A detour on Sobolev inequalities in the Heisenberg group and applications to critical subelliptic problems

Patrizia Pucci

Università degli Studi di Perugia

Ghent Analysis & PDE Center

Ghent University

M. Ruzhansky, K. Van Bockstal, M. Chatzakou, J. Restrepo & B. Torebek

I. Ali, J. Huang, D. Santiago Gómez Cobos

Ms K. Verbeeck

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Backgrounds on the Heisenberg group

The study of critical equations in the context of the stratified Lie groups is a fast growing and fascinating topic. There are multiple reasons behind this interest. On one hand, it is well known that the Heisenberg group appears in various areas of physics and science, such as quantum theory (uncertainty principle, commutation relations), cf.

P. Cartier, Quantum mechanical commutation relations and theta functions, *Proc. Sympos. Pure Math.* **9** (1966), 361–383.

P.P. Divakaran, Quantum theory as the representation theory of symmetries, *Physical Review Letters* **79** (1997), 2159–2163.

in signal theory,

W. Schempp, *Harmonic analysis on the Heisenberg nilpotent Lie group, with applications to signal theory*, Longman Scientific and Technical, Harlow, Essex, 1986, 199 pp.

and theory of theta functions, cf. Cartier above and

S. Zelditch, Index and dynamics of quantized contact transformations, *Ann. Inst. Fourier* **47** (1997), 305–363.

From a mathematical point of view, the main reason of the interest in studying critical equations in this context is the strong connection with the Yamabe problem on Cauchy–Riemann (CR) manifolds. We refer for further details on this subject, e.g., to

- B. Bianchini, L. Mari, M. Rigoli, Yamabe type equations with sign–changing nonlinearities on the Heisenberg group, and the role of Green functions, Recent trends in nonlinear partial differential equations, I. Evolution problems, 115–136, Contemp. Math., 594, Amer. Math. Soc., Providence, RI, 2013.
- N. Garofalo, Gradient bounds for the horizontal p–Laplacian on a Carnot group and some applications *Manuscripta Math.* **130** (2009), 375–385.
- N. Garofalo, E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, *Ann. Inst. Fourier* **40** (1990), 313–356.
- N. Garofalo, E. Lanconelli, Existence and nonexistence results for semilinear equations on the Heisenberg group, *Indiana Univ. Math. J.* **41** (1992), 71–98.
- N. Garofalo, D. Vassilev, Symmetry properties of positive entire solutions of Yamabe–type equations on groups of Heisenberg type, *Duke Math. J.* **106** (2001), 411–448.

N. Garofalo, D. Vassilev, The non–linear Dirichlet problem and the CR Yamabe problem, *Boundary value problems for elliptic and parabolic operators* (Catania, 1998), Matematiche (Catania) **54** (1999), suppl., 75–93.

D. Jerison, J.M. Lee, *A subelliptic, nonlinear eigenvalue problem and scalar curvature on CR manifolds*, Microlocal analysis (Boulder, Colo., 1983), 57–63, Contemp. Math. **27**, Amer. Math. Soc., Providence, RI, 1984.

D. Jerison, J.M. Lee, The Yamabe problem on CR manifolds, *J. Differential Geom.* **25** (1987), 167–197.

A. Kristály, Nodal solutions for the fractional Yamabe problem on