Dynamics of concentrated vorticities in 2 and 3 dimensional Euler flows

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The **vorticity-stream formulation**

$$\begin{cases} \omega_t + \nabla^\perp \psi \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi = (-\Delta)^{-1} \omega & \text{in } \mathbb{R}^2 \times (0, T), \\ \omega(\cdot, 0) = \omega_0 & \text{in } \mathbb{R}^2 \end{cases} \qquad (V2)$$

$$\psi = (-\Delta)^{-1} \omega = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \, \omega(y,t) dy \, dy.$$

The velocity field is (Biot-Savart Law)

$$u(x,t) = \nabla^\perp \psi = \int_{\mathbb{R}^2} k(x-y) \omega(y,t) \, dy, \quad k(z) = \frac{z^\perp}{2\pi |z|^2}$$

Fact: There exists a unique, globally defined weak solution of (V2) for any bounded, integrable initial datum. Solution are class C^k is the initial datum is.

- Conservation L^p-norm. Test (V2) against $p|\omega|^{p-2}\omega$.

$$\partial_t|\omega|^p + u\cdot\nabla|\omega|^p = 0 \implies$$
$$\partial_t\int_{\mathbb{R}^2}|\omega|^p\,dx = -\int_{\mathbb{R}^2}u\cdot\nabla|\omega|^p\,dx = \int_{\mathbb{R}^2}(\nabla\cdot u)|\omega|^p\,dx = 0.$$

Hence

$$\|\omega(\cdot,t)\|_{L^p(\mathbb{R}^2)} = \|\omega_0\|_{L^p(\mathbb{R}^2)} \quad \text{for all} \quad 1\le p\le\infty.$$

In particular, mass is conserved in time on any component of the support of ω.

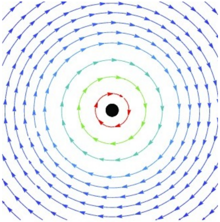
Radial functions are steady States of (V2)

$$u \cdot \nabla \omega = 0, \quad u = \nabla^{\perp}(-\Delta)^{-1}\omega.$$

- If $W \in L^{\infty}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ is radial: $W = W(|y|)$ then

$$\Psi(|y|) = (-\Delta)^{-1} W(y) = C + \int_0^{|y|} \frac{dr}{r} \int_0^r W(s) s ds.$$

$\omega = W(|y|)$ is a steady state since $\nabla^{\perp}\Psi \cdot \nabla W = 0$.

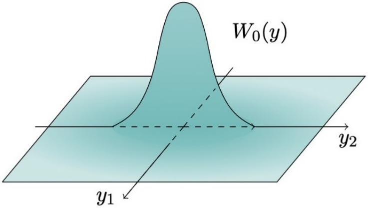


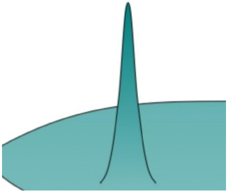
Radial vortex steady state

- Example: The *Kaufmann-Scully vortex*.

$$W_0(y) = \frac{8}{(1+|y|^2)^2}, \quad \Psi_0(y) = \log \frac{8}{(1+|y|^2)^2}.$$

It satisfies $\int_{\mathbb{R}^2} W_0 = 8\pi$, $-\Delta_y \Psi_0 = e^{\Psi_0} = W$.





$$W_\varepsilon(x) = \frac{\kappa}{\varepsilon^2} W_0\left(\frac{x-\xi}{\varepsilon}\right), \quad \Psi_\varepsilon(x) = \kappa \Psi_0\left(\frac{x-\xi}{\varepsilon}\right) - 4\log\varepsilon.$$

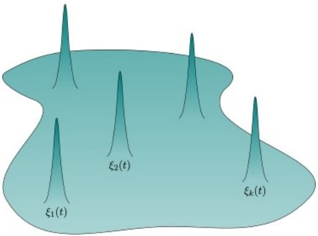
Steady state of (V2) with concentrated vorticity.

$$W_\varepsilon(x) \rightharpoonup 8\pi\kappa\delta(x-\xi), \quad \Psi_\varepsilon(x) \rightharpoonup 4\kappa \log\frac{1}{|x-\xi|} \quad \text{as } \varepsilon \to 0.$$

Solutions with highly concentrated vorticies around more than one point? Superposition:

$$W_\varepsilon = \sum_{j=1}^{N} \frac{\kappa_j}{\varepsilon^2} W_0\left(\frac{x-\xi_j}{\varepsilon}\right), \quad \Psi_\varepsilon = \sum_{j=1}^{N} \kappa_j \Psi_0\left(\frac{x-\xi_j}{\varepsilon}\right) - 4\kappa_j \log\varepsilon.$$

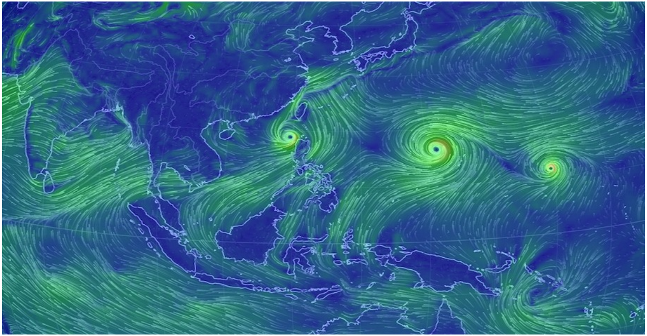
NOT a steady state. Higher hopes if $\xi_j = \xi_j(t)$.



Dynamics of the centers $\xi_j(t)$ if a solution with that shape exists?

