Some harmonic analysis in a general Gaussian setting – Lecture 1

Valentina Casarino

Università degli Studi di Padova

Summer School "Modern Problems in PDEs and Applications"
23 August – 2 September 2023
Ghent Analysis & PDE Center, Ghent University

Plan of the lectures

1. First lecture. Some motivations; symmetric vs nonsymmetric case; spectrum and Mehler kernel for the symmetric O.U. semigroup.
2. Second lecture. Spectrum and Mehler kernel for the nonsymmetric O.U. semigroup; orthogonality of eigenspaces.
3. Third lecture. Discussion of a problem from harmonic analysis: functional calculus in the nonsymmetric context.

First lecture

Some history

In 1997 P. Sjögren published a survey[1] describing the state of art about the study of some operators (like maximal operators, Riesz transforms, multiplier operators,...) in $\mathbb{R}^n$, where Lebesgue measure was replaced by a suitable normalized Gaussian measure $d\gamma$.

Some years later, a group formed by J. Garcìa-Cuerva, G. Mauceri, S. Meda, P. Sjögren and J. L. Torrea carried out this analysis, by studying in a series of papers (the weak type $(1,1)$) of some operators in $\mathbb{R}^n$ endowed with $d\gamma$.[2]

---

[1] P. Sjögren, Operators Associated with the Hermite Semigroup - A Survey, *Journal of Fourier Analysis and Applications* (1997)

[2] –J. García-Cuerva, G. Mauceri, P. Sjögren and J. Torrea, Higher-order Riesz operators for the Ornstein-Uhlenbeck semigroup, *Potential Anal.* (1999)
–J. García-Cuerva, G. Mauceri, P. Sjögren and J. Torrea, Spectral multipliers for the Ornstein-Uhlenbeck semigroup, *J. Anal. Math.* (1999)
–J. Garcìa-Cuerva, G. Mauceri, S. Meda, P. Sjögren and J. L. Torrea, Maximal Operators for the Holomorphic Ornstein– Uhlenbeck Semigroup, *J. London Math.* (2003)
–J. Garcìa-Cuerva, G. Mauceri, S. Meda, P. Sjögren and J. L. Torrea, Functional calculus for the Ornstein–Uhlenbeck operator, *J. Funct. Anal.* (2001)

Their setting was symmetric.
To be more precise, one can associate with $d\gamma$ in a rather natural way a Laplacian L, called the Ornstein–Uhlenbeck operator, which turns out to be the infinitesimal generator of a (heat) semigroup.
In the context studied by Garcìa-Cuerva, Mauceri, Meda, Sjögren and Torrea the Laplacian was self-adjoint and any operator in the semigroup generated by L was self-adjoint as well.

In the last years, in collaboration with P. Ciatti and P. Sjögren we generalized the results, previously obtained by Garcìa-Cuerva, Mauceri, Meda, Sjögren and Torrea, to a nonsymmetric context.

We will briefly explain what we mean when we talk about a nonsymmetric context.
First of all, the symmetric Laplacian L, introduced by Garcìa-Cuerva, Mauceri, Meda, Sjögren and Torrea, is replaced by a sort of nonsymmetric Laplacian.

To define a nonsymmetric Ornstein–Uhlenbeck operator $\mathcal{L}$ we need two matrices:

- Q (covariance) is a real, symmetric and positive definite $N \times N$ matrix;
- B (drift) is a real $N \times N$ matrix whose eigenvalues have negative real parts; here $N \geqslant 1$.

Then

$$\mathcal{L}f = \mathcal{L}^{Q,B}f = \frac{1}{2}\operatorname{tr}\left(Q\nabla^2 f\right) + \langle Bx, \nabla f\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^N).$$

Here $Q\nabla^2 f$ denotes the product of Q and the Hessian matrix of f.

As in the symmetric context, it is interesting to study $\mathcal{L}^{Q,B}$ and some related operators in $\mathbb{R}^N$ endowed with a particular Gaussian measure, denoted by $d\gamma_\infty$.
We are going to motivate the choice of this particular measure and also the study of Gaussian harmonic analysis, both in a symmetric and in a nonsymmetric context.

