Introduction to hypercomplex analysis

3rd lecture

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Right Clifford modules

- Analysis in terms of 1*st* order operators

$$\text{Dirac} \quad (\mathbb{R}^n) \qquad\qquad \text{Cauchy} - \text{Riemann} \quad (\mathbb{R}^{n+1})$$

$$\partial = \sum_j \mathbf{e}_j \partial_{x_j} \qquad\qquad D = \partial_{x_0} + \partial$$

- Linearity w.r.t. $\mathbb{R}_{p,q}$-scalars: if $f(x) = \sum_A f_A(x)\mathbf{e}_A, \ x \in \Omega \subset \mathbb{R}^n$ then

$$\partial f = 0 \quad \Rightarrow \quad \partial(\lambda f) = \sum_{j,i,A} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_A \lambda_i \partial_{x_j} f_A = (?).$$

Definition 3.1.

A right unitary module over $\mathbb{R}_{p,q}$ (right Clifford module) is a vector space $\mathbb{V}$ with an algebra morphism $R : \mathbb{R}_{p,q} \to \mathrm{End}(\mathbb{V}), \ a \mapsto R(a)$, s.t.

- $R(ab + c) = R(b)R(a) + R(c)$;
- $R(1) = Id$;

Examples

1. $\mathbb{V} = \mathbb{C}_n$;
$$R(a)x = xa, \qquad x, a \in \mathbb{C}_n.$$
2. $\mathbb{V}$ is a vector space of functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}_{p,q}$;
$$R(a)f = g \qquad x \mapsto R(a)f(x) = f(x)a =: g(x), \qquad x \in \Omega, a \in \mathbb{R}_{p,q}.$$

Functions spaces of $\mathbb{R}_{p,q}$-valued functions are right Clifford modules.

Banach right Clifford modules

Definition 3.2.

A function space $\mathbb{V}$ is a Banach right Clifford module when

- $\mathbb{V}$ is right Clifford module;
- $\mathbb{V}$ is a Banach space endowed with a norm $\|\cdot\|$;
- there exists a constant $C > 0$ s.t. for all $f \in \mathbb{V},\ a \in \mathbb{R}_{p,q}$,

$$\|R(a)f\| = \|fa\| \leq C|a|\|f\|,$$

where $|a|^2 = \sum_A |a_A|^2$.

Example: $L^p(\Omega, \mathbb{C}_n)$ spaces, $1 \leq p < \infty$, with norm

$$\|f\|_p := \sum_A \left(\int_\Omega |f_A(x)|^p dx \right)^{1/p} < \infty$$

Hilbert right Clifford modules

Definition 3.3.

Given a Hilbert space $\mathcal{H}$ endowed with a inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, we say that

$$\mathbb{V} := \mathcal{H} \otimes_{\mathbb{R}} \mathbb{R}_{0,n}$$

is a Hilbert right Clifford module.

Two types of "inner products" on $\mathbb{V}$:

- Hermitean form

$$\langle\!\langle f, g \rangle\!\rangle = \langle\!\langle \sum_A f_A \mathbf{e}_A, \sum_B g_B \mathbf{e}_B \rangle\!\rangle := \sum_{A,B} \langle f_A, g_B \rangle_{\mathcal{H}} \overline{\mathbf{e}}_A \mathbf{e}_B.$$

  Remark that this form is sesquilinear, linear on the 2nd argument, but not positive definite;

- Inner product

$$\langle f, g \rangle := [\langle\!\langle f, g \rangle\!\rangle]_0 = \sum_A \langle f_A, g_A \rangle_{\mathcal{H}} \in \mathbb{C}.$$

Functionals

Lemma 3.4. (Schwarz' formula)

For all $f, g \in \mathbb{V} = L^2(\Omega) \otimes \mathbb{R}_{0,n}$, a Hilbert right Clifford module, then

$$\left| \langle\langle f, g \rangle\rangle \right| \leq 2^{n/2} \|f\|_2 \|g\|_2.$$

Given a Banach right Clifford module $\mathbb{V}$ we obtain its dual $\mathbb{V}^*$ as the set of all $\varphi : \mathbb{V} \to \mathbb{C}_n$ for which

$$f = \sum_A f_A \mathbf{e}_A \mapsto \varphi(f) := \int_\Omega \overline{\varphi(x)} f(x) dx.$$

