FACULTY OF ENGINEERING
AND ARCHITECTURE

Inverse Source Problems for Space-Dependent Sources in Thermoelastic Systems

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Modern Problems in PDEs and Applications

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Thermoelasticity

- **Isotropic thermoelastic system** (of type-III) for $\boldsymbol{u}$ (displacement) and θ (temperature)
- $\Omega \subset \mathbb{R}^d$ open and bounded domain with Lipschitz continuous boundary Γ,
  $Q_T = \Omega \times (0, T)$, $\Sigma_T = \Gamma \times (0, T)$.

$$\begin{cases} \rho \partial_{tt} \boldsymbol{u} - \mu \Delta \boldsymbol{u} - (\lambda + \mu) \nabla (\nabla \cdot \boldsymbol{u}) + \gamma \nabla \theta = \boldsymbol{p} & \text{in } Q_T \\ \rho C_s \partial_t \theta - \kappa \Delta \theta - (k * \Delta \theta) + T_0 \gamma \nabla \cdot \partial_t \boldsymbol{u} = h & \text{in } Q_T \end{cases} \qquad (\bigstar)$$

- Convolution product

$$(k * \theta)(\mathbf{x}, t) = \int_0^t k(t - s) \theta(\mathbf{x}, s) \, \mathrm{d}s, \quad (\mathbf{x}, t) \in Q_T$$

with (typically) $k(t) = a \exp(-bt)$, $a, b > 0$.

Biot, M. A. Thermoelasticity and irreversible thermodynamics. *J. Appl. Phys. 27(3):240-253, 1956.*

Inverse Source Problems

- $\boxed{\textbf{ISP 1.}}$ Let $\boldsymbol{p}(\mathbf{x},t) = g(t)\boldsymbol{f}(\mathbf{x}) + \widetilde{\boldsymbol{p}}(\mathbf{x},t)$ with $g(t)$ and $\widetilde{\boldsymbol{p}}(\mathbf{x},t)$ given. Find $\boldsymbol{f}(\mathbf{x})$ from the final time measurement

$$\boldsymbol{u}(\mathbf{x},T) = \boldsymbol{\xi}_T(\mathbf{x}), \quad \mathbf{x} \in \Omega, \qquad (M_1)$$

or from the time-average measurement

$$\int_0^T \boldsymbol{u}(\mathbf{x},t)\,\mathrm{d}t = \boldsymbol{\xi}_T(\mathbf{x}), \quad \mathbf{x} \in \Omega. \qquad (M_2)$$

- $\boxed{\textbf{ISP 2.}}$ Let $h(\mathbf{x},t) = g(t)f(\mathbf{x}) + \widetilde{h}(\mathbf{x},t)$ with $g(t)$ and $\widetilde{h}(\mathbf{x},t)$ given. Find $f(\mathbf{x})$ from the time-average measurement

$$\int_0^T \theta(\mathbf{x},t)\,\mathrm{d}t = \psi_T(\mathbf{x}), \quad \mathbf{x} \in \Omega. \qquad (M_3)$$

Well-posedness forward problem

Let $k \in C^2([0,T])$ be strongly positive definite,

(i) if $\mathbf{p} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right)$, $h \in L^2\left((0,T), L^2(\Omega)\right)$, $\boldsymbol{U}_0 \in \mathbf{H}_0^1(\Omega)$, $\boldsymbol{U}_1 \in \mathbf{L}^2(\Omega)$ and $V_0 \in L^2(\Omega)$, then there is a unique weak solution $(\mathbf{u}, \theta)$ to $(\bigstar)$

$$\mathbf{u} \in C\left([0,T], \mathbf{L}^2(\Omega)\right) \cap L^2\left((0,T), \mathbf{H}_0^1(\Omega)\right), \quad \partial_t \mathbf{u} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right),$$
$$\partial_{tt}\mathbf{u} \in L^2\left((0,T), \mathbf{H}_0^1(\Omega)^*\right),$$
$$\theta \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left((0,T), H_0^1(\Omega)\right), \quad \partial_t \theta \in L^2\left((0,T), H_0^1(\Omega)^*\right)$$

(ii) if $\mathbf{p} \in H^1\left((0,T), \mathbf{L}^2(\Omega)\right)$, $h \in H^1\left((0,T), L^2(\Omega)\right)$, $\boldsymbol{U}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, $\boldsymbol{U}_1 \in \mathbf{H}_0^1(\Omega)$ and $V_0 \in H_0^1(\Omega)$ and the PDEs are satisfied for $t = 0$, then there is a unique weak solution $(\mathbf{u}, \theta)$ to $(\bigstar)$

$$\mathbf{u} \in C\left([0,T], \mathbf{H}_0^1(\Omega)\right), \quad \partial_t \mathbf{u} \in C\left([0,T], \mathbf{L}^2(\Omega)\right) \cap L^2\left((0,T), \mathbf{H}_0^1(\Omega)\right),$$
$$\partial_{tt}\mathbf{u} \in L^2\left((0,T), L^2(\Omega)\right),$$
$$\theta \in C\left([0,T], H_0^1(\Omega)\right), \quad \partial_t \theta \in L^2\left((0,T), H_0^1(\Omega)\right)$$

