Some harmonic analysis in a general Gaussian setting – Lecture 2

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Plan of the lectures

1. First lecture. Some motivations; classical vs nonsymmetric case; spectrum and Mehler kernel for the classical O.U. semigroup.
2. Second lecture. Spectrum and Mehler kernel for a general, nonsymmetric O.U. semigroup; orthogonality of eigenspaces.
3. Third lecture. Discussion of a problem from harmonic analysis: functional calculus in the nonsymmetric context.

Second lecture

In the first lecture we have dealt with the spectrum of $L = \mathcal{L}^{I,-I}$ and the Mehler kernel of $(H_t)_{t>0}$ in the classical case ($Q = -B = I$).

Now we come back to the general case, that is, we consider an Ornstein–Uhlenbeck operator $\mathcal{L}^{Q,B}$

$$\mathcal{L}^{Q,B} f = \frac{1}{2} \operatorname{tr}\left(Q\nabla^2 f\right) + \langle Bx, \nabla f\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^N),$$

where

- Q (covariance) is a real, symmetric and positive definite $N \times N$ matrix;
- B (drift) is a real $N \times N$ matrix whose eigenvalues have negative real parts; here $N \geqslant 1$.

Here $Q\nabla^2 f$ denotes the product of Q and the Hessian matrix of f.

Just a remark. Between the classical case and the general one there are some intermediate cases. For instance,

- one could consider the case when $Q = I$ and $B = -\lambda I$, for some $\lambda > 0$;
- one could consider the symmetric framework (that is, the case when any operator in $(\mathcal{H}_t)_{t>0}$ is symmetric in $L^2(d\gamma_\infty)$. This is true if and only if $QB^* = BQ$ [1]. In the symmetric framework there are sparse results.
- Mauceri-Noselli[2] and then C.-Ciatti-Sjögren[3] studied the normal framework (where any operator in $(\mathcal{H}_t)_{t>0}$ is normal).

Anyway, these intermediate cases are often troublesome, so we might as well study the general case directly.

[1] A. Chojnowska-Michalik – B. Goldys, Symmetric Ornstein-Uhlenbeck semigroups and their generators, *Probab. Theory Related Fields* (2002)

[2] – G. Mauceri and L. Noselli, The maximal operator associated to a non symmetric Ornstein-Uhlenbeck semigroup, *J. Fourier Anal. Appl.* (2009)
–Riesz transforms for a non symmetric Ornstein-Uhlenbeck semigroup, *Semigroup Forum* (2008)

[3] C.-Ciatti-Sjögren, The maximal operator of a normal Ornstein–Uhlenbeck semigroup is of weak type $(1,1)$, *Annali SNS Pisa, (2020)*

– The nonsymmetric case

In the general case we define the O. U. semigroup by means of an integral formula, as follows.

Definition. $\mathcal{H}_t^{Q,B}$ is defined for all bounded continuous functions f on $\mathbb{R}^N$ and all $t > 0$ by the Kolmogorov formula[4]

$$\boxed{\mathcal{H}_t^{Q,B} f(x) = \int f(e^{tB}x - y)\, d\gamma_t(y)\,, \quad x \in \mathbb{R}^N\,.}$$

[4] A. N. Kolmogorov, Zufällige bewegungen, Ann. of Math. (1934).

$$\mathcal{H}_t^{Q,B} f(x) = \int f(e^{tB}x - y) d\gamma_t(y) , \quad x \in \mathbb{R}^N .$$

It is easy to check that

- the semigroup law holds for $\mathcal{H}_t^{Q,B}$ (usually, we check it on the Schwartz space, and then we deduce that it holds also in L^p spaces with respect to Lebesgue or Gaussian measures);
- this semigroup is generated by $\mathcal{L}^{Q,B}$.

We call $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ Ornstein–Uhlenbeck semigroup.

$d\gamma_t$ are normalized Gaussian measures, $t \in (0, +\infty]$, defined as follows.

We introduce the covariance matrices

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*}\, ds\,, \qquad t \in (0, +\infty],$$

all symmetric and positive definite. Then we define the family of normalized Gaussian measures in $\mathbb{R}^N$ by

$$d\gamma_t(x) = (2\pi)^{-\frac{N}{2}} (\det Q_t)^{-\frac{1}{2}}\, e^{-\frac{1}{2}\langle Q_t^{-1}x, x\rangle}\, dx\,, \qquad t \in (0, +\infty].$$

[5] Prop. 9.3.1 in M. Bertoldi-L. Lorenzi, Analytical Methods for Markov semigroups, 2007, Chapman & Hall.