Theorem 3.5. (Riesz' representation theorem - BDS, 1982)

For every continuous linear functional $\varphi \in \mathbb{V}^*$ there exists a unique $g_\varphi \in \mathbb{V}$ such that

$$f \mapsto \varphi(f) := \langle\langle g_\varphi, f \rangle\rangle = \sum_{A,B} \overline{\mathbf{e}_A} \mathbf{e}_B \left[\int_\Omega \overline{\varphi_A(x)} f_B(x) dx \right].$$

Reproducing kernels in Hilbert right Clifford modules

Let $K = K(\cdot,\cdot)$ be a reproducing kernel of $\mathcal{H} \otimes \mathbb{R}_{0,n}$, i.e.

- $K_x = K(x,\cdot) \in \mathcal{H} \otimes \mathbb{R}_{0,n}$;
- for all $f \in \mathcal{H} \otimes \mathbb{R}_{0,n}$,

$$x \mapsto f(x) = \langle\langle K_x, f \rangle\rangle = \int_\Omega \overline{K(x,y)} f(y) dy, \quad \text{for all } x \in \Omega.$$

Reconstruction via interpolation spaces

1. $\mathbb{V}_M := \mathrm{Span}\{K_{x_j}, \quad x_j \in \Omega, j = 1, \ldots, M\}$ with

$$\mathbb{V}_M \ni u = \sum_j K_{x_j} c_j, \qquad \underline{c} = (c_1, \ldots, c_M) \in (\mathbb{C}_n)^M.$$

2. $\mathbb{K} := [\langle\langle K_{x_i}, K_{x_j} \rangle\rangle]_{i,j=1}^M$
   - allows for $\langle\langle u, u \rangle\rangle = \underline{c}^* \mathbb{K} \underline{c}$;
   - but what does it means $\mathbb{K}$ positive define?

Quasi-determinants and Schur complements

The Gram matrix $\mathbb{K}$ is hermitean and we have the positivity of the all sub-matrices $\mathbb{K}_m = [\langle\langle K_{x_i}, K_{x_j}\rangle\rangle]_{i,j=1}^m$ where $m \leq M$.

This positiviness condition implies that the corresponding Schur complements

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & -A^{-1}B \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & \underbrace{D - CA^{-1}B}_{quasi-determinant} \end{pmatrix}.$$

are positive and, consequently, so are their quasi-determinants; further, they satisfy the heredity principle.

This ensures that the quasi-determinant of the $\mathbb{K}$ is non-zero and the system

$$\mathbb{K}\underline{c} = \underline{f}$$

has a unique solution.

Kreĭn spaces

What about the case of signature spaces (real case)? $\mathbb{R}_{p,q} \ \ (p, q \neq 0)$.

Quadratic space $\mathbb{R}^{p,q}$ can be split into $\mathbb{R}^{p,q} = \mathbb{R}^{p,0} \oplus \mathbb{R}^{0,q}$.

Definition 3.6.

Given a right-linear module $(\mathbb{V}, \langle\langle \cdot, \cdot \rangle\rangle)$, that is, $\mathbb{V}$ endowed with a sesquilinear form $\langle\langle \cdot, \cdot \rangle\rangle$, we define its anti-module as $(\mathbb{V}, -\langle\langle \cdot, \cdot \rangle\rangle)$.

- Obviously, if $\langle\langle \cdot, \cdot \rangle\rangle$ is a positive form, i.e.

$$[\langle\langle x, x \rangle\rangle]_0 > 0, \quad \text{for all non-zero } x \in \mathbb{V},$$

then its anti-module is endowed with a negative form

$$[\langle\langle x, x \rangle\rangle]_0 < 0, \quad \text{for all non-zero } x \in \mathbb{V}.$$

Clifford-Krein modules

Definition 3.7.