ISP 1

- Final time measurement $\mathbf{u}(\mathbf{x}, T) = \xi_T(\mathbf{x})$

Theorem (ISP1 with measurement M_1)

Under conditions (ii), assume $g \in \mathrm{C}^1([0,T])$ *fulfills*

$$g(t) > 0 \quad \textit{and} \quad g'(t) > 0, \quad \forall t \in [0,T]$$

(or $g(t) < 0, g'(t) < 0$*). Given* $\xi_T \in \mathbf{L}^2(\Omega)$, *then there exists at most one triple*

$$(\mathbf{u}, \theta, \boldsymbol{f}) \in \mathrm{C}\left([0,T], \mathbf{H}_0^1(\Omega)\right) \times \mathrm{C}\left([0,T], \mathrm{H}_0^1(\Omega)\right) \times \mathbf{L}^2(\Omega)$$

with $\partial_t \boldsymbol{u} \in \mathrm{C}\left([0,T], \mathbf{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T), \mathbf{H}_0^1(\Omega)\right)$ *solving* $(\bigstar)$ *for which* M_1 *holds.*

Van Bockstal, K., Slodička, M. Recovery of a space-dependent vector source in thermoelastic systems. *Inverse Probl. Sci. Eng. 23(6):956–968, 2015*

Van Bockstal, K., Marin, L. Recovery of a space-dependent vector source in anisotropic thermoelastic systems. *Comput. Methods Appl. Mech. Eng. 321:269–293, 2017*

ISP 1

- Time-average measurement $\int_0^T \mathbf{u}(\mathbf{x}, t)\, \mathrm{d}t = \boldsymbol{\xi}_T(\mathbf{x})$.

Theorem (ISP1 with measurement M_2)

Under conditions (i), assume that $g \in \mathrm{C}([0, T])$ fulfills $g(t) > 0$ or $g(t) < 0$. Given $\boldsymbol{\xi}_T \in \mathbf{L}^2(\Omega)$, then there exists at most one triple

$$(\mathbf{u}, \theta, \boldsymbol{f}) \in \mathrm{L}^2\left((0, T), \mathbf{H}_0^1(\Omega)\right) \times \mathrm{L}^2\left((0, T), \mathrm{H}_0^1(\Omega)\right) \times \mathbf{L}^2(\Omega)$$

solving ($\bigstar$) for which M_2 holds.

ISP 2

- Time-average measurement $\int_0^T \theta(\mathbf{x}, t)\,\mathrm{d}t = \psi_T(\mathbf{x})$

Theorem (ISP2 with measurement M_3)

Under conditions (ii), assume that $g \in \mathrm{C}^1([0, T])$ satisfies either

$$g(t) > 0 \quad \textit{and} \quad g'(t) \geq 0, \quad \textit{for all } t \in [0, T]$$

(or $g(t) < 0, g'(t) \leq 0$). Given $\psi \in \mathrm{L}^2(\Omega)$, then there exists at most one triple

$$(\mathbf{u}, \theta, f) \in \mathrm{C}\left([0, T], \mathbf{H}_0^1(\Omega)\right) \times \mathrm{C}\left([0, T], \mathrm{H}_0^1(\Omega)\right) \times \mathrm{L}^2(\Omega)$$

with $\partial_t \mathbf{u} \in \mathrm{C}\left([0, T], \mathbf{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0, T), \mathbf{H}_0^1(\Omega)\right)$ solving ($\bigstar$) for which M_3 holds.

Combined problem

Source functions

$$\boldsymbol{p}(\mathbf{x}, t) = g_{\mathbf{u}}(t)\boldsymbol{f}_{\mathbf{u}}(\mathbf{x}) + \widetilde{\boldsymbol{p}}(\mathbf{x}, t)$$
$$h(\mathbf{x}, t) = g_{\theta}(t) f_{\theta}(\mathbf{x}) + \widetilde{h}(\mathbf{x}, t)$$

- **Extension 1.** Find $\boldsymbol{f}_{\mathbf{u}}$ and f_θ simultaneously from the measurements M_1, M_3. Here we use additional assumption that $g_{\mathbf{u}}(t) = c g_\theta(t)$ for some $c > 0$.
- **Extension 2.** Find $\boldsymbol{f}_{\mathbf{u}}$ and f_θ simultaneously from the measurements M_2, M_3 and the additional measurement
$$\widetilde{\psi}_T(\mathbf{x}) = \int_0^T t\theta(\mathbf{x}, t)\,\mathrm{d}t. \tag{M_4}$$