Analysis of solutions with highly concentrated vorticities:

A mathematical subject with a long history: it traces back to Helmholtz (1858), Kirchhoff (1876), Routh (1881), Lagally (1921) C.C. Lin (1941). $\xi(t) = (\xi_1(t), \ldots, \xi_N(t))$ centers.



$$W_\varepsilon^\xi(x, t;) = \sum_{j=1}^{N} \frac{\kappa_j}{\varepsilon^2} W_0\left(\frac{x - \xi_j(t)}{\varepsilon}\right), \quad \Psi_\varepsilon^\xi(x, t) = \sum_{j=1}^{N} \kappa_j \Psi_0\left(\frac{x - \xi_j(t)}{\varepsilon}\right) - 4\kappa_j \log \varepsilon.$$

$$W_\varepsilon^\xi(x,t;)=\sum_{j=1}^N \frac{\kappa_j}{\varepsilon^2} W_0\left(\frac{x-\xi_j(t)}{\varepsilon}\right), \quad \Psi_\varepsilon^\xi(x,t)=\sum_{j=1}^N \kappa_j \log \frac{8}{(\varepsilon^2+|x-\xi_j|^2)^2}.$$

Computation of error of approximation:

$$\varepsilon^4\left[\partial_t W_\varepsilon^\xi + \nabla^\perp \Psi_\varepsilon^\xi \cdot \nabla W_\varepsilon^\xi\right] = -\sum_{j=1}^N \varepsilon\kappa_j \nabla W_0\left(\frac{x-\xi_j}{\varepsilon}\right)\cdot \dot{\xi}_j$$

$$+\sum_{i,j=1}^N \varepsilon\kappa_i\kappa_j \nabla^\perp \Psi_0\left(\frac{x-\xi_i}{\varepsilon}\right)\cdot \nabla W_0\left(\frac{x-\xi_j}{\varepsilon}\right)$$

$$= -\sum_{j=1}^N \varepsilon\kappa_j \underbrace{[\dot{\xi}_j + \nabla_x^\perp(\sum_{i\neq j} 2\kappa_i \log(\varepsilon^2+|x-\xi_i|^2))]}_{error\ \ reduced\ \ if\ \ imposed\ =0\ at\ \ x=\xi_j,\ \varepsilon=0} \cdot \nabla W_0\left(\frac{x-\xi_j}{\varepsilon}\right).$$

Thus W_ε^ξ will be a good approximate solution of (V2) if $(\xi_1, \dots, \xi_N)$ solves the N-body problem

$$\dot{\xi}_j(t) = \sum_{i \neq j} 4\kappa_i \frac{(\xi_i(t) - \xi_j(t))^\perp}{|\xi_i(t) - \xi_j(t)|^2}, \quad j = 1, \dots, N. \qquad (K)$$

The Kirchhoff-Routh system (1876-1881).

(K) is the equation satisfied by formal "vortex solutions"

$$\omega_*(x,t) = \sum_{j=1}^N 8\pi\kappa_j \delta(x - \xi_j(t)), \quad \Psi_*(x,t) = \sum_{j=1}^N 4\kappa_j \log \frac{1}{|x - \xi_j(t)|}$$

$$\partial_t \omega_* + \nabla^\perp \Psi_* \cdot \nabla \omega_* =$$

$$-\sum_{j=1}^N \kappa_j \nabla\delta(x - \xi_j) \cdot \dot{\xi}_j + \sum_{j=1}^N 4\kappa_j\kappa_i \sum_{i=1}^N \nabla\delta(x - \xi_j) \cdot \nabla^\perp \log \frac{1}{|x - \xi_i|}$$

$$= \sum_{j=1}^N \kappa_j \nabla\delta(x - \xi_j) \cdot [-\dot{\xi}_j + \sum_{i \neq j} 4\kappa_i \frac{(\xi_i - x)^\perp}{|x - \xi_i|^2}] = 0 \iff (K) \text{ holds.}$$

A Natural question:

The Vortex desingularization problem. Are there (true) smooth solutions of (V2) with vorticities highly concentrated around a finite set of points which evolve by a dynamics approximated by (K)?

• Marchioro and Pulvirenti (1993). Given a colissionless solution $\xi(t)$ of System (K) in $[0, T]$ and an initial condition with supports contained in balls of radius ε around the points $\xi_j(0)$, there is a solution ω_ε of (V2) with

$$\omega_\varepsilon(x,t) \rightharpoonup \omega_*(x,t) = \sum_{j=1}^{N} 8\pi\kappa_j \delta(x - \xi_i(t)),$$

in the distributional sense.

Their proof provides very little information on the behaviour of the solution or its velocity field near the vortices as ε becomes smaller

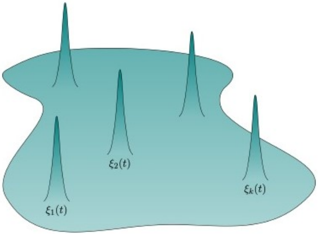
$$W_\varepsilon^\xi(x,t) = \sum_{j=1}^N \frac{\kappa_j}{\varepsilon^2} W_0\left(\frac{x-\xi_j(t)}{\varepsilon}\right), \quad W_0(y) = \frac{8}{(1+|y|^2)^2}$$

$$\Psi_\varepsilon^\xi(x,t) = \sum_{j=1}^N \kappa_j \log \frac{8}{(|x-\xi_j(t)|^2+\varepsilon^2)^2}$$

We prove that for a given colisionless solution of the system

$$\boxed{\dot\xi_j(t) = \sum_{i\neq j} 4\kappa_i \frac{(\xi_i(t)-\xi_j(t))^\perp}{|\xi_i(t)-\xi_j(t)|^2}, \quad j=1,\dots,N.} \qquad (K)$$

$\exists$ a solution $(\Psi_\varepsilon, \omega_\varepsilon)$ of (V2) that differs little from $(\Psi_\varepsilon^\xi, W_\varepsilon^\xi)$.