A right-linear module $(\mathbb{V}, \langle\langle\cdot,\cdot\rangle\rangle)$, (where $\langle\langle\cdot,\cdot\rangle\rangle$ is a sesquilinear form) is called a Clifford-Krein right-module if

(a) it admits a decomposition into a positive and a negative Hilbert modules

$$\mathbb{V} = \mathbb{V}_+ \oplus \mathbb{V}_-; \tag{1}$$

(b) the decomposition is orthogonal with respect to the sesquilinear form, i.e.

$$\langle\langle v_+, v_- \rangle\rangle = 0, \quad \text{for every } (v_+, v_-) \in \mathbb{V}_+ \times \mathbb{V}_-. \tag{2}$$

- The decomposition (1) is called a fundamental decomposition.
- The positivity is induced from the secondary linear form $\langle\cdot,\cdot\rangle := [\langle\langle\cdot,\cdot\rangle\rangle]_0$;
- A Clifford-Krein module is an inner product space which is non-degenerate, decomposable, and complete. In general the positive and negative modules $\mathbb{V}_\pm$ are infinite-dimensional right Hilbert modules which are orthogonal to each other w.r.t. $\langle\langle\cdot,\cdot\rangle\rangle$.

A word of caution

Theorem 3.8. (Moore-Aronszajn Theorem)

A $K : \Omega \times \Omega \to \mathbb{R}_{0,n}$ be a Hermitean positive kernel. Then there exists a unique reproducing kernel Hilbert module which has K as its reproducing kernel.

In general, a Hermitean kernel is not a reproducing kernel of a Clifford-Krein module.

Also, a given reproducing kernel may be associated to different Clifford-Krein modules.

A counter-example (courtesy of L. Schwartz)

Let $\mathbb{V}$ be a Banach bi-module that does not allow for a Hilbert structure, and consider $\mathbb{V}' \times \mathbb{V}$.

Then the kernel acting on $(\mathbb{V}' \times \mathbb{V}) \times (\mathbb{V}' \times \mathbb{V})$ as

$$k((\mathbf{e}_1', \mathbf{e}_1), (\mathbf{e}_2', \mathbf{e}_2)) \quad = \quad \mathbf{e}_1'(\mathbf{e}_2) + \mathbf{e}_2'(\mathbf{e}_1)$$

cannot be written as $k = k_+ - k_-$.

If it would be possible, then $\mathbb{V} \times \mathbb{V}' = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathbb{V}$ could be endowed with a Hilbert structure.

This means that a sesquilinear form can only induces a kernel if it can be written as $k = k_1 - k_2$ with k_1, k_2 positive kernels.

Kolmogorov decomposition

When does a Hermitean kernel is a reproducing kernel of a Clifford-Krein module?

Definition 3.9.

A Hermitean kernel $K = [K_{i,j}]_{i,j\in\mathbb{J}}, K_{i,j} \in \mathcal{L}(\mathbb{V}_i, \mathbb{V}_j)$, with $K_{i,j} = K_{j,i}^*$, admits a Kolmogorov decomposition if there exists a Clifford Krein-module $\mathcal{K}$ and operators $V_j \in \mathcal{L}(\mathbb{V}_j, \mathcal{K}),\ j \in \mathbb{J}$, such that

1. $K_{i,j} = V_i^* V_j, \quad i, j \in \mathbb{J}$;
2. $\mathcal{K} = \vee_j V_j \mathbb{V}_j$.

Theorem 3.10.

Let K be a Hermitean kernel then the following statements are equivalent

1. K has a Kolmogorov decomposition;
2. K has a nonnegative majorant;
3. $K = K_+ - K_-$ for some Hermitean kernels $K_+ \geq 0, K_- \geq 0$.

In this case, the decomposition in point (3) can be chosen such that the only Hermitean kernel M such that $0 \leq M \leq M_\pm$ is $M = 0$.

Unique factorisation property

Definition 3.11.

A self-adjoint operator $C \in \mathcal{L}(\mathcal{K})$ on a Clifford-Krein module has the unique factorisation property if for any two factorisations

$$C = A_1 A_1^* = A_2 A_2^*,$$

where $A_j \in \mathcal{L}(\mathcal{K}_j, \mathcal{K})$, with $\ker A_j = \{0\}$, for Clifford-Krein modules $\mathcal{K}_j, j = 1, 2,$ there exists an isomorphism $U \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $A_1 = A_2 U$.

Theorem 3.12.

Let K be a Hermitean kernel with nonnegative majorant ℓ and Gram operator $\mathbb{K}$. Any two ℓ-continuous Kolmogorov decompositions are equivalent if and only if K has the unique factorisation property.

Evaluations mappings

Theorem 3.12.

