Introduction to hypercomplex analysis

1st lecture

P. Cerejeiras

CIDMA and Departamento de Matemática
Universidade de Aveiro
pceres@ua.pt

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CENTRO DE I&D EM MATEMÁTICA E APLICAÇÕES
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Higher dimensional equivalent of Complex Analysis

For the initial part of these lectures I will follow the book of Gilbert and Murray (1991).

Complex analysis - combines geometric insight with analytic concepts; for example, the Laplacian Δf coincides with the Laplace-Beltrami operator $\Delta_{LB} f = \nabla \cdot (\nabla f)$.

Types of known generalisations:

- Several complex variables - $\mathbb{C}^n$;
- Clifford algebras;
- more "exotic algebras".

Quadratic spaces

Let $\mathbb{V}$ be a vectorial space of dimension n over a field $\mathbb{F}$ ($\mathbb{R}$, or $\mathbb{C}$).

Given a quadratic form $Q : \mathbb{V} \mapsto \mathbb{F}$, that is,

1. $Q(\lambda x) = \lambda^2 Q(x)$, for all $\lambda \in \mathbb{F},\ x \in \mathbb{V}$;
2. $Q(x) + Q(y) - Q(x-y) = 2\mathcal{B}(x,y)$, for all $x, y \in \mathbb{V}$, where $\mathcal{B}$ is a bilinear form,

we say that $(\mathbb{V}, Q)$ is a quadratic space

Some remarks:

- the quadratic form Q induces a type of norm in $\mathbb{V}$;
- the bilinear form $\mathcal{B} : \mathbb{V} \times \mathbb{V} \mapsto \mathbb{F}$ induces a type of inner product in $\mathbb{V}$.

Quadratic spaces - real case

Examples (for $\mathbb{V} = \mathbb{R}^n$)

- "easiest example": take $Q \equiv 0$; this gives raise to Grassmannian algebras;
- for $Q_{p,q}(x_1, \ldots x_n) = \sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{p+q} x_j^2$, where $p + q = n$, we obtain a quadratic space with "signature" (p, q), i.e.

$$(\mathbb{R}^n, Q_{p,q}) = \mathbb{R}^{p,q}.$$

In particular, we have

- $\mathbb{R}^{0,n}$ or $\mathbb{R}^{n,0}$ as Euclidean spaces

$$Q_{0,n}(x_1, \ldots, x_n) = -\sum_{j=1}^{n} x_j^2 = -\|(x_1, \ldots, x_n)\|^2,$$

  and
- $\mathbb{R}^{1,3}$ or $\mathbb{R}^{3,1}$ as Minkowski spaces (time-space),

$$Q_{1,3}(t, x, y, z) = t^2 - (x^2 + y^2 + z^2).$$

Quadratic spaces - complex case

For $\mathbb{V} = \mathbb{C}^n$, we can assign

$$Q_n(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$$

and we obtained a (complex) quadratic space $(\mathbb{C}^n, Q_n)$.

In this case, both the quadratic form and the corresponding complex bilinear form $\mathcal{B}$,

$$\mathcal{B}(z, w) = z_1 w_1 + \cdots + z_n w_n,$$

require an additional operation (conjugation) in order to be identified with a norm or an inner product in $\mathbb{C}^n$.

Radical part of $(\mathbb{V}, Q)$

Consider now $\{\mathbf{e}_j, j = 1, \ldots, n\}$ a basis for $\mathbb{V}$.