Theorem (Dávila, del Pino, Musso, Wei, ARMA 2020)

Let $T > 0$ *and* $\xi(t)$ *be a colisionless solution of* (K) *in* $[0, T]$*. There exists a solution* $(\omega_\varepsilon, \Psi_\varepsilon)$ *of Problem* (V2) *such that for all* $(x, t) \in \mathbb{R}^2 \times [0, T]$.

$$\omega_\varepsilon(x,t) = \sum_{j=1}^{k} \frac{\kappa_j}{\varepsilon^2} W_0\left(\frac{x-\xi_j}{\varepsilon}\right) + \phi(x,t)$$

$$\Psi_\varepsilon(x,t) = \sum_{j=1}^{k} \kappa_j \log \frac{1}{(\varepsilon^2 + |x-\xi_j|^2)^2} + \psi(x,t)$$

$$|\phi(x,t)| \ \le \ C_T \sum_{j=1}^{k} \frac{\varepsilon^2}{\varepsilon^2 + |x-\xi_j|^2},$$

$$|\psi(x,t)| + \sum_{j=1}^{k} (\varepsilon + |x-\xi_j|)|D_x \psi(x,t)| \ \le \ C_T \varepsilon^2.$$

In particular:

$$\omega_\varepsilon \rightharpoonup \sum_{j=1}^{k} 8\pi\kappa_j \delta(x-\xi_i) \quad \frac{1}{|\log\varepsilon|}|\nabla\Psi_\varepsilon|^2 \rightharpoonup \sum_{j=1}^{k} 8\pi\kappa_j^2 \delta(x-\xi_j).$$

Ingredients in our construction:

• Improvement of the approximation in powers of ε using elliptic and transport equations.

• Setting up the problem as a coupled system of inner problems near the singularities and and an outer problem more regular (the inner-outer gluing scheme)

• A priori estimates to solve by a continuation (degree) argument.

Improving the approximation. Let $\Psi_0(y) := \log \frac{8}{(1+|y|^2)^2}$,

$$\Psi_\varepsilon^\xi = \sum_{j=1}^{k} \kappa_j \log \frac{1}{(\varepsilon^2 + |x-\xi_j|^2)^2} = \sum_{j=1}^{k} \kappa_j \Psi_0\left(\frac{x-\xi_j}{\varepsilon}\right) - \kappa_j \log 8\varepsilon^4$$

We want to solve the equation $E(\omega, \Psi) = 0$, where

$$E(\omega, \Psi) := \omega_t + \nabla_x^\perp \Psi \cdot \nabla_x \omega, \quad -\Delta_x \Psi = \omega.$$

Near $\xi_j(t)$ write $y = \frac{x-\xi_j(t)}{\varepsilon}$. We look for a solution of the form

$$\Psi = \Psi_\varepsilon^\xi(x,t) + \kappa_j \psi(y,t), \quad \omega = \frac{\kappa_j}{\varepsilon^2} W_0(y) + \frac{\kappa_j}{\varepsilon^2} \phi(y,t).$$

In terms of the y-variable we get the expression

$$
\begin{aligned}
\varepsilon^4 E(\omega,\Psi) =& \varepsilon^2\phi_t \ + \ (-\varepsilon\dot{\xi} + \nabla_y^\perp \Psi_\varepsilon^\xi + \kappa_j \nabla_y^\perp \psi)\cdot\nabla_y(W_0+\phi),\\
-\Delta_y\psi =& \phi
\end{aligned}
$$

We have

$$
\Psi_\varepsilon^\xi(x,t) = \kappa_j \Psi_0(y) + \varphi(x) + O(\varepsilon^2) + constant, \quad y = \frac{x-\xi_j}{\varepsilon},
$$

$$
\varphi(x) = \sum_{i\neq j} \kappa_i \Gamma(x-\xi_i), \quad \Gamma(z) = 4\log\frac{1}{|z|}.
$$

By assumption $\dot{\xi}_j = \nabla_x^\perp \varphi(\xi_j)$, hence we get

$$
-\varepsilon\dot{\xi}_j + \nabla_y^\perp \Psi_\varepsilon^\xi(\xi_j+\varepsilon y) = \kappa_j \nabla^\perp(\Psi_0 + \mathcal{R})
$$

with $\mathcal{R} = O(\varepsilon^2|y|^2)$.

$$
\boxed{\begin{aligned}
\varepsilon^4 E(\omega,\Psi) =& \varepsilon^2\phi_t \; + \; \kappa_j \nabla_y^\perp(\Psi_0(y) + \mathcal{R} + \psi)\cdot\nabla_y(W_0+\phi),, \\
-\Delta_y\psi =& \phi \\
\mathcal{R} =& O(\varepsilon^2|y|^2)
\end{aligned}}
$$

Let $f(u) = e^u$. Since $W_0 = f(\Psi_0)$ we find

$$
\boxed{\begin{aligned}
\varepsilon^4 E(\omega,\Psi) =& \varepsilon^2\phi_t \; - \; \kappa_j \nabla_y^\perp \Psi_0 \cdot \nabla(\Delta\psi + f'(\Psi_0)\psi) \\
&+ \kappa_j \nabla^\perp\mathcal{R}\cdot\nabla W_0 + \kappa_j\nabla^\perp\mathcal{R}\nabla\phi + \nabla^\perp\psi\nabla\phi.
\end{aligned}}
$$

The 0-error term:

$$
\varepsilon^4 E(W_\varepsilon^\xi, \Psi_\varepsilon^\xi) = \nabla^\perp\mathcal{R}\cdot\nabla W_0 = O(\varepsilon^2|y|^{-4}).
$$

We obtain a reduction in the error by solving the elliptic equation

$$
-\nabla_y^\perp\Psi_0\cdot\nabla(\Delta\psi + W_0\psi) + \nabla^\perp\mathcal{R}\cdot\nabla W_0 = 0
$$

Polar coordinates $y = \rho e^{i\theta}$

$$\nabla^{\perp}\Psi_0 \cdot \nabla(\Delta\psi + W_0\psi) = E(y) \iff$$

$$\frac{1}{1+\rho^2}\frac{\partial}{\partial\theta}(\Delta\psi + W_0\psi) = E(y) = E_0(\rho) + \sum_{k\in\mathbb{Z}} E_k(\rho)e^{ik\theta}.$$