Let $\mathcal{K}$ be a Clifford-Krein module of functionals defined on a set Ω and taking values in a Clifford-Krein module $\mathbb{V}$. Then

1. $\mathcal{K}$ has a reproducing kernel if and only if all evaluation mappings $E(x),\ x \in \Omega$ belong to $\mathcal{L}(\mathcal{K}, \mathbb{V})$;
2. the reproducing kernel is given by

$$K(x,y) = E(x)E(y)^*, \qquad x, y \in \Omega,$$

and uniquely determined by the module $\mathcal{K}$.

Remarks:

- the existence of a reproducing kernel is equivalent to the existence of a Kolmogorov decomposition with $V_j = E_j^*$.
- The reproducing kernel Clifford-Krein module is not necessarily unique, i.e. we can have two Clifford-Krein modules with the same reproducing kernel.

Uniqueness

Uniqueness of the RK Clifford-Krein module is obtained by imposing that the Krein module has an essentially unique Kolmogorov decomposition.

Theorem 3.13.

If $K(x, y), x, y \in \Omega$ is a Hermitean kernel with values in $\mathcal{L}(\mathcal{K})$ (where $\mathcal{K}$ is a Krein module), then the following statements are equivalent:

1. $K(x, y)$ is the RK for some Krein module $\mathbb{V}$ of functions on Ω.
2. $K(x, y)$ has a nonnegative majorant $\ell(x, y)$ on $\Omega \times \Omega$.
3. $K(x, y) = K_+(x, y) - K_-(x, y)$ for some nonnegative kernels $K_\pm(x, y)$ on $\Omega \times \Omega$.

Furthermore, under the above conditions we have

i. For a given nonnegative majorant $\ell(x, y)$ for $K(x, y)$ there is a Krein module $\mathbb{V}$ with RK $K(x, y)$ which is contained continuously in the Hilbert module $\mathcal{H}_\ell$ where K has a nonnegative majorant $\ell(x, y)$.
ii. There is a continuous self-adjoint operator J on $\mathcal{H}_\ell$ such that $J : \ell(x, \cdot)f \mapsto k(x, \cdot)f, x \in \Omega, f \in \mathcal{K}$. The module $\mathcal{V}$ is unique if and only if J has the unique factorization property.

An example

The Clifford module $L^2(\Omega;\mathbb{R}_{p,q}) := L^2(\Omega)\otimes_{\mathbb{R}}\mathbb{R}_{p,q}$, where $\Omega\subset\mathbb{R}^n$, endowed with the sesquilinear form

$$\langle\!\langle f,g\rangle\!\rangle_2 := \sum_{A,B}\langle f_A,g_B\rangle_{L^2(\Omega)}\,\overline{e}_A e_B$$
$$=\sum_{A,B}\left(\int_\Omega f_A(x)g_B(x)dx\right)\overline{e}_A e_B$$

is a Krein module, with inner product

$$\langle f,g\rangle_2 := [\langle\!\langle f,g\rangle\!\rangle_2]_0$$
$$=\sum_{A:\#A^+\,even}\langle f_A,g_A\rangle_{L^2(\Omega)}-\sum_{A:\#A^+\,odd}\langle f_A,g_A\rangle_{L^2(\Omega)}\,.$$

For each $f\in L^2(\Omega;\mathbb{R}_{p,q}) := L^2(\Omega)\otimes_{\mathbb{R}}\mathbb{R}_{p,q}$ we have the fundamental decomposition

$$f=\underbrace{\sum_{A:\#A^+\,even} f_A\mathbf{e}_A}_{=f_+}-\underbrace{\sum_{A:\#A^+\,odd} f_A\mathbf{e}_A}_{=f_-}\,.$$

Ultrahyperbolic Dirac operator

Consider the ultrahyperbolic Dirac operator

$$\partial_{p,q} = \partial_T - \partial_X,$$

which factorizes the ultrahyperbolic Laplace operator

$$\partial_{p,q}^2 = \Delta_T - \Delta_X,$$

where $T = \sum_{i=1}^{p} e_i x_i$ and $X = \sum_{i=p+1}^{p+q} e_i x_i$.