For $x = \sum_j x_j \mathbf{e}_j \in \mathbb{V}$ we have

$$Q(x) = \mathcal{B}(x, x) = \sum_{i,j} \mathcal{B}(\mathbf{e}_i, \mathbf{e}_j) x_i x_j = x^t \, \mathbb{B} x.$$

The basis $\{\mathbf{e}_j\}$ is said $\mathcal{B}$-orthogonal if $\mathcal{B}(\mathbf{e}_i, \mathbf{e}_j) = 0$ for $i \neq j$. In that case,

$$Q(x) = \sum_i \mathcal{B}(\mathbf{e}_i, \mathbf{e}_i) x_i^2 = \sum_i Q(\mathbf{e}_i) x_i^2.$$

We define the radical part of $(\mathbb{V}, Q)$ as

$$\mathrm{Rad}(\mathbb{V}, Q) := \{x \in \mathbb{V} : \; \mathcal{B}(x, y) = 0, \text{for all } y \in \mathbb{V}\}.$$

The quadratic space $(\mathbb{V}, Q)$ is called non-degenerated if $\mathrm{Rad}(\mathbb{V}, Q) = \{0\}$, and degenerated otherwise. In that case,

$$\mathbb{V} = \mathrm{Rad}(\mathbb{V}, Q) \oplus_{\mathcal{B}} \mathrm{Rad}(\mathbb{V}, Q)^{\perp}.$$

Radical part of $(\mathbb{V}, Q)$

Furthermore, for the non-degenerated part $\mathrm{Rad}(\mathbb{V}, Q)^{\perp}$ one can construct normalised basis s.t.

$$Q(\mathbf{e}_j) = \pm 1.$$

Theorem 1.1

Let $(\mathbb{V}, Q)$ be a quadratic space with

$$\mathbb{V} = \mathrm{Rad}(\mathbb{V}, Q) \oplus_{\mathcal{B}} \mathrm{Rad}(\mathbb{V}, Q)^{\perp}.$$

Then

1. $Q \equiv 0$ on $\mathrm{Rad}(\mathbb{V}, Q)$;
2. $\mathrm{Rad}(\mathbb{V}, Q)^{\perp}$ is isomorphic to
    1. $\mathbb{R}^{p,q}$ if $\mathbb{F} = \mathbb{R}$, with p, q depending only on $Q_{p,q}$ (Sylvester's Theorem)
    2. $(\mathbb{C}^n, Q_n)$ if $\mathbb{F} = \mathbb{C}$,

with $p + q = n$ the dimension of $\mathrm{Rad}(\mathbb{V}, Q)^{\perp}$.

Sylvester's law of inertia

Property 2.1 is called Sylvester's law of inertia since it implies for the matrix $\mathbb{B}$

$$\mathbb{B} := [\mathcal{B}(\mathbf{e}_i, \mathbf{e}_j)]_{i,j} = \underbrace{\begin{bmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}}_{}$$

$$\underbrace{\qquad\qquad}_{p \text{ entries}} \quad \underbrace{\qquad\qquad}_{q \text{ entries}}$$

Clifford algebras

Let $\mathbb{V}$ a finite dimensional vector space over $\mathbb{F}$ and $(\mathbb{V}, Q)$ a quadratic space.

Let $\mathcal{A}$ be an associative algebra over $\mathbb{F}$ with identity $1_{\mathcal{A}}$ and a $c : \mathbb{V} \mapsto \mathcal{A}$ an $\mathbb{F}$-linear embedding of $\mathbb{V}$ into $\mathcal{A}$.

Definition 1.2

The pair $(\mathcal{A}, c)$ is a Clifford algebra associated to $(\mathbb{V}, Q)$ if

1. $\mathcal{A}$ is generated as an algebra by $\{c(x), x \in \mathbb{V}\}$;
2. $\mathbb{F} \cong \{\lambda 1_{\mathcal{A}}, \lambda \in \mathbb{F}\}$;
3. $[c(x)]^2 = -Q(x)1_{\mathcal{A}}$.

First example: Grassmann algebra

1. Let $Q \equiv 0$ on $\mathbb{V}$.

The exterior algebra $\mathcal{A} = \wedge^*(\mathbb{V}) = \sum_{k=0}^{n} \wedge^k(\mathbb{V})$, with n the dimension of the vector sp. $\mathbb{V}$, satisfy

- $\wedge^0(\mathbb{V}) \cong \mathbb{F}$;
- $\wedge^1(\mathbb{V}) \cong \mathbb{V}$;
- let $c : \mathbb{V} \mapsto \wedge^1(\mathbb{V})$; since we have $x_1 \wedge \cdots \wedge x_k \in \wedge^k(\mathbb{V})$ then $\mathcal{A} = \wedge^*(\mathbb{V})$ is generated by $\{c(x), x \in \mathbb{V}\}$;
- $[c(x)]^2 = x \wedge x = 0 = -Q(x)1_{\mathcal{A}}$ for all $x \in \mathbb{V}$.

