Some harmonic analysis in a general Gaussian setting – Lecture 3

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Plan of the lectures

1. First lecture. Some motivations; classical vs nonsymmetric case; spectrum and Mehler kernel for the classical O.U. semigroup.
2. Second lecture. Spectrum and Mehler kernel for a general, nonsymmetric O.U. semigroup; orthogonality of eigenspaces.
3. Third lecture. Discussion of a problem from harmonic analysis: functional calculus in the nonsymmetric context.

Third lecture

Just a remark. Between the classical case and the general one there are some intermediate cases. For instance,

- one could consider the case $Q = I$ and $B = -\lambda I$, $\lambda > 0$;
- one could consider the symmetric framework (that is, the case when any operator in $(\mathcal{H}_t)_{t>0}$ is symmetric in $L^2(d\gamma_\infty)$. This is true if and only if $QB^* = BQ$ [1] (sparse results).
- Mauceri-Noselli[2] and then C.-Ciatti-Sjögren[3] studied the normal framework (where any operator in $(\mathcal{H}_t)_{t>0}$ is normal).

Anyway, these intermediate cases are often troublesome, so we might as well study the general case directly.

[1] A. Chojnowska-Michalik – B. Goldys, Symmetric Ornstein-Uhlenbeck semigroups and their generators, *Probab. Theory Related Fields* (2002)

[2] – G. Mauceri and L. Noselli, The maximal operator associated to a non symmetric Ornstein-Uhlenbeck semigroup, *J. Fourier Anal. Appl.* (2009)
–Riesz transforms for a non symmetric Ornstein-Uhlenbeck semigroup, *Semigroup Forum* (2008)

[3] C.-Ciatti-Sjögren, The maximal operator of a normal Ornstein–Uhlenbeck semigroup is of weak type $(1, 1)$, *Annali SNS Pisa, (2020)*

Recently, in collaboration with P. Ciatti and P. Sjögren, we started a program concerning harmonic analysis in a general, nonsymmetric Ornstein–Uhlenbeck setting. We approached different problems (maximal operators, Riesz transforms, multipliers, variational bounds). In this third lecture we shall try to explain our general strategy, focusing on a particular problem: the multiplier theorem.

For the sake of simplicity, we shall denote $\mathcal{L}^{Q,B}$ by $\mathcal{L}$.

We are interested in $m(\mathcal{L})$. We focus on multipliers of Laplace transform type. This class of multipliers was introduced some fifty years ago by E. M. Stein, in the context of the Littlewood–Paley theory for a sublaplacian on a connected Lie group G.

Def.. A function m of a real variable $\lambda > 0$ is said to be of Laplace transform type if

$$m(\lambda) = \lambda \int_0^{+\infty} \varphi(t) e^{-t\lambda}\, dt = -\int_0^{+\infty} \varphi(t)\, \frac{d}{dt}\, e^{-t\lambda}\, dt, \qquad \lambda > 0,$$

for some $\varphi \in L^\infty(0, +\infty)$. Observe that *such a function m can be extended to an analytic function in the half-plane* $\operatorname{Re} z > 0$.

Example: By choosing $\varphi(t) = \mathrm{const.}\, t^{-i\gamma}$, we obtain $m(\lambda) = \lambda^{i\gamma}$, with $\gamma \in \mathbb{R}\setminus\{0\}$. For other examples see

- B.Wróbel, Laplace Type Multipliers for Laguerre Expansions of Hermite Type, Mediterr. J. Math. (2013).

If $\mathcal{L}$ is a self-adjoint operator on $L^2(\mathbb{R}^N, d\gamma_\infty)$, and if E denotes a spectral resolution of $\mathcal{L}$ on $\mathbb{R}$, one can define $m(\mathcal{L})$ (for many functions m) as

$$m(\mathcal{L}) = \int_{\mathbb{R}} m(\lambda)\, dE(\lambda).$$

Problem: to find minimal assumptions on the multiplier m that will ensure the boundedness of $m(\mathcal{L})$ on $L^p(\mathbb{R}^N, d\gamma_\infty)$, both in a strong and in a weak sense, when $p \neq 2$.