Some motivations

1) Also in the general, nonsymmetric context, $\mathcal{L}^{Q,B}$ is the infinitesimal generator of a semigroup $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$, called Ornstein–Uhlenbeck semigroup, given for all bounded continuous functions f in $\mathbb{R}^N$, and all $t > 0$ by the Kolmogorov formula

$$\mathcal{H}_t^{Q,B} f(x) = \int f(e^{tB}x - y) d\gamma_t(y)\,, \quad x \in \mathbb{R}^N\,.$$

$d\gamma_t$ are normalized Gaussian measures, $t \in (0, +\infty]$, which will be defined in the following. For the moment, we may forget $d\gamma_t$ and focus only on $d\gamma_\infty$, given by

$$d\gamma_\infty(x) = (2\pi)^{-\frac{N}{2}} (\det Q_\infty)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle Q_\infty^{-1}x,x\rangle}\, dx$$

(Here $Q_\infty = \int_0^\infty e^{sB}\, Q\, e^{sB^*}\, ds$ is a positive definite and symmetric matrix)

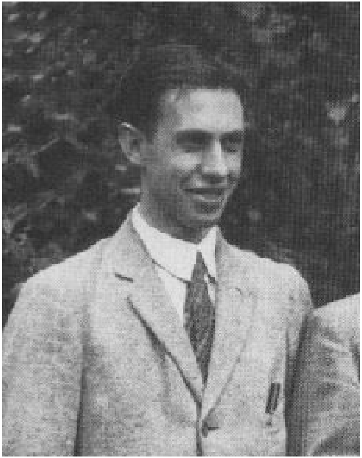
$$d\gamma_\infty(x) = (2\pi)^{-\frac{N}{2}}(\det Q_\infty)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle Q_\infty^{-1}x,x\rangle}\, dx$$

It is possible to prove that $d\gamma_\infty$ is the **unique** invariant (probability) measure of the Ornstein–Uhlenbeck semigroup, that is,

$$\int \mathcal{H}_t^{Q,B} f(x)d\gamma_\infty(x) = \int f(x)d\gamma_\infty(x) \qquad \forall t > 0.$$

2)The Ornstein–Uhlenbeck operator $\mathcal{L}^{Q,B}$ and the Ornstein–Uhlenbeck semigroup $\mathcal{H}_t^{Q,B}$ play the role of the Laplacian and of the heat semigroup in $\mathbb{R}^N$ if the Lebesgue measure dx is replaced by $d\gamma_\infty$.

3) The relevance of the semigroup $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ is also due to the fact that it is associated to the Ornstein-Uhlenbeck process describing the random motion of a particle subject to friction.



Leonard Ornstein (1880-1941) and George Uhlenbeck (1900-1988)

4) Since Ornstein and Uhlenbeck's seminal work in the '30, the O.U. theory has been widely applied in quantum physics, stochastic analysis, control theory, partial differential equations.

Example. *"Evolution equations driven by Ornstein–Uhlenbeck operators are the Kolmogorov equations of linear stochastic ODEs, and they are one of the few examples of multidimensional linear parabolic equations for which a resolvent kernel is explicitly known."*[3]

[3] Lunardi, Metafune, Pallara, The Ornstein–Uhlenbeck semigroup in finite dimension, Phil. Trans. R. Soc. A (2020).

We close this little digression and come back to the main point, the study of $\mathcal{L}^{Q,B}$ and some related operators.
We distinguish between two cases.

– **The nonsymmetric case**

We assume:

Q real, symmetric and positive definite $N \times N$ matrix;

B real $N \times N$ matrix whose eigenvalues have negative real parts.

In this context, in general

$$\mathcal{L}^{Q,B} f = \frac{1}{2} \operatorname{tr}\left(Q \nabla^2 f\right) + \langle Bx, \nabla f \rangle$$

has no self-adjoint or normal extension to $L^2(\mathbb{R}^N, d\gamma_\infty)$ and many problems arise.

Example. In order to study multipliers, if L is a self-adjoint operator on $L^2(\mathbb{R}^n, d\gamma)$, and if E denotes a spectral resolution of L on $\mathbb{R}$, one can define $m(L)$ (for many functions m) as

$$m(L) = \int_{\mathbb{R}} m(\lambda)\, dE(\lambda).$$

But in order to study $m(\mathcal{L}^{Q,B})$ one cannot invoke spectral theorem to define $m(\mathcal{L}^{Q,B})$. Some subtler tools are required.