Hence, $\mathcal{A} = \wedge^*(\mathbb{V})$ is a (degenerated) Clifford algebra for $(\mathbb{V}, Q)$.

Realisations of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ in terms of Pauli matrices

2. Consider the associated Pauli matrices in $\mathbb{C}^{2\times 2}$

$$\mathbf{e}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

We have

$$\mathbf{e}_0^2 = \mathbb{I}, \quad \mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -\mathbb{I}.$$

Then we obtain the following Clifford algebras:

- $\mathbb{R}_{0,0} = \{\lambda \mathbf{e}_0, \lambda \in \mathbb{R}\} \cong \mathbb{R}$, associated to the vector space $\mathbb{V} = \mathbb{R}$;
- For $\sigma_3 = -i\mathbf{e}_3$ we have $\sigma_3^2 = \mathbf{e}_0$ so that

  $$\mathbb{R}_{1,0} = \left\{ x\mathbf{e}_0 + y\sigma_3 = \begin{bmatrix} x & y \\ y & x \end{bmatrix}, x, y \in \mathbb{R} \right\} \cong \mathbb{R} \oplus \mathbb{R}, \text{ also}$$

  associated to the vector space $\mathbb{V} = \mathbb{R}$, and with

  $$[c(y\sigma_3)]^2 = \begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix}^2 = y^2\mathbf{e}_0.$$

Realisations of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ in terms of Pauli matrices

- $\mathbb{R}_{0,1} = \left\{ x\mathbf{e}_0 + y\mathbf{e}_2 = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}, x, y \in \mathbb{R} \right\} \cong \mathbb{C}$, also

  associated to the vector space $\mathbb{V} = \mathbb{R}$, and with

$$[c(y\mathbf{e}_2)]^2 = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix}^2 = -y^2\mathbf{e}_0.$$

- $\mathbb{R}_{0,2} = \{x\mathbf{e}_0 + y\mathbf{e}_1 + x'\mathbf{e}_2 + y'\mathbf{e}_3, x, y, x', y' \in \mathbb{R}\} \cong \mathbb{H}$, where

$$x\mathbf{e}_0 + y\mathbf{e}_1 + x'\mathbf{e}_2 + y'\mathbf{e}_3 = \begin{bmatrix} x + iy & x' + iy' \\ -x' + iy' & x - iy \end{bmatrix} = \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}.$$

  This Clifford algebra is associated to $\mathbb{V} = \mathbb{R}^2$ and we have

$$[c(y\mathbf{e}_1 + x'\mathbf{e}_2)]^2 = \begin{bmatrix} iy & x' \\ -x' & -iy \end{bmatrix}^2 = -(y^2 + x'^2)\mathbf{e}_0.$$

A complex Clifford algebra

3. For $\mathbb{V} = \mathbb{C}^2$, and $Q_2(z, w) = z^2 + w^2$, we take $c : \mathbb{C}^2 \mapsto \mathbb{C}^{2\times 2}$ as

$$c(z, w) = \begin{bmatrix} 0 & z - iw \\ z + iw & 0 \end{bmatrix},$$

so that $\mathbb{C}_2 = \mathbb{C}^{2\times 2}$ is a Clifford algebra associated to $(\mathbb{C}^2, Q_2)$.

Universal Clifford algebras

Some simplifications:

1. we will make no distinction between $\lambda 1_{\mathcal{A}} \in \mathcal{A}$ and $\lambda \in \mathbb{F}$;
2. similarly, no distinction between $c(x) \in \mathcal{A}$ and $x \in \mathbb{V}$.