In order to solve for a bounded solution we need $E_0 = 0$ and the solvability condition (that sees only modes $k = \pm 1$)

$$\int_{\mathbb{R}^2} E(y)y_l dy = 0, \ \ l = 1, 2.$$

Main term in first error $\mathcal{R}$ only has mode two. Then ξ needs to be adjusted. to solve cubic term. We need to solve transport equations to get further decay far away.

$$\varepsilon^2\phi_t + \nabla\Psi_0(y)\cdot\nabla\phi = E(y,t), \quad \phi(y,0) = 0.$$

After sufficiently improving the approximation we solve the problem by a degree argument, which near each ξ_j roughly reads as

$$\boxed{\begin{aligned} \varepsilon^2\phi_t - \nabla^\perp\Psi_0\cdot\nabla(\Delta\psi + W_0\psi) + Q(\phi) + E(y,t) = 0 \\ -\Delta\psi = \phi \quad \text{in } \mathbb{R}^2\times[0,T] \end{aligned}}$$

with $E = O(\varepsilon^5\rho^{-3})$, $Q(\phi) = \nabla^\perp\psi\nabla\phi$, quadratic term.

A basic ingredient: A priori estimates for the linear problem

$$\begin{cases} \varepsilon^2\phi_t - \nabla^\perp\Psi_0 \cdot \nabla(\Delta\psi + W_0\psi) + E(y,t) = 0 \\ -\Delta\psi = \phi \quad \text{in } \mathbb{R}^2 \times [0,T], \quad \phi(y,0) = 0, \\ \int_{B(0,\delta\varepsilon^{-1})} y\phi dy = 0, \quad \int_{\mathbb{R}^2} \phi dy = 0 = \int_{\mathbb{R}^2} \phi \frac{1-2|y|^2}{1+|y|^2} W_0(y) dy. \end{cases}$$

Then the following estimate holds:

$$\|\phi(\cdot,t) W_0^{-\frac{1}{2}}\|_{L^2} \lesssim \varepsilon^{-2}|\log\varepsilon| \sup_{t\in[0,T]} \|E W_0^{-\frac{1}{2}}\|_{L^2}$$

This allows a fixed point scheme to work when $E = O(\varepsilon^5\rho^{-3})$.

.

$$\varepsilon^2\phi_t + \nabla^\perp\Psi_0\cdot\nabla(\Delta\psi + f'(\Psi_0)\psi) + E(y,t) = 0, \quad \phi(y,0) = 0$$

We use the test function $g = \frac{\phi}{W_0} - (-\Delta)^{-1}\phi$, so

$$W_0 g = -(\Delta\psi + f'(\Psi_0)\psi).$$

and

$$\varepsilon^2\partial_t\int_{\mathbb{R}^2}\phi g = \int_{\mathbb{R}^2} W_0^{-1}\nabla^\perp\Psi_0\nabla(W_0^2 g^2) + 2\int_{\mathbb{R}^2} Eg$$

The second integral is zero for

$$\int_{\mathbb{R}^2} W_0^{-1}\nabla^\perp\Psi_0\nabla(W_0^2 g^2) = -\int_{\mathbb{R}^2}\nabla\cdot(W_0^{-1}\nabla^\perp\Psi_0)\nabla(W_0^2 g^2)$$

and since Ψ_0 and W_0 are radial,

$$\nabla\cdot(W_0^{-1}\nabla^\perp\Psi_0) = 0.$$

Thus

$$\varepsilon^2 \partial_t \int_{\mathbb{R}^2} \phi g = 2 \int_{\mathbb{R}^2} Eg \le C \| E W_0^{-\frac{1}{2}} \|_{L_2} \| \| g W_0^{\frac{1}{2}} \|_{L^2}$$

and integrating,

$$\varepsilon^2 \int_{\mathbb{R}^2} \phi g(\cdot, t) \le \max_{t \in (0,T)} C \| E(\cdot, t) W_0^{-\frac{1}{2}} \|_{L_2} \| \| g(\cdot, t) W_0^{\frac{1}{2}} \|_{L^2}.$$

Under the orthogonality conditions assumed on ϕ we can prove the following Poincare inequality:

$$\frac{\gamma}{|\log \varepsilon|} \int_{\mathbb{R}^2} \phi^2 W_0^{-1} \le \int_{\mathbb{R}^2} \phi g$$

while we always have

$$\int_{\mathbb{R}^2} g^2 W_0 \le C \int_{\mathbb{R}^2} \phi^2 W_0^{-1}.$$

From these inequalities the desired estimate follows.

To prove the Poincare inequality

$$\frac{\gamma}{|\log \varepsilon|} \int_{\mathbb{R}^2} \phi^2 W_0^{-1} \leq \int_{\mathbb{R}^2} \phi g$$

we set $\tilde{\phi} = W_0^{-1}\phi$. Using stereographic projection we see that

$$\int_{S^2} \tilde{\phi}^2 = \int_{\mathbb{R}^2} \phi^2 W_0^{-1}, \quad \int_{S^2} \tilde{\phi} = \int_{\mathbb{R}^2} \phi = 0.$$

Besides

$$\int_{\mathbb{R}^2} \phi g = \int_{S^2} \tilde{\phi}(\tilde{\phi} - 2(-\Delta_{S^2})^{-1}\tilde{\phi}).$$

Expanding $\tilde{\phi}$ in the orthonormal basis in $L^2(S^2)$ of spherical harmonics we get

$$\tilde{\phi} = \sum_{j=0}^{\infty} \tilde{\phi}_j e_j(z) = \sum_{j=0}^{3} \tilde{\phi}_j e_j + \tilde{\phi}^{\perp},$$

where $-\Delta_{S^2} e_j = \lambda_j e_j$.