Notice that self-adjointness and normality may fail also for the semigroup $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$, generated by $\mathcal{L}^{Q,B}$.

The nonsymmetric case has been largely studied in PDE's setting

- Chill, Da Prato, Fašangová, Lunardi, Metafune, Pallara, Priola, Prüss, Rhandi, Schnaubelt,...

much less in harmonic analysis.
In the field of harmonic analysis the focus has been on the **classical case**, which was studied, in particular, by J. Garcìa-Cuerva, G. Mauceri, S. Meda, P. Sjögren and J. L. Torrea.

– **The classical (symmetric) case**:

$Q = I$ and $B = -I$

(here $I = I_N$ is the identity matrix of order N)

Recall that the invariant measure $d\gamma_\infty$ is

$$d\gamma_\infty(x) = (2\pi)^{-\frac{N}{2}} (\det Q_\infty)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle Q_\infty^{-1} x, x\rangle} \, dx.$$

Since in this case

$$Q_\infty = \int_0^\infty e^{sB} \, Q \, e^{sB^*} ds = \int_0^\infty e^{-sI} \, e^{-sI} ds = \frac{1}{2} I,$$

one has

$$d\gamma_\infty(x) = \pi^{-\frac{N}{2}} e^{-\langle x, x\rangle} \, dx.$$

Recall that $\mathcal{L}^{Q,B}$ is given by

$$\mathcal{L}^{Q,B} f = \frac{1}{2} \operatorname{tr}\left(Q \nabla^2 f\right) + \langle Bx, \nabla f \rangle, \qquad f \in \mathcal{S}(\mathbb{R}^N),$$

and therefore

$$\boxed{\mathcal{L}^{I,-I} f = \tfrac{1}{2} \Delta f - \langle x, \nabla f \rangle, \qquad f \in \mathcal{S}(\mathbb{R}^N).}$$

For the sake of simplicity, we denote

$$d\gamma = d\gamma_\infty(x) = \pi^{-\frac{N}{2}} e^{-\langle x, x \rangle} \, dx$$

and

$$Lf = \mathcal{L}^{I,-I} f = \frac{1}{2} \Delta f - \langle x, \nabla f \rangle, \qquad f \in \mathcal{S}(\mathbb{R}^N).$$

For a while, we will work with $d\gamma$ and L.

The operator given by

$$L := \mathcal{L}^{I,-I} f = \frac{1}{2}\Delta f - \langle x, \nabla f \rangle, \qquad f \in \mathcal{S}(\mathbb{R}^N),$$

is self-adjoint **with respect to** $d\gamma = d\gamma_\infty$; $\left(\mathcal{H}_t\right)_{t>0}$ is also symmetric.

We will prove this fact.

Lemma 1. The symmetric Ornstein–Uhlenbeck operator $L = \mathcal{L}^{I,-I}$ is self-adjoint **with respect to** $d\gamma$.

Proof. We denote ∂_{x_k} by ∂_k.

We shall compute the adjoint operator ∂_k^* in $L^2(d\gamma)$. Let $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. Then

$$\begin{aligned}
\langle \partial_k f, g\rangle_{L^2(d\gamma)} &= \int_{\mathbb{R}^N} \partial_k f(x)\, g(x)\, d\gamma(x) \\
&= -\pi^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x)\, \partial_k\big(g(x)\, e^{-|x|^2}\big)\, dx \\
&= -\pi^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x)\, \big(\partial_k g(x)\, e^{-|x|^2} - 2x_k\, g(x)\, e^{-|x|^2}\big)\, dx \\
&= -\pi^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x)\, \big(\partial_k g(x) - 2x_k g(x)\big)\, e^{-|x|^2}\, dx \\
&= \pi^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x)\, \big(2x_k g(x) - \partial_k g(x)\big)\, e^{-|x|^2}\, dx \\
&= \int_{\mathbb{R}^N} f(x)\, \big(2x_k - \partial_k\big) g(x)\, d\gamma(x).
\end{aligned}$$