This machinery may be applied, for instance, in the classical case, that is, when $Q = I$ and $B = -I$

In fact we recall that in this case the O. U. operator, given by $Lf = \frac{1}{2}\Delta f - \langle x, \nabla f \rangle$, $f \in \mathcal{S}(\mathbb{R}^N)$, is self-adjont; $\left(\mathcal{H}_t\right)_{t>0}$ is also symmetric.

In the classical context ($Q = -B = I$) strong boundedness of $m(L)$ on $L^p(\mathbb{R}^N, d\gamma_\infty)$, $1 < p < \infty$, follows from general results by E. M. Stein:

- E. M. Stein, Topics in harmonic analysis related to the Littlewood-Paley theory, Princeton University Press,

Other contributions in the symmetric case:

- M. Kemppainen, An L^1-estimate for certain spectral multipliers associated with the Ornstein–Uhlenbeck operator, *J. Fourier Anal. Appl.* (2016)
- —, Admissible decomposition for spectral multipliers on Gaussian L^p,*Math. Z.* (2018)

Weak type $(1, 1)$ for $m(L)$ was proved in

- J. García-Cuerva, G. Mauceri, P. Sjögren and J. Torrea, Spectral multipliers for the Ornstein-Uhlenbeck semigroup, *J. Anal. Math.* (1999)

The nonsymmetric case

We assume:

Q real, symmetric and positive definite $N \times N$ matrix;

B real $N \times N$ matrix whose eigenvalues have negative real parts.

In this context, in general $\mathcal{L}$ has no self-adjoint or normal extension to $L^2(\mathbb{R}^N, d\gamma_\infty))$ and one cannot invoke spectral theorem to define $m(\mathcal{L})$.
Self-adjointness and normality may fail also for the semigroup $(\mathcal{H}_t)_{t>0}$, generated by $\mathcal{L}$.

To define $m(\mathcal{L})$, we follow the method proposed by McIntosh (Operators which have an H^∞ functional calculus, Proc. Centre Math. Analysis, ANU, 1986) and by Cowling, Doust, McIntosh and Yagi (Banach space operators with a bounded H^∞ functional calculus, *J. Austral. Math. Soc.*, (1996))

They give a general definition for $m(T)$ when T is a linear operator acting in a Banach space, one-to-one, whose spectrum lies within some sector, and which satisfies certain resolvent bounds, and when m is holomorphic on a larger sector.

We choose $H = L^2_0(d\gamma_\infty)$ and $T = \mathcal{L}_{|L^2_0(d\gamma_\infty)}$.

In fact, 0 is an eigenvalue of $\mathcal{L}$ and $\ker \mathcal{L}$ consists of the constant functions. This eigenspace is orthogonal to all other generalized eigenfunctions of $\mathcal{L}$. We denote by $L^2_0(d\gamma_\infty)$ the orthogonal complement of $\ker \mathcal{L}$ in $L^2(d\gamma_\infty)$.

Our main contribution concerns the weak type $(1,1)$ of $m(\mathcal{L})$. We will prove this result with $\mathcal{L}$ replaced by T. The weak type $(1,1)$ of $m(\mathcal{L})$ then follows, since obviously $\mathcal{L}$ vanishes on $\ker \mathcal{L}$.

In this nonsymmetric context strong boundedness of $m(\mathcal{L})$ on $L^p(\mathbb{R}^N, d\gamma_\infty)$ for $1 < p < \infty$ follows, also for more general multipliers m, from

- A. Carbonaro and O. Dragičević, Bounded holomorphic functional calculus for nonsymmetric Ornstein-Uhlenbeck operators, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (2019)

Theorem (C.-Ciatti-Sjögren, `arXiv:2202.01547`)
If

- Q is a real, symmetric and positive definite $N \times N$ matrix
- B is a real $N \times N$ matrix whose eigenvalues have negative real parts,

and if the function m is of Laplace transform type, then the multiplier operator $m(\mathcal{L})$ associated to a general Ornstein–Uhlenbeck operator $\mathcal{L}$ is of weak type $(1, 1)$ with respect to the invariant measure $d\gamma_\infty$.