Here $\lambda_0 = 0$ and e_0 is constant, while $\lambda_1 = \lambda_2 = \lambda_3 = 2$, with $e_j(z) = z_j$. Thus $\tilde{\phi}_0 = 0$ and also $\tilde{\phi}_3 = 0$ because of our orthogonality condition:

$$\int_{\mathbb{R}^2} \phi(y,t)dy = \int_{\mathbb{R}^2} \phi(y,t) \frac{1-2|y|^2}{1+|y|^2} W_0(y)dy = 0,$$

$$\int_{\mathbb{R}^2} \phi g = \sum_{j=4}^{\infty} \left(1 - \frac{2}{\lambda_j}\right) \tilde{\phi}_j^2 \sim \|\tilde{\phi}^{\perp}\|^2_{L^2(S^2)}$$

We also have, $j = 2, 3$

$$0 = \int_{B_R} \phi y_j = c\tilde{\phi}_j + O(\|\tilde{\phi}^{\perp}\|_{L^2(S^2)}) |\log R|^{\frac{1}{2}}$$

with $R = \delta\varepsilon^{-1}$ which gives

$$\tilde{\phi}_j = O(\|\tilde{\phi}^{\perp}\|_{L^2(S^2)}) |\log \varepsilon|^{\frac{1}{2}}.$$

From here it follows that

$$\int_{\mathbb{R}^2} \phi g \geq \gamma |\log \varepsilon|^{-1} \int_{S^2} \tilde{\phi}^2$$

as we wanted.

The inner-outer gluing scheme

Let us assume $\kappa_j = 1$. We consider smooth cut-off functions

$$\eta_j(x,t) = \eta_0\left(\frac{|x-\xi_j(t)|}{\delta}\right)$$

where $\eta_0(s) = 1$ if $s < 1$, $\eta_0(s) = 0$ for $s > 2$. We look for a solution the form

$$\begin{aligned} \omega(x,t) =& \omega^*_\varepsilon(x,t) + \varepsilon^{-2}\sum_{j=1}^{k} \eta_j\, \phi_j\left(\frac{x-\xi_j(t)}{\varepsilon}, t\right) + \phi^{out}(x,t) \\ \Psi(x,t) =& \Psi^*_\varepsilon(x,t) + \sum_{j=1}^{k} \eta_j\, \psi_j\left(\frac{x-\xi_j(t)}{\varepsilon}, t\right) + \psi^{out}(x,t) \end{aligned}$$

The inner-outer gluing system

$$(I) \quad \begin{cases} \varepsilon^2 \partial_t \phi_j + \nabla_y^{\perp} \Psi_0 \cdot \nabla(\Delta \psi_j + f'(\Psi_0)\psi_j) \\ + \nabla_y^{\perp} \psi^{out} \cdot \nabla W_0 + Q_j + E_j = 0 \quad \text{in } \mathbb{R}^2 \times [0, T] \\ - \Delta_y \psi_j = \phi_j, \end{cases}$$

for $j = 1, \ldots, k$, coupled with $\phi^{out}(\cdot, 0) = 0$ and

$$(O) \quad \begin{cases} \partial_t \phi^{out} + \nabla_x^{\perp}[\Psi_* + \eta_j \psi_j + \psi^{out})] \cdot \nabla_x \phi^{out} \\ + \varepsilon^{-2} \phi_j \partial_t \eta_j + \nabla_x^{\perp}(\eta_j \psi_j + \psi^{out}) \cdot \nabla_x \eta_j \\ + Q_{out} + E_{out} = 0 \quad \text{in } \mathbb{R}^2 \times [0, T] \\ \\ \Delta_x \psi^{out} + \phi^{out} + \psi_j \Delta_x \eta_j + 2\nabla_x \eta_j \cdot \nabla \psi_j = 0. \end{cases}$$

We solve System (I)-(O) by a continuation (degree) argument extablishing uniform a priori estimates for small solutions of its solutions.

For the **inner problem** we need to solve in $\mathbb{R}^2$

$$\varepsilon^2\phi_t - \nabla^{\perp}\Psi_0 \cdot \nabla(\Delta\psi + f'(\Psi_0)\psi) + E(y,t) = 0, \quad \phi(y,0) = 0$$
$$-\Delta\psi = \phi \quad \text{in } \mathbb{R}^2 \times [0,T]$$

with $E = O(\varepsilon^5(1+|y|)^{-3})$.

The central ingredient is the L^2-a priori estimate assuming

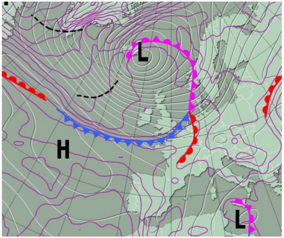
$$\int_{B(0,\delta\varepsilon^{-1})} y\phi dy = 0, \quad \int_{\mathbb{R}^2} \phi dy = 0 = \int_{\mathbb{R}^2} \phi \frac{1-2|y|^2}{1+|y|^2} W_0(y) dy.$$

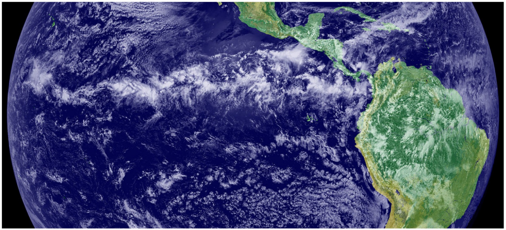
which yields

$$\|\phi(\cdot,t) W_0^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)} \le C\varepsilon^{-2}|\log\varepsilon| \sup_{t\in[0,T]} \|E(\cdot,t) W_0^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}$$

These conditions are achieved by adjusting parameters in the basic ansatz (in particular small remainders in the ξ_j's).