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Among all $d\gamma_t$ we choose
$d\gamma_\infty(x) = (2\pi)^{-\frac{N}{2}} (\det Q_\infty)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle Q_\infty^{-1}x, x\rangle}\, dx$, which is the **unique** invariant (probability) measure of the O.U. semigroup[5], that is,

$$\int \mathcal{H}_t^{Q,B} f(x) d\gamma_\infty(x) = \int f(x) d\gamma_\infty(x) \qquad \forall t > 0.$$

[5] Prop. 9.3.1 in M. Bertoldi-L. Lorenzi, Analytical Methods for Markov semigroups, 2007, Chapman & Hall.

In the nonsymmetric case as well, one can prove that any operator $\mathcal{H}_t^{Q,B}$ is an integral operator, with a kernel $K_t(x,u) \in L^2(d\gamma_\infty \times d\gamma_\infty)$, which is called again **Mehler kernel** (in fact, Mehler's computations are valid only in the case $Q = -B = I$).

In other words, for all $t > 0$

$$\mathcal{H}_t^{Q,B} f(x) = \int_{\mathbb{R}^N} K_t^{Q,B}(x,u)\, f(u)\, d\gamma_\infty(u).$$

Starting from Kolmogorov's formula, we may deduce a closed expression for the Mehler kernel $K_t = K_t^{Q,B}$.

Proposition 4. Let $p \in [1,\infty]$. For all $f \in L^p(d\gamma_\infty)$, $t > 0$, $x \in \mathbb{R}^N$, one has

$$\mathcal{H}_t^{Q,B} f(x) = \int_{\mathbb{R}^N} K_t(x,u) f(u)\, d\gamma_\infty(u), \qquad \text{where}$$

$$K_t(x,u) = \left(\frac{\det Q_\infty}{\det Q_t}\right)^{1/2} \exp\left(\frac{1}{2}\left\langle Q_\infty^{-1}x, x\right\rangle\right) \times \exp\left[-\left\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t x)\,,\, u - D_t x\right\rangle\right],$$

with $D_t x = Q_\infty e^{-tB^*} Q_\infty^{-1} x, \qquad t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Remark: compare with the classical Mehler kernel (corresponding to the choice $Q = -B = I$), which is given by

$$M_t(x,u) = \frac{1}{(1-e^{-2t})^{\frac{n}{2}}}\, e^{|x|^2/2} \exp\left(-\frac{1}{2}\frac{|u - e^t x|^2}{1-e^{-2t}}\right),$$

Proof. The proof [6] follows from

- Kolmogorov's formula

$$\mathcal{H}_t^{Q,B} f(x) = \int f(e^{tB}x - y)\, d\gamma_t(y)\,, \quad x \in \mathbb{R}^N\,,$$

- definition of $d\gamma_t$

$$d\gamma_t(x) = (2\pi)^{-\frac{N}{2}} (\det Q_t)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle Q_t^{-1} x, x\rangle}\, dx,$$

- some algebraic properties of Q_t.

□

We now focus on the spectrum of $\mathcal{L}^{Q,B}$ in $L^p(d\gamma_\infty)$ in the nonsymmetric setting.

[6] V.C., P. Ciatti, P. Sjögren, On the maximal operator of a general Ornstein–Uhlenbeck semigroup, Math. Z., (2022).

Recall that in the classical framework ($Q = I$ and $B = -I$), the spectrum of L in $L^p(d\gamma)$, $1 < p < \infty$, consists of the negative integers and the Hermite polynomials form a complete system of eigenfunctions in $L^2(d\gamma)$.

In the nonsymmetric framework, the following facts are well-known:

1. the spectrum of $\mathcal{L}^{Q,B}$ in $L^p(\gamma_\infty)$ is contained in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant 0\}$, since $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ is a contraction semigroup in $L^p(\gamma_\infty)$.
2. 0 is an eigenvalue and every eigenfunction corresponding to the eigenvalue 0 is constant (trivial, since $\mathcal{L}^{Q,B} f = \frac{1}{2}\operatorname{tr}\left(Q\nabla^2 f\right) + \langle Bx, \nabla f\rangle$).

In addition, Metafune, Pallara and Priola proved the following facts.