In other words, we proved the inequality

$$\gamma_\infty\{x \in \mathbb{R}^N : m(\mathcal{L})\, f(x) > \alpha\} \leqslant \frac{C}{\alpha}\, \|f\|_{L^1(\gamma_\infty)}, \qquad \alpha > 0,$$

for all functions $f \in L^1(\gamma_\infty)$, with $C = C(N, Q, B)$.

An idea of the proof. Recall that m satisfies

$$m(\lambda) = \lambda \int_0^{+\infty} \varphi(t) e^{-t\lambda}\, dt = -\int_0^{+\infty} \varphi(t) \frac{d}{dt} e^{-t\lambda}\, dt, \qquad \lambda > 0,$$

for some $\varphi \in L^\infty(0, +\infty)$.
Replacing λ by $\mathcal{L}$ we obtain

$$m(\mathcal{L}) = -\int_0^{+\infty} \varphi(t) \frac{d}{dt} e^{t\mathcal{L}}\, dt$$

The integral kernel of $\mathcal{H}_t$ is the Mehler kernel $K_t(x, u)$, that is, for each $f \in L^p(\mathbb{R}^N)$ and all $t > 0$

$$\mathcal{H}_t f(x) = \int K_t(x, u)\, f(u)\, d\gamma_\infty(u)\,.$$

This makes it plausible that the off-diagonal kernel of $m(\mathcal{L})$ is

$$\mathcal{M}_\varphi(x, u) = -\int_0^{+\infty} \varphi(t) \frac{d}{dt} K_t(x, u)\, dt.$$

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In the proof, first of all we verify this heuristic deduction of the kernel of $m(\mathcal{L})$, proving that the off-diagonal kernel of $m(\mathcal{L})$ is

$$\mathcal{M}_\varphi(x,u) = -\int_0^{+\infty} \varphi(t)\, \frac{d}{dt}\, K_t(x,u)\, dt,$$

where

$$K_t(x,u) = \Big(\frac{\det Q_\infty}{\det Q_t}\Big)^{1/2} \exp\Big(\frac{1}{2}\big\langle Q_\infty^{-1}x, x\big\rangle\Big) \\ \times \exp\Big[-\big\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t x)\,,\, u - D_t x\big\rangle\Big]$$

for $x, u \in \mathbb{R}^N$. Here

$$D_t x = Q_\infty e^{-tB^*} Q_\infty^{-1} x, \qquad t \in \mathbb{R} \text{ and } x \in \mathbb{R}^N.$$

Then we look for an explicit expression the kernel of $m(\mathcal{L})$, given by

$$\mathcal{M}_\varphi(x,u) = -\int_0^{+\infty} \varphi(t)\,\frac{d}{dt}\,K_t(x,u)\,dt,$$

When computing $\frac{d}{dt}\,K_t(x,u)$, we find bounds which are very different for small and large times. Thus, to prove the theorem, we distinguish between $t \geqslant 1$ and $t \leqslant 1$.

Remark. The same distinction between small and large times is necessary also in the study of maximal operators, Riesz transforms and variational bounds.

$t \geqslant 1$

We use a method developed by C.-Ciatti-Sjögren in the study of O.U. maximal operator, based on a suitable system of polar coordinates.
More precisely, any $x \in \mathbb{R}^N$, $x \neq 0$, can be written uniquely as

$$x = D_s\, \tilde{x},$$

for some $s \in \mathbb{R}$ and $\tilde{x}$ belonging to some ellipsoidal surface E_α (*recall that*

$$D_s x = Q_\infty e^{-sB^*} Q_\infty^{-1} x, \qquad s \in \mathbb{R} \textit{ and } x \in \mathbb{R}^N).$$

Here,

$$E_\alpha = \left\{ x \in \mathbb{R}^N : \langle Q_\infty^{-1} x\,,\, x \rangle = \log \alpha \right\}.$$

We call $\tilde{x}$ and s the *polar coordinates of* x.