The generalized surface quasigeostrophic equation (SQG)



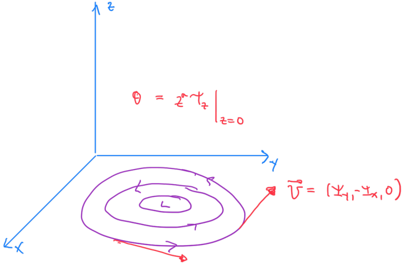


A model for Earth's atmosphere, rotating sphere. $0 < s < 1$.

$$\begin{cases} \theta_t + \nabla^{\perp}\psi \cdot \nabla\theta = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi = (-\Delta)^{-s}\theta & \text{in } \mathbb{R}^2 \times (0, T), \end{cases}$$

Locally approximating the sphere by the plane $z = 0$, velocity field without vertical component is given for a stream function $\psi(x, y, z, t)$ by $v(x, y, t) = (\psi_y(x, y, 0), -\psi_x(x, y, 0))$. $-1 < a < 1$

$$\begin{cases} \partial_x^2 \psi + \partial_y^2 \psi + \partial_z(z^a \partial_z \psi) = 0, \quad (x, y, z) \in \mathbb{R}^3,\ z > 0 \\ \theta(x, y.t) := z^a \partial_z \psi(x, y, z)|_{z=0}, \\ \partial_t \theta + v \cdot \nabla \theta = 0. \end{cases}$$



$$\theta = (-\Delta)^s \psi, \quad s = \frac{1-a}{2}$$

Let $0 < s < 1$.

$$\begin{cases} \theta_t + \nabla^\perp \psi \cdot \nabla \theta = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi = (-\Delta)^{-s} \theta & \text{in } \mathbb{R}^2 \times (0, T), \end{cases}$$

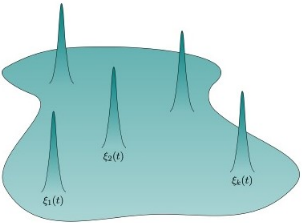
$$(-\Delta)^{-s} f(y) = c_s \int_{\mathbb{R}^2} \frac{1}{|x-y|^{2-2s}} f(y)\, dy$$

• Visually similar to Euler but harder to treat.

• No Yudovich global well-posedness theory for the initial value problem is available.

• $s = 1/2$ carries similar features as Euler 3d

Vortex evolution in this case ?



Formal asymptotics for

$$\theta(x,t) = \sum_{j=1}^{N} k_j \delta(x - \xi_j(t))$$

leads to the law

$$\boxed{\dot{\xi}_j(t) = \sum_{i \neq j} \kappa_i d_s \frac{(\xi_i(t) - \xi_j(t))^{\perp}}{|\xi_i(t) - \xi_j(t)|^{4-2s}}, \quad j = 1, \ldots, N.} \qquad (K_s)$$

The generalized Kaufmann-Scully vortex solves the fractional Yamabe equation in $\mathbb{R}^2$,

$$(-\Delta)^s \Psi_0(y) = \Psi_0(y)^{\frac{2+2s}{2-2s}} = W_0(y).$$

$$\Psi_0(y) = \frac{d_s}{(1+|y|^2)^{1-s}}, \quad W_0(y) = \frac{c_s}{(1+|y|^2)^{1+s}}.$$

Theorem (M. del Pino, Antonio Fernandez)

Let $0.937 \leq s < 1$*. For a colisionless solution* $\xi(t)$ *of the N-body problem* (K_s) *in* $[0, T]$ *there exists a smooth solution of (SQG) such that*

$$\theta(x,t) = \sum_{j=1}^{N} \frac{k_j}{\varepsilon^2} W_0\left(\frac{x-\xi_j}{\varepsilon}\right) + o(1)$$

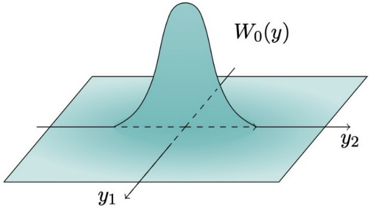
- **Special case:** A travelling wave solution for a vortex pair: Ao, Dávila, del Pino, Musso, Wei (TAMS, 2021).

- **Previous partial results** Cavallaro,Garra and Marchioro (2013) Geldhauser-Romito (2020). Rosenzweig, (2020). Unlike Euler, those constructions are **conditional** to the existence of regular solutions.

- Similar scheme as in Euler, but technically harder.

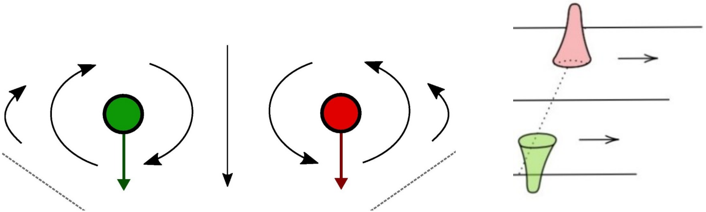
Long-time asymptotics. An open problem: Stability of a radially stationary vortex as $t \to +\infty$: If $\omega(x,t) = W_0(x) + \phi(x,t)$. For small $\phi(x,0)$, do we have $\|\phi(\cdot,t)\|$ small at all times?
Partial answers:

- Arnold (1998) A form of L^2 orbital Stability
- Bedrossian, Coti-Zelati and Vicol (2019) Linear L^2-stability
- Ionescu-Jia (2022) Linear, Gevrey spaces.



Multiple-vortex configurations:

Simplest example steady solutions known: Vortex pair travelling-wave solution $\omega(x,t) = W(x_1 - ct, x_2)$



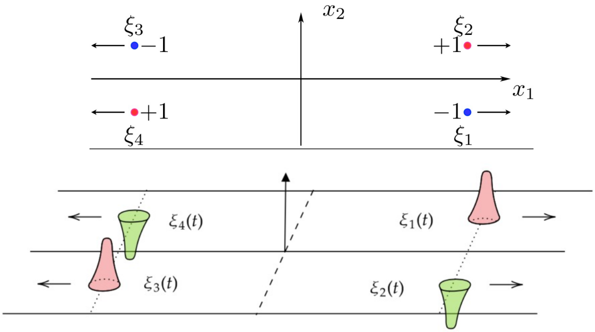
$$-\Delta_x \psi(x) = f(\psi(x) - cx_2) = W(x)$$

$$W(y) = \varepsilon^{-2} W_0((x_1, x_2 - d)/\varepsilon) - \varepsilon^{-2} W_0((x_1, x_2 + d)/\varepsilon) + o(1)$$

- Existence: Norbury (1975)
- Orbital stability results: Burton-Lopes-Nussenweig (2013).