Hào, D. N., Thanh, P. X., Lesnic, D., Ivanchov, M. Determination of a source in the heat equation form integral observations. *J. Comput. Appl. Math. 264:82–98, 2014*

Extension 1

- Measurements $\mathbf{u}(\mathbf{x}, T) = \xi_T(\mathbf{x})$ and $\int_0^T \theta(\mathbf{x}, t)\,\mathrm{d}t = \psi_T(\mathbf{x})$

Theorem

Under conditions (ii), assume that $g_{\mathbf{u}}, g_\theta \in \mathrm{C}^1([0, T])$ satisfy $g_{\mathbf{u}} = c g_\theta$ for some $c > 0$ and the monotonicity condition

$$g_\theta(t) > 0 \quad \text{and} \quad g_\theta'(t) > 0, \qquad \text{for all } t \in [0, T].$$

Given $\xi_T \in \mathbf{L}^2(\Omega)$ and $\psi_T \in \mathrm{L}^2(\Omega)$, then there exists at most one solution

$$(\mathbf{u}, \theta, \mathbf{f}_{\mathbf{u}}, f_\theta) \in \mathrm{C}\left([0, T], \mathbf{H}_0^1(\Omega)\right) \times \mathrm{C}\left([0, T], \mathrm{H}_0^1(\Omega)\right) \times \mathbf{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$$

to ($\bigstar$), with $\partial_t \mathbf{u} \in \mathrm{C}\left([0, T], \mathbf{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0, T), \mathbf{H}_0^1(\Omega)\right)$ for which M_1 and M_3 hold.

Extension 2

- Measurements $\int_0^T \mathbf{u}(\mathbf{x},t)\,\mathrm{d}t = \boldsymbol{\xi}_T(\mathbf{x})$, $\int_0^T \theta(\mathbf{x},t)\,\mathrm{d}t = \psi_T(\mathbf{x})$ and $\int_0^T t\theta(\mathbf{x},t)\,\mathrm{d}t = \widetilde{\psi}_T(\mathbf{x})$

Theorem

Under the conditions (i), assume that $g_{\mathbf{u}}, g_\theta \in \mathrm{C}(]0,T])$ satisfy $g_{\mathbf{u}} = c g_\theta$ for some $c > 0$. Given $\boldsymbol{\xi}_T \in \mathbf{L}^2(\Omega)$ and $\psi_T, \widetilde{\psi}_T \in \mathrm{L}^2(\Omega)$, then there exists at most one solution

$$(\mathbf{u}, \theta, \boldsymbol{f}_{\mathbf{u}}, f_\theta) \in \mathrm{L}^2\left((0,T), \mathbf{H}_0^1(\Omega)\right) \times \mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)\right) \times \mathbf{L}^2(\Omega) \times \mathrm{L}^2(\Omega),$$

to ($\bigstar$) for which M_2, M_3 and M_4 hold.

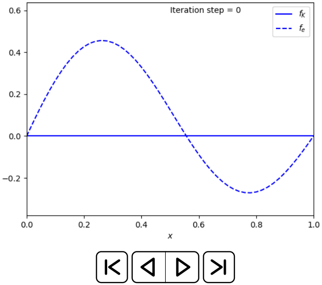
Numerical reconstruction

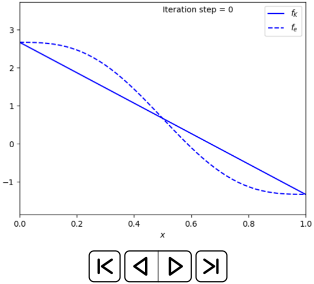
- Implementation of **ISP2** with measurement M_3, $\psi_T(\mathbf{x}) = \int_0^T \theta(\mathbf{x}, t)\,\mathrm{d}t$
- **Goal**: find the unknown $f(\mathbf{x})$ in the source term

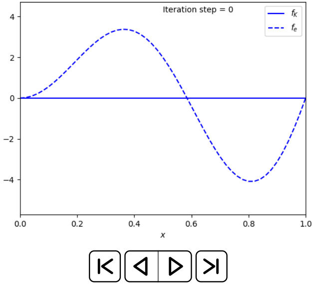
  $$h(\mathbf{x}, t) = g(t)f(\mathbf{x}) + r(\mathbf{x}, t) \quad \text{with } g, r \text{ known}$$

  under the time-average measurement ψ_T.
- FEniCSx computing platform (open-source) `https://fenicsproject.org/`
- Docker image with a Jupyter Lab environment with the latest stable release of DOLFINx `https://github.com/FEniCS/dolfinx#installation`

Numerical reconstruction







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