We denote by $m_1(\mathcal{L})$ the part of the multiplier operator corresponding to $t \geqslant 1$, with kernel

$$\mathcal{M}_1(x,u) = -\int_1^{+\infty} \varphi(t)\, \frac{d}{dt}\, K_t(x,u)\, dt.$$

By writing $x = D_s\, \tilde{x}$ and $u = D_\sigma\, \tilde{u}$, for some $s, \sigma \in \mathbb{R}$ and $\tilde{x}, \tilde{u} \in E_\alpha$, by means of some estimates we prove that

$$\left|\mathcal{M}_1(x,u)\right| \lesssim e^{\frac{1}{2}\left\langle Q_\infty^{-1}\, x, x\right\rangle} \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right).$$

Lemma (C.-Ciatti-Sjögren, 2020)

Let $f \geqslant 0$ be normalized in $L^1(\gamma_\infty)$. For $\alpha > 2$

$$\gamma_\infty\Big\{x = D_s\,\tilde{x} :\ e^{\frac{1}{2}\left\langle Q_\infty^{-1}\, x, x\right\rangle}\int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u) > \alpha\Big\} \lesssim \frac{C}{\alpha\sqrt{\log \alpha}}.$$

The estimate in this lemma means that for large α one has a slightly stronger estimate than the classical weak type $(1,1)$ bound. In fact, by means of the lemma, we obtain for $t \geqslant 1$

$$\gamma_\infty\{x \in \mathbb{R}^N : m_1(\mathcal{L})\, f(x) > \alpha\} \leqslant \frac{C}{\alpha\sqrt{\log \alpha}}\, \|f\|_{L^1(\gamma_\infty)}, \qquad \alpha > 0,$$

for all functions $f \in L^1(\gamma_\infty)$, with $C = C(n, Q, B)$.
This stronger estimate does not hold for the full multiplier operator.

This phenomenon (a stronger bound for $t \geqslant 1$) was already observed for the Ornstein–Uhlenbeck maximal operator, for the first-order Riesz transforms and for variational bounds for the Ornstein–Uhlenbeck semigroup in previous works by C.-Ciatti-Sjögren.

$t \leqslant 1$ The multiplier operator corresponding to small values of t is more involved and requires a further distinction between local and global regions, according that the Mehler kernel $K_t(x,u)$ lives close to or away from the diagonal $\{(x,u) \in \mathbb{R}^N \times \mathbb{R}^N : x = u\}$. We only recall that the decomposition in local and global parts in a Gaussian context was used by B. Muckenhoupt in

- B. Muckenhoupt, Hermite conjugate expansions, *Trans. Amer. Math. Soc.* **139** (1969), 243–260.

The local part is the region where the Mehler kernel $K_t(x,u)$ is close to the diagonal $\{(x,u) \in \mathbb{R}^N \times \mathbb{R}^N : x = u\}$.
The idea is that the local part of $m(\mathcal{L})$ behaves precisely as classical operators on Lebesgue spaces, since in the local region $d\gamma_\infty \simeq dx$.
This is true for many operators related to Ornstein–Uhlenbeck operator (like maximal operators, Riesz transforms of any order, variation operators,...) and was verified by S. Pérez in

- S. Pérez, The local part and the strong type for operators related to the Gaussian measure, *J. Geom. Anal.* **11**, 491–507 (2001)

Bearing in mind this idea, we prove the weak type $(1,1)$ by means of standard Calderón–Zygmund techniques.

The remaining, global part is more delicate (and much more technical). For its kernel, that is, for

$$m_0(\mathcal{L}) = -\int_0^1 \varphi(t)\, \frac{d}{dt}\, K_t(x,u)\, dt, \quad (\varphi \in L^\infty(0,+\infty))$$

with (x,u) far from the diagonal, we have a bound

$$\left|m_0(\mathcal{L})\right| \leqslant \int_0^1 \left|\frac{d}{dt}\, K_t(x,u)\right| dt \leqslant \sum \left|\int \frac{d}{dt}\, K_t(x,u)\, dt\right|,$$

where the integrals in the sum are taken between consecutive zeros of $\frac{d}{dt}\, K_t$.
Therefore, we need an estimate of the number of zeros of $\frac{d}{dt}\, K_t(x,u)$ as t runs through the interval $(0,1]$.

Proposition (C.-Ciatti-Sjögren)

For $(x, u) \in \mathbb{R}^N \times \mathbb{R}^N$, the number of zeros in $I = (0, 1]$ of the function $t \mapsto \frac{d}{dt} K_t(x, u)$ is bounded by a positive integer depending only on N and B.

The proof is quite long and divided in several steps.