An infinite-time construction. Two vortex pair travelling in opposite directions glued: there exists a solution or (V2) that follows this configuration as $t \to +\infty$. (J. Dávila, M. del Pino, M. Musso, Shrish Parmeshwar)

$$\omega(x,t) = \sum_{j=1}^{4} (-1)^j \frac{1}{\varepsilon^2} W_0\left(\frac{x-\xi_j(t)}{\varepsilon}\right) + o(1).$$



Nearly singular solutions for Euler in $\mathbb{R}^3$?



Open question: Solutions with concentrated vorticities near curves (filaments): *the Vortex filament conjecture* (Helmholtz, Da Rios, Levi-Civita 1858-1906-1931).

We consider the Euler equation in $\mathbb{R}^3$ in stream-vorticity formulation

$$\begin{cases} \omega_t + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0 \quad , \\ u = \nabla \times \psi, \quad \psi(x,t) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \dfrac{x-y}{|x-y|^2} \times \omega(y,t)dy. \end{cases} \qquad (V3)$$

($\omega = \nabla \times u$ in $\mathbb{R}^3$). We want to find solutions with vorticity concentrated on a time evolving curve (filament) $\Gamma(t)$ parametrized by arclength as $\gamma(s,t)$ in $\mathbb{R}^3$.

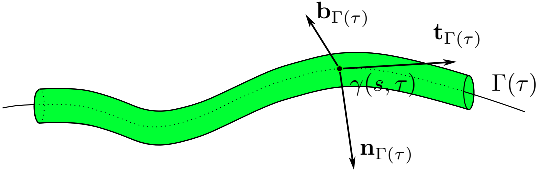
Vortex filament dynamics. (After Helmholtz and Kelvin) is a solution $\omega_\varepsilon(x,t)$ of (V) concentrated in a tube radius ε so that

$$\omega_\varepsilon(\cdot,t) \approx c\delta_{\Gamma(t)}\mathbf{t}_{\Gamma(t)} \quad \text{as } \varepsilon \to 0,$$

$\mathbf{t}_{\Gamma(t)}$ tangent vector field, $\delta_{\Gamma(t)}$ the uniform curve Dirac measure.
1904, Da Rios formal law: Letting $\tau = t|\log\varepsilon|$, $\gamma(s,\tau)$ parametrization by arclength of $\Gamma(\tau)$, κ curvature, then

$$\boxed{\gamma_\tau = \frac{c}{4\pi}(\gamma_s \times \gamma_{ss}) = \frac{c}{4\pi}\kappa \mathbf{b}_{\Gamma(\tau)}},$$

$\mathbf{b}_{\Gamma(\tau)}$ binormal vector. This is the *binormal flow of curves*.



The *vortex filament conjecture*:

Let $\Gamma(\tau)$ *be a solution curve of the binormal flow defined in* $[0, T]$ *for some* $c > 0$, $T > 0$. *For each* $\varepsilon > 0$ *there exists a smooth solution* $\omega_\varepsilon(x,t)$ *to (V3) satisfying in the distributional sense,*

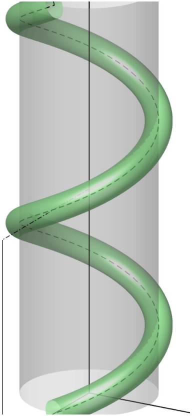
$$\omega_\varepsilon(\cdot, \frac{\tau}{|\log \varepsilon|}) \rightharpoonup c\delta_{\Gamma(\tau)}\mathbf{t}_{\Gamma(\tau)} \quad \textit{as } \varepsilon \to 0, \quad \textit{for all} \quad 0 \le \tau \le T.$$

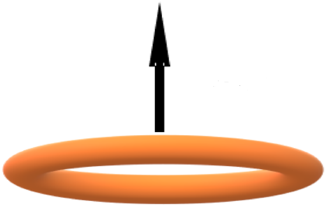
Natural: To look for a solution of the form

$$\omega_\varepsilon(x,\tau) = \frac{1}{\varepsilon^2} W_0\left(\frac{z}{\varepsilon}\right)\mathbf{t}_{\Gamma(\tau)} + o(1), \quad x = \gamma(\tau, s) + z_1\mathbf{b}_{\Gamma(\tau)} + z_2\mathbf{n}_{\Gamma(\tau)},$$

This statement is only known for special curves associated to travelling wave solutions: the thin vortex ring first found by Fraenkel, and recently a helicoidal filament.

Examples: a helix whose horizontal section rotates at a constant angular speed or a vertically translating circle are solutions of the bi-normal flow of curves.





Thin Vortex ring travelling-wave solution: Fraenkel, 1970
(Axisymmetric, no swirl, problem reduces to an elliptic equation)

Solutions $\omega(x,y,z,t)$ of $3d$-Euler with Helicoidal symmetry can be obtained from a scalar function $w(x+iy,t)$ in the form

$$\omega(x,y,z,t) = w(e^{-iz}(x+iy), t/|\log\varepsilon|)\begin{bmatrix} i(x+iy) \\ b \end{bmatrix}$$

where $w(x,\tau)$ solves

$$\begin{cases} |\log\varepsilon| w_\tau + \nabla^\perp\psi\cdot\nabla w = 0 \\ -\nabla\cdot(K\nabla\psi) = w \end{cases}$$

$$K(x,y) = \frac{1}{\kappa^2+x^2+y^2}\begin{pmatrix} \kappa^2+y^2 & -xy \\ -xy & \kappa^2+x^2 \end{pmatrix}$$