Under these conventions, condition 3. of Definition 1.2. becomes

$$Q(x) = -x^2, \quad \text{for all } x \in \mathbb{V}.$$

Lemma 1.3

For all $x, y \in \mathbb{V}$, it holds

$$\mathcal{B}(x, y) = -\frac{1}{2}(xy + yx).$$

Immediate, since

$$-(x+y)^2 = Q(x+y) = 2\mathcal{B}(x,y) + Q(x) + Q(y) = 2\mathcal{B}(x,y) - x^2 - y^2.$$

Basis for a Clifford algebra - 1

Lemma 1.4

Let $\{\mathbf{e}_j, j = 1, \ldots, n\}$ be an $\mathcal{B}$-orthonormal basis of $\mathbb{V}$. Then,

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2\mathbf{e}_j^2\delta_{i,j}, \quad i, j = 1, \ldots, n.$$

Theorem 1.5

Let $(\mathcal{A}, c)$ be a Clifford algebra associated to the quadratic space $(\mathbb{V}, Q)$, and $\{\mathbf{e}_j, j = 1, \ldots, n\}$ be an $\mathcal{B}$-orthonormal basis of $\mathbb{V}$. Then,

1. $\mathcal{A}$ is spanned by all products

$$\mathbf{e}_1^{\alpha_1} \cdots \mathbf{e}_n^{\alpha_n}, \qquad \alpha_j = 0, 1;$$

2. $\dim(\mathcal{A})$ is at most $2^{\dim(\mathbb{V})}$.

Basis for a Clifford algebra - 2

Since for an $\mathcal{B}$-orthonormal basis we have $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i,\ i \neq j$, we can re-order these basis elements as

$$\mathbf{e}_\emptyset = 1, \quad \mathbf{e}_A = \mathbf{e}_{i_1} \cdots \mathbf{e}_{i_k}, \quad 1 \leq i_1 < \cdots < i_k \leq n = \dim(\mathbb{V}).$$

By convention, $A = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} := N$.

Remark: as a Clifford algebra, $\mathcal{C}\ell_{0,2}$ has dimension $4 = 2^2$ although associated to $\mathbb{V} = \mathbb{R}^3$. (Recall, $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_{1,2} = \mathbf{e}_3$).

Theorem 1.5

The dimension of a Clifford algebra $\mathbb{R}_{p,q} = (\mathcal{A}, c)$ associated to $\mathbb{R}^{p,q} = (\mathbb{R}^n, Q_{p,q})$ is

1. $\dim(\mathbb{R}_{p,q}) = 2^{p+q-1}$ if
    - $p - q - 1 \equiv 0 \pmod 4$;
    - $p + q$ is odd;
    - $\mathbf{e}_N = \mathbf{e}_1 \cdots \mathbf{e}_n \in \mathbb{R}$.
2. $\dim(\mathbb{R}_{p,q}) = 2^{p+q}$, otherwise.

Universal Clifford algebras

Whenever $\dim(\mathcal{A}) = 2^{\dim(\mathbb{V})}$ we say $\mathcal{A}$ is a universal Clifford algebra. This implies that each product

$$\mathbf{e}_i\mathbf{e}_j$$

is a new element in the basis of the algebra.

$\mathbb{H}$ is *not* a universal Clifford algebra as $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3$.

For every quadratic space $(\mathbb{V}, Q)$ there exists a universal Clifford algebra $(\mathcal{A}, c)$ (i.e., $\dim(\mathcal{A}) = 2^n$).