3. If $\lambda_1, \dots, \lambda_r$ are the (distinct) eigenvalues of B, then

$$\sigma(\mathcal{L}^{Q,B}) = \{\sum_{j=1}^{r} n_j \lambda_j \ : n_j \in \mathbb{N} \cup \{0\}\}$$

   (and this is true for $L^p(d\gamma_\infty)$ for all $1 < p < \infty$); moreover, the eigenvalues have finite multiplicity;

4. the spectrum of $\mathcal{L}^{Q,B}$ in $L^1(d\gamma_\infty)$ is the left half-plane and all complex numbers λ with negative real part are eigenvalues;

5. $\mathcal{L}^{Q,B}$ admits a complete system of generalized eigenfunctions (which are polynomials) in $L^2(d\gamma_\infty)$. [7]

[7] G.Metafune, D. Pallara and E. Priola, Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures, *J. Funct. Anal.* (2002)

We shall now define a generalized eigenfunction.
Def. A number $\lambda \in \mathbb{C}$ is a generalized eigenvalue of $\mathcal{L}$ if there exists a nonzero $u \in L^2(d\gamma_\infty)$ such that

$$(\mathcal{L} - \lambda I)^k u = 0$$

for some positive integer k.
Then u is called a generalized eigenfunction, and those u span the generalized eigenspace corresponding to λ.

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For $k = 1$ one has

$$(\mathcal{L} - \lambda I) u = 0,$$

that is, u is an eigenfunction.

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Theorem (Metafune-Pallara-Priola, 2002)
The Ornstein–Uhlenbeck operator $\mathcal{L} = \mathcal{L}^{Q,B}$ admits a complete system of generalized eigenfunctions, that is, the linear span of the generalized eigenfunctions is dense in $L^2(d\gamma_\infty)$. Moreover, the generalized eigenfunctions are polynomials.

An (incomplete) overview on spectra

Space	$\sigma(\mathcal{L}^{Q,B})$		
$L^p(d\gamma_\infty)$	$\{\sum_{j=1}^r n_j\lambda_j \; : n_j \in \mathbb{N}\}$	$1 < p < \infty$	MPP (2002)
$L^1(d\gamma_\infty)$	$\{z \; : \operatorname{Re} z \leqslant 0\}$	$p = 1$	FMPS (2020)
$L^p(dx)$	$\{z : \operatorname{Re} z \leqslant -\operatorname{tr} B/p\}$	$1 \leqslant p < \infty$	Metafune (2001), Fornaro-Metafune- -Pallara-Schnaubelt (2020)
$\mathcal{C}_0(\mathbb{R}^N)$	some partial results		Metafune (2001), Lorenzi (2001)

Above, FMPS=Fornaro-Metafune-Pallara-Schnaubelt, and MPP=Metafune-Pallara-Priola

Orthogonality of eigenspaces

Obviously, in the classical case the eigenspaces of $L = \mathcal{L}^{I,-I}$ are orthogonal, since $\mathcal{L}^{I,-I}$ is self-adjoint.
But in the general case, orthogonality of eigenspaces of $\mathcal{L}^{Q,B}$ is in general not guaranteed. Thus, we shall investigate orthogonality of eigenspaces of $\mathcal{L}^{Q,B}$. As expected, the spectral properties of B play a prominent role here.

Motivation

- Definition of negative powers of $-\mathcal{L}$ ($\Rightarrow$ definition of Riesz transforms)
- Definition of spectral multipliers for the Ornstein–Uhlenbeck operator. We cannot invoke spectral theorem. One possibility is to define the restriction of $m(\mathcal{L})$ to each finite-dimensional generalized eigenspace $\mathcal{E}_\lambda$. Then, $m(\mathcal{L})$ is determined by these restrictions, since the $\mathcal{E}_\lambda$ together span $L^2(\gamma_\infty)$.

We start with a positive result. [8]

Proposition 1 (C.-Ciatti-Sjögren) The kernel of $\mathcal{L}^{Q,B}$ is orthogonal to the other generalized eigenspaces of $\mathcal{L}^{Q,B}$ in $L^2(d\gamma_\infty)$.

Proof Proposition 1

This property will play an essential role in the definition of multiplier operators $m(\mathcal{L}^{Q,B})$.

[8] C.-Ciatti-Sjögren, On the orthogonality of generalized eigenspaces for the Ornstein–Uhlenbeck operator, *Archiv Math.* (2021)

Anyway, the question of orthogonality between generalized eigenspaces associated to nonzero eigenvalues is more delicate and strongly depends on the spectral properties of B.
We distinguish between two cases:

1) B has only one eigenvalue

$\Downarrow$

Orthogonality

2) B has at least two distinct eigenvalues

$\Downarrow$

?? (It depends)

The case when B has only one eigenvalue

Proposition 2 (C.-Ciatti-Sjögren) If the drift matrix B has only one eigenvalue λ, then any two generalized eigenfunctions of $\mathcal{L}$ with different eigenvalues are orthogonal with respect to γ_∞.

- Proof

First, one proves the following fact.

Lemma. Let u be a generalized eigenfunction of $\mathcal{L}$ which is a polynomial of degree $n \geqslant 0$. Then the corresponding eigenvalue is $n\lambda$.