We have obtained

$$\begin{aligned}\langle \partial_k f, g\rangle_{L^2(d\gamma)} &= \int_{\mathbb{R}^N} f(x)\left(2x_k - \partial_k\right) g(x)\, d\gamma(x) \\ &= \langle f, \left(2x_k - \partial_k\right) g\rangle_{L^2(d\gamma)} \\ &= \langle f, \partial_k^* g\rangle_{L^2(d\gamma)}\end{aligned}$$

where

$$\partial_k^* = 2x_k - \partial_k;$$

the first term here is a multiplication operator. We observe that

$$\begin{aligned}-\frac{1}{2}\sum_{k=1}^{N}\partial_k^*\partial_k &= -\frac{1}{2}\sum_{k=1}^{N}\left(2x_k - \partial_k\right)\partial_k \\ &= \frac{1}{2}\sum_{k=1}^{N}(\partial_k)\partial_k - \frac{1}{2}\sum_{k=1}^{N}2x_k\partial_k \\ &= \frac{1}{2}\Delta - \langle x, \nabla\rangle = \mathcal{L}^{I,-I},\end{aligned}$$

that is, $L = \mathcal{L}^{I,-I}$ is a negative operator.

Moreover, $L = \mathcal{L}^{I,-I}$ is self-adjoint in $L^2(\gamma)$, since

$$\begin{aligned}\langle Lf, g\rangle &= -\frac{1}{2}\sum_{k=1}^{N}\langle \partial_k^* \partial_k f, g\rangle \\ &= -\frac{1}{2}\sum_{k=1}^{N}\langle \partial_k f, \partial_k g\rangle = -\frac{1}{2}\sum_{k=1}^{N}\langle f, \partial_k^* \partial_k g\rangle \\ &= \langle f, Lg\rangle.\end{aligned}$$

□

Remark. Since, in particular,

$$L = -\frac{1}{2}\sum_{k=1}^{N}\partial_k^* \partial_k,$$

$L = \mathcal{L}^{I,-I}$ plays the role of the Laplacian in $L^2(d\gamma)$.

In the general case, when

$$\mathcal{L}^{Q,B}f = \frac{1}{2}\,\mathrm{tr}\left(Q\nabla^2 f\right) + \langle Bx, \nabla f\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^N),$$

an analogous result does not hold and $\mathcal{L}^{Q,B}$ is in general not self-adjoint.

Example.
In $\mathbb{R}^2$ consider $Q = I_2$ and

$$B = \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix}. \tag{3.1}$$

Exercise 1. Prove that the corresponding O.U. operator is not self-adjoint.

The spectrum of L

We are going to study the spectrum of $\mathcal{L}^{Q,B}$, first of all in the classical case $Q = I$ and $B = -I$.
A little digression about Hermite polynomials is necessary.

Hermite polynomials

Definition. The n^{th} Hermite polynomial H_n is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad x \in \mathbb{R}.$$

By differentiation, we see that H_n is of the form

$$H_n(x) = 2^n x^n + \text{lower order terms.}$$

It also holds $H_0(x) \equiv 1$.
Moreover,

$$\frac{d}{dx} H_n(x) = 2n H_{n-1} \qquad n \geqslant 1.$$

The following result is well-known.

Proposition 2. The polynomials H_n, $n \in \mathbb{N}$, form a complete orthogonal system in $L^2(\mathbb{R}, d\gamma_1)$. Moreover,

$$\|H_n\|_{L^2(\mathbb{R},d\gamma_1)} = 2^{n/2}\sqrt{n!}, \qquad n \in \mathbb{N}.$$

(Here $d\gamma_1(x) = \pi^{-\frac{1}{2}} e^{-|x|^2}\, dx$)

Hermite polynomials exist also in dimensione $N > 1$.

Definition. The Hermite polynomial H_α on $\mathbb{R}^N$, with $\alpha \in \mathbb{N}^N$ multiindex, is defined by the tensor product

$$H_\alpha = \otimes_{j=1}^N H_{\alpha_j},$$

that is, $H_\alpha(x) = \Pi_{j=1}^N H_{\alpha_j}(x_j), \qquad x = (x_1, \ldots, x_N) \in \mathbb{R}^N.$

Then H_α is a polynomial of degree $|\alpha| = \sum_{j=1}^N \alpha_j$.