$\mathbb{Z}_2$-grading

We have for $x \in \mathcal{A}$ that

$$x = \sum_{A \subset N} x_A \mathbf{e}_A, \quad x_A \in \mathbb{F}.$$

We now decompose $\mathcal{A}$ into

$$\mathcal{A}^+ : \sum_{|A| \ \mathit{even}} x_A \mathbf{e}_A, \qquad \mathcal{A}^- : \sum_{|A| \ \mathit{odd}} x_A \mathbf{e}_A.$$

Then

- $\mathcal{A}^+$ is again a Clifford algebra, with $\dim(\mathcal{A}^+) = 2^{n-1}$. Furthermore, $\mathbb{F} \subset \mathcal{A}^+$;
- $\mathcal{A}^+$ is not anymore an algebra but $\mathbb{V} \subset \mathcal{A}^-$;
- $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ and the following multiplication rules hold

$$\mathcal{A}^+\mathcal{A}^+ = \mathcal{A}^-\mathcal{A}^- = \mathcal{A}^+, \qquad \mathcal{A}^+\mathcal{A}^- = \mathcal{A}^-\mathcal{A}^+ = \mathcal{A}^-;$$

- $x = \sum_{k=0}^{2^n} [x]_k$, where $[x]_k := \sum_{|A|=k} x_A \mathbf{e}_A$.

Universal real Clifford algebra $\mathbb{R}_{p,q}$

In what follows, let $\mathbb{R}_{p,q}$ be the universal real Clifford algebra associated to the quadratic space $\mathbb{R}^{p,q}$.

- $\mathbb{R}_{n,0}, \mathbb{R}_{0,n}$ are real Clifford algebras associated to the Euclidean space $\mathbb{R}^n$;
- $\mathbb{R}_{p,q}$, $(p, q \neq 0)$ are called Lorentzian algebras;
- $\mathbb{H}$ is not a universal Clifford algebra, since

$$\mathbb{H} \cong \mathbb{R}_{0,2} \cong \mathbb{R}^{+}_{0,3}.$$

Witt basis in $\mathbb{R}^{2n}$ (Sommen's trick)

We use a quadratic form $Q_{n,n}$ and associate an $\mathcal{B}$-orthonormal basis $\{\mathbf{e}_j, \epsilon_j,\ j = 1, \ldots, n\}$, that is

$$\mathbf{e}_j^2 = -1, \quad \epsilon_j^2 = +1,$$

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i, \quad \epsilon_i\epsilon_j + \epsilon_j\epsilon_i, \quad i \neq j$$

$$\mathbf{e}_i\epsilon_j + \epsilon_j\mathbf{e}_i.$$

Construct

$$\mathfrak{f}_j = \frac{1}{2}(\epsilon_j + \mathbf{e}_j), \quad \mathfrak{f}_j^\dagger = \frac{1}{2}(\epsilon_j - \mathbf{e}_j).$$

Then

- $\mathfrak{f}_j^2 = (\mathfrak{f}_j^\dagger)^2 = 0$;
- $\mathfrak{f}_j + \mathfrak{f}_j^\dagger = \epsilon_j, \quad \mathfrak{f}_j - \mathfrak{f}_j^\dagger = \mathbf{e}_j$;

Usage:

- Super-algebras: $\partial_{z_j}\mathfrak{f}_j + \partial_{\bar{z}_j}\mathfrak{f}_j^\dagger, \quad z_j = x_j\epsilon_j + y\mathbf{e}_j$;
- Factorisation of the heat operator $\partial_t - \Delta$ in terms of 1st order operators.

Ideas of other (future) applications

Lemma 1.6

$$\mathbb{R}_{1,n+1} = \mathbb{R}_{0,n} \otimes_{\mathbb{R}} \mathbb{R}_{1,1},$$

where $\mathbb{R}_{1,1}$ is spanned by $\{1, \mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_+\mathbf{e}_-\}$.

This allows for the imbedding of the vectorial space $\mathbb{R}^n$ into the quadratic sp. $\mathbb{R}^{1,n+1}$:

$$(x_1, \cdots, x_n) \quad \mapsto \quad \left(x_1, \cdots, x_n, \frac{1-|x|^2}{2}, \frac{1+|x|^2}{2}\right)$$

with

$$ds^2 = \sum_j dx_j^2 \quad \leftrightarrow \quad dT^2 = \sum_j dX_j^2$$

Euclidean metric hyperbolic metric

whereas $X_j = x_j,\ X_{n+1} = \frac{1-|x|^2}{2}$, and $T = \frac{1+|x|^2}{2}$.

References

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