Additionally, we assume the ultrahyperbolic monogenic functions to be $\alpha-$homogeneous, that is,

$$\begin{cases} \partial_{p,q} u = 0, \\ \mathbb{E} u = \alpha u, \end{cases}$$

where $\mathbb{E} = \sum_{i=1}^{p+q} x_i \partial_i$ is the Euler operator. Then

$$u(T, X) = \partial_{p,q} \left[|T|^{\alpha+1} f(|X|^2) \epsilon V_\lambda(\epsilon) V_\kappa(\omega) \right]$$

where $f(\tau) = f(|X|^2)$ satisfies a hypergeometric differential equation.

RK Clifford-Krein module

Consider the reproducing kernel

$$k_c(z,w) = \sum_{(\lambda,\kappa)\in \operatorname{supp}(c):|\lambda|+|\kappa|=0}^{\infty} \overline{\Psi_{\lambda,\kappa}(\omega_z,\epsilon_z)}\Psi_{\lambda,\kappa}(\omega_w,\epsilon_w)c_{\lambda,\kappa},$$

where $\Psi_{\lambda,\kappa}(\omega,\epsilon) := [A(r_X) + \omega\epsilon B(r_X)]\, V_\lambda(\epsilon) V_\kappa(\omega)$.

This kernel k_c defines the RK Clifford-Krein module $\mathcal{K}_c$ containing all functions

$$f(z) = \sum_{(\lambda,\kappa)\in \operatorname{supp}(c):|\lambda|+|\kappa|=0}^{\infty} \Psi_{\lambda,\kappa}((\omega_z,\epsilon_z),(\omega_w,\epsilon_w))f_{\lambda,\kappa},$$

with $f_{\lambda,\kappa} \in \mathbb{R}_{p,q}$, for which it holds

$$\|f\|_c^2 := \sum_{(\lambda,\kappa)\in \operatorname{supp}(c):|\lambda|+|\kappa|=0}^{\infty} \frac{\|f_{\lambda,\kappa}\|^2_{|\mathbb{R}_{p,q}|}}{c_{\lambda,\kappa}} < \infty.$$

Interpolation

We now consider the interpolation problem: *Given the nodes and values* $(z_j, w_j), j \in \mathbb{J}$ *we want to construct a function* $f \in \mathcal{K}_c$, *such that* $f(z_j) = w_j$, *for all* $j \in \mathbb{J}$.

$$w_j = f(z_j) = \sum_{(\lambda,\kappa)\in \text{supp}(c):|\lambda|+|\kappa|=0}^{\infty} \Psi_{\lambda,\kappa}(\omega_{z_j}, \epsilon_{z_j}) f_{\lambda,\kappa}, \quad f_{\lambda,\kappa} \in \mathbb{R}_{p,q}.$$

Theorem 3.14.

The interpolation problem has a unique solution $f \in \mathcal{H}_c$ whereby the coefficients $f_{\lambda,\kappa}$ satisfy $f_{\lambda,\kappa} = f^+_{\lambda,\kappa} - f^-_{\lambda,\kappa}$ and $f^+_{\lambda,\kappa}, f^-_{\lambda,\kappa}$ are solutions of

$$\underline{w}^+ = (\Psi^+_{\lambda_l,\kappa_l}(\omega_{z_j}, \epsilon_{z_j}))_{j,l} \underline{f^+_{\lambda,\kappa}}, \quad \underline{w}^- = (\Psi^-_{\lambda_l,\kappa_l}(\omega_{z_j}, \epsilon_{z_j}))_{j,l} \underline{f^-_{\lambda,\kappa}}.$$

References

[1] Brackx, F. Delanghe R., and Sommen, *Clifford Analysis*, Research Notes in Mathematics, **76**, Pitman Advanced Publishing, 1982.

[2] O. Teichmüller, *Operatoren im Wachsschen Raum*, J. Reine Angew. Math., **174**. (1936), 73-124.

[3] L. Schwartz, *Sous espaces Hilbertiens d'espaces vectoriels topologiques et noyaux associés (noyaux reproduisants)*, J. Analyse Math., **13**, (1964),115-256.

[4] T. Ya. Azizov and I.S. Iohvidov, *Linear operators in spaces with an indefinite metric*, John Wiley, New York, 1989.

[5] I. Gelfand, S. Gelfand, V. Retakh, and R. L. Wilson, *Quasideterminants, Advances in Mathematics*, **193**, (2005), 56-141.

[6] D. Alpay, PC, U. Kähler, *Krein reproducing kernel modules in Clifford analysis*, J. Analyse Math., **143**, (2021), 253-288.

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