Proposition 3. $(H_\alpha)_{\alpha\in\mathbb{N}^N}$ is a complete orthogonal system in $L^2(\mathbb{R}^N, d\gamma)$.

We are interested in Hermite polynomials because of the following result.

Theorem 4. The Hermite polynomials are eigenvectors for the classical Ornstein–Uhlenbeck operator L. Moreover, for any multi-index $\alpha \in \mathbb{N}^N$,

$$LH_\alpha = -|\alpha|\, H_\alpha.$$

Sketch of the proof in dimension 1. In this case, we have to prove that $\mathcal{L}^{I,-I} H_n = -n\, H_n$ for all $n \in \mathbb{N}$.
Recall that

$$\mathcal{L}^{I,-I} = -\frac{1}{2} \sum_{k=1}^{N} \partial_k^* \partial_k.$$

If $N = 1$, in particular,

$$\mathcal{L}^{I,-I} f(x) = -\frac{1}{2} \Big(\frac{d}{dx}\Big)^* \frac{d}{dx} f.$$

Keeping in mind that

$$\frac{d}{dx}H_n(x) = 2nH_{n-1} \qquad n \geqslant 1,$$

we shall compute the operator $\left(\frac{d}{dx}\right)^*$ on the Hermite polynomials. We have:

$$\langle \left(\frac{d}{dx}\right)^* H_{n-1}, H_j \rangle = \langle H_{n-1}, \left(\frac{d}{dx}\right) H_j \rangle = 2j \, \langle H_{n-1}, H_{j-1} \rangle$$

and the last term is $\neq 0$ if and only if $n = j$. In this case one has

$$\begin{aligned} \langle \left(\frac{d}{dx}\right)^* H_{n-1}, H_n \rangle &= 2n \, \langle H_{n-1}, H_{n-1} \rangle = 2n \, 2^{n-1} \, (n-1)! \\ &= 2^n \, n! = \langle H_n, H_n \rangle. \end{aligned}$$

Thus

$$\left(\frac{d}{dx}\right)^* H_{n-1} = H_n.$$

We may now compute $\mathcal{L}^{I,-I} H_n$. We have

$$\mathcal{L}^{I,-I} H_n = -\frac{1}{2}\Big(\frac{d}{dx}\Big)^* \frac{d}{dx} H_n = -\frac{1}{2}\, 2n\,\Big(\frac{d}{dx}\Big)^* H_{n-1}$$
$$= -n\,\Big(\frac{d}{dx}\Big)^* H_{n-1} = -n\, H_n.$$

This concludes the proof in dimension 1.

To prove the assertion in dimension $N > 1$, we shall use the following result.

Exercise 2. Taking into account that

$$\frac{\partial}{\partial x_k} H_\alpha(x) = 2\alpha_k H_{\alpha - e_k}(x) \qquad n \geqslant 1,$$

with $\{e_j\}$ canonical basis of vectors in R^N, prove that

$$\Big(\frac{\partial}{\partial x_k}\Big)^* H_{\alpha - e_k} = H_\alpha.$$

We may now compute $\mathcal{L}^{I,-I} H_\alpha$. Recall that

$$\mathcal{L}^{I,-I} = -\frac{1}{2} \sum_{k=1}^{N} \partial_k^* \partial_k .$$

Then

$$\begin{aligned} \mathcal{L}^{I,-I} H_\alpha &= -\frac{1}{2} \sum_{k=1}^{N} \partial_k^* \partial_k H_\alpha = -\frac{1}{2} \sum_{k=1}^{N} 2\alpha_k \, \partial_k^* H_{\alpha - e_k} \\ &= - - \sum_{k=1}^{N} \alpha_k \, H_\alpha = -|\alpha| H_\alpha . \end{aligned}$$

This conclude the proof in dimension $N > 1$. □

One can prove more. In fact, the $L^2(d\gamma)$ spectrum of $L = \mathcal{L}^{I,-I}$ is

$$\sigma(L) = \{-n : n \in \mathbb{N}\}$$

and the eigenfunctions for arbitrary N are tensor products $H_\alpha = \otimes_{j=1}^N H_{\alpha_j}$, where α is a multiindex, and the corresponding eigenvalue is the length $|\alpha|$.

