)^{\ell} + |x|^{2k}), \quad \text{for } j \geq L_0. \tag{8}$$

Thus, the eigenvalues $\lambda_j((-\Delta)^{\ell} + |x|^{2k})$ have a growth of order at least

$$j^{\frac{1}{r}}, \text{ as } j \to \infty. \tag{9}$$

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THANK YOU ¡