Rotating helicoidal solutions:

$$w(x+iy,\tau) = w(e^{i\alpha\tau}(x+iy)), \quad \psi((x+iy),\tau) = \psi(e^{i\alpha\tau}(x+iy)).$$

$$\boxed{-\nabla\cdot(K\nabla\psi) = f(\psi - \frac{\alpha}{2}|\log\varepsilon|(x^2+y^2)) = w \quad \text{in } \mathbb{R}^2}$$

Special case $f(u) = \varepsilon^2 e^u$. we prove:

Theorem (Dávila, del Pino, Musso, Wei (2022))

There exists a solution ψ_ε to the equation

$$-\nabla \cdot (K \nabla \psi) = \varepsilon^2 e^{\psi + \lambda(x^2+y^2)} \quad \text{in } \mathbb{R}^2$$

such that $\varepsilon^2 e^{\psi - \frac{\alpha}{2}|\log \varepsilon|(x^2+y^2)} \rightharpoonup 8\pi\delta_{(x_0,0)}$, $x_0 > 0$, *for a suitable choice of* α.

α is precisely the number that makes the "rotating helix"

$$\gamma(s,\tau) = \begin{pmatrix} e^{i\left(\frac{s}{\sqrt{b^2+x_0^2}} - \alpha\tau\right)}(x_0 + iy_0) \\ \frac{bs}{\sqrt{b^2+x_0^2}} \end{pmatrix}$$

a solution of the binormal flow

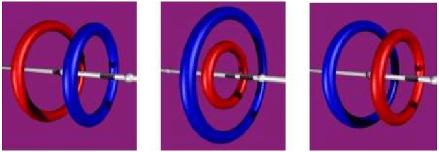
Axisymmetric Euler no-swirl:

$$\omega(r,z,\tau) = W(r,z,\tau)(-y,x).$$

After rescaling time $t = \tau/|\log\varepsilon|$, we get

$$(AN) \quad \begin{cases} |\log\varepsilon| r W_\tau + \nabla^\perp(r^2\psi)\nabla W = 0 \\ -(\psi_{rr} + \frac{3}{r}\psi_r + \psi_{zz}) := -\Delta_5\psi = W \\ \psi_r(0,z,\tau) = 0. \end{cases}$$

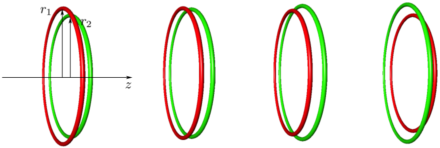
Interaction of multiple vortex rings:



Leapfrogging Vortex-Rings Helmholtz 1858: predicted the way two identical, coaxial vortex rings interact.

• The rings travel in the same direction. Due to their mutual interaction, the rear ring shrinks and accelerates, and the leading ring widens and decelerates. The rear ring then passes through the leading ring, with this process of *leapfrogging* then repeating again and again.

Aim: To justify rigorously the leapfrogging dynamics for the 3d axisymmetric Euler flow without swirl.



• The leapfrogging motion was justified in the Gross-Pitaievskii equation $iu_t = \Delta u + \varepsilon^{-2}(1 - |u|^2)u = 0$ by Jerrard and Smets (2018)

Incompressible Euler: Outstanding open problem.

Theorem (Dávila, del Pino, Musso, Wei (2023))

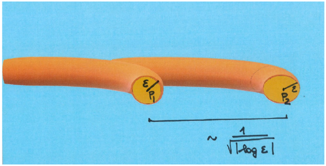
Let $b(\tau) = (b_1(\tau), \ldots, b_N(\tau))$ *be a colisionless solution of the system*

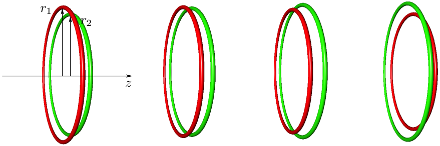
$$\dot{b}_i(\tau) = \sum_{j \neq i} \frac{(b_i - b_j)^{\perp}}{|b_i - b_j|^2} - \frac{b_i^1}{r_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \textit{in } (0, T).$$

$$a_i(\tau) = \Big(r_0, z_0\Big) + \frac{1}{\sqrt{|\log \varepsilon|}} b_i(\tau)$$

Then there exists a solution W_ε *of the form* $W_0(y) = \frac{8}{(1+|y|^2)^2}$

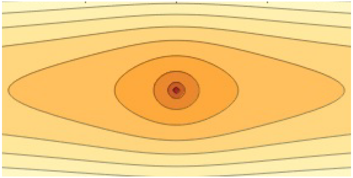
$$W_\varepsilon(x, \tau) = \sum_{j=1}^{N} \frac{1}{\varepsilon^2} W_0 \left(\frac{(r, z - r_0^{-1}\tau) - a_j(\tau)}{\varepsilon \sqrt{a_j^1(\tau)}} \right) + o(1)$$

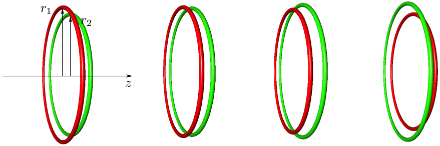




Restricting ourselves to $b_1 = -b_2 = b = (b^1, b^2)$

$$\dot{b}(\tau) = \frac{1}{2}\frac{b(\tau)^\perp}{|b(\tau)|^2} - \frac{b^1(\tau)}{r_0^2}\begin{pmatrix}0\\1\end{pmatrix} \implies \log|b| + |b^1|^2 = \mathit{constant}$$





$$\boxed{W_\varepsilon(x,t) = \sum_{j=1}^{2}\frac{1}{\varepsilon^2}W_0\left(\frac{(r, z - r_0^{-1}\tau) - a_j(\tau)}{\varepsilon\sqrt{a^1(\tau)}}\right) + o(1)}$$

$$a_j(\tau) = (r_0, z_0) + (-1)^j\frac{b(\tau)}{\sqrt{|\log\varepsilon|}}$$

Thanks for your attention