It may be proved that the $L^2(d\gamma)$ spectrum of L coincides with the $L^p(d\gamma)$ spectrum of L for $p > 1$.

The $L^1(d\gamma)$ spectrum of L is different (it coincides with the left half-plane).

Results are known also for the spectrum of L in $L^p(dx)$[4].

This spectral information may be used to define in a rigorous way the O.U. semigroup in the classical case.

---

[4] For results, also holding in the nonsymmetric case, see:
– G. Metafune, L^p-spectrum of Ornstein-Uhlenbeck operators. Ann. Sc. Norm. Sup. Pisa 30, (2001)
– S. Fornaro, G. Metafune, D. Pallara, R. Schnaubelt, L^p-spectrum of degenerate hypoelliptic Ornstein-Uhlenbeck operators. J. Funct. Anal., (2022).

For each nonnegative integer k, we denote by P_k the orthogonal projection of $L^2(d\gamma)$ onto the subspace generated by the Hermite polynomials of degree k.

We define the Ornstein-Uhlenbeck semigroup $\left(H_t\right)_{t>0} = \left(e^{tL}\right)_{t>0}$ in a spectral sense as

$$\boxed{H_t = \sum_{k=0}^{\infty} e^{-tk} P_k.}$$

Its infinitesimal generator is the operator $L = \mathcal{L}^{I,-I}$.

In other words, $H_t = e^{tL}$ is the bounded operator on $L^2(d\gamma)$ which maps

$$H_\alpha \quad \mapsto \quad e^{-t|\alpha|} H_\alpha.$$

In order to study harmonic analysis, it is a very useful fact that any operator of the semigroup $H_t = e^{tL}$ may be written in integral form, as

$$H_t f(x) = \int_{\mathbb{R}^N} M_t(x,u) f(u) d\gamma(u), \qquad t > 0,$$

for some function $M_t \in L^2(d\gamma \times d\gamma)$ known as **Mehler kernel** (since it was found already already in 1866 by Mehler[5]).

[5] F. G. Mehler, Über die Entwicklung einer Funktion von beliebig vielen Variablen nach Laplaceschen Funktionenh höherer Ordnung, J. reine angew. Math. 66, 161-176, (1866).

The Mehler kernel M_t, that is, the function $M_t \in L^2(d\gamma \times d\gamma)$ such that

$$H_t f(x) = \int_{\mathbb{R}^N} M_t(x,u) f(u) d\gamma(u), \tag{4.1}$$

may be expressed in different ways.

Exercise 3. Prove that

$$M_t(x,u) = \sum_{\alpha \in \mathbb{N}^n} e^{-t|\alpha|} h_\alpha(x) h_\alpha(u),$$

where $h_\alpha = H_\alpha / \|H_\alpha\|_{L^2(d\gamma_\infty)}$.

Anyway, we shall use in particular the closed expression

$$M_t(x,u) = \frac{1}{(1-e^{-2t})^{\frac{n}{2}}} e^{|x|^2/2} \exp\left(-\frac{1}{2}\frac{|u-e^t x|^2}{1-e^{-2t}}\right),$$

(integration is meant with respect to Gaussian measure $d\gamma$).

Remark. Integration against M_t is well defined for $f \in L^1(d\gamma)$, so we use (4.1) to extend the domain of $e^{-tL} = H_t$ to $L^1(d\gamma)$, which of course contains $L^p(d\gamma)$ for $1 \leqslant p \leqslant \infty$.

In fact, we have the following result.

Proposition 4. Let $1 \leqslant p \leqslant \infty$. For all $f \in L^p(d\gamma)$, $t > 0$, $x \in \mathbb{R}^N$, one has

$$H_t f(x) = \int_{\mathbb{R}^N} M_t(x,u) f(u) d\gamma(u),$$

where

$$M_t(x,u) = \frac{1}{(1-e^{-2t})^{\frac{n}{2}}} e^{|x|^2/2} \exp\left(-\frac{1}{2}\frac{|u-e^t x|^2}{1-e^{-2t}}\right)$$

Notice that $M_t(x,u)$ is symmetric, that is,

$$M_t(x,u) = M_t(u,x).$$