Proof. Let u be a generalized eigenfunction of $\mathcal{L}$, that is, $(\mathcal{L} - \mu)^k\, u = 0$ for some $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$.
We shall prove that $u \in \mathrm{H}_n$ (space of Hermite polynomials of degree n in suitable coordinates).

By the classical Hermite expansion,

$$u = \sum_j u_j,$$

with $u_j \in \mathrm{H}_j$ and convergence in $L^2(\gamma_\infty)$. This sum is finite. Then

$$u = \sum_j u_j \implies \sum_j (\mathcal{L} - \mu)^k u_j = 0$$

Since H_j is invariant under $\mathcal{L}$, each term $(\mathcal{L} - \mu)^k u_j \in \mathrm{H}_j$.
Thus all the terms are 0, that is, for all j

$$(\mathcal{L} - \mu)^k u_j = 0.$$

But this is compatible with the lemma only if there is only one nonzero term in the decomposition of u. Since u is a polynomial of degree n, $u \in \mathrm{H}_n$.
By the lemma two generalized eigenfunctions with different eigenvalues are of different degrees and thus belong to different H_n.
The desired orthogonality now follows from that of the H_n. □

The case when B has two distinct eigenvalues

Example 1

Generalized eigenspaces of the Ornstein–Uhlenbeck operator may be orthogonal even in the case when B has more than one eigenvalue. In two dimensions, we let

$$Q = I_2 \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

whose eigenvalues are $-1 \pm i$.

Proposition 3 (C.-Ciatti-Sjögren) With $N = 2$, let Q and B be as above. Then each generalized eigenfunction of $\mathcal{L}$ is an eigenfunction. Moreover, any two eigenfunctions of $\mathcal{L}$ with different eigenvalues are orthogonal with respect to γ_∞.

The case when B has has two distinct eigenvalues

Example 2

We exhibit a class of drift matrices B with two different eigenvalues (which, in contrast with those in the previous example, are real), but such that the generalized eigenspaces associated to the corresponding Ornstein–Uhlenbeck operator $\mathcal{L}$ are not orthogonal.
In $\mathbb{R}^2$ we consider $Q = I_2$ and

$$B = \begin{pmatrix} -\lambda & 0 \\ c & -\mu \end{pmatrix},$$

with $\lambda, \mu > 0$, $\lambda \neq \mu$, and $c \neq 0$.
We rewrite B as

$$B = \begin{pmatrix} -a+d & 0 \\ c & -a-d \end{pmatrix},$$

with $a > d > 0$ and $c \neq 0$.

The invariant measure γ_∞ is thus proportional to

$$\exp\left(-(a-d)\,x_1^2\right)\ \exp\left(-\frac{a+d}{c^2+4a^2}\,(cx_1-2ax_2)^2\right)dx.$$

To find some eigenfunctions of $\mathcal{L}$, we consider polynomials in x_1, x_2 of degree 2. One finds that

$$\begin{aligned} v_1 &= x_1^2 - \frac{1}{2(a-d)}, \\ v_2 &= x_1^2 - \frac{2d}{c}\,x_1x_2 - \frac{1}{2a}, \\ v_3 &= x_1^2 - \frac{4d}{c}\,x_1x_2 + \frac{4d^2}{c^2}\,x_2^2 - \frac{c^2+4d^2}{2c^2(a+d)} \end{aligned}$$

are eigenfunctions, with eigenvalues $-2(a-d)$, $-2a$ and $-2(a+d)$, respectively.

Any two of these polynomials turn out not to be orthogonal with respect to the invariant measure (straightforward computations).

Remark. Given $B = \begin{pmatrix} -\lambda & 0 \\ c & -\mu \end{pmatrix}$, it is easily seen that the eigenspaces corresponding to $-\lambda, -\mu$ are not orthogonal in $\mathbb{R}^2$. This turns out to be related to the non-orthogonality of the eigenspaces of $\mathcal{L}$, at least in dim. 2.

Proposition 5 (C.-Ciatti-Sjögren) Let $N = 2$ and $Q = I$, and assume that B has two different, real eigenvalues. Then the generalized eigenspaces of $\mathcal{L}$ are orthogonal in $L^2(d\gamma_\infty)$ if and only if the two eigenspaces of B are orthogonal in $\mathbb{R}^2$.

The “if" part easily extends to arbitrary dimension N.

Proposition 6 (C.-Ciatti-Sjögren) Let $Q = I$, and assume that B has N different, real eigenvalues, with mutually orthogonal eigenspaces. Then the generalized eigenspaces of $\mathcal{L}$ are orthogonal in $L^2(d\gamma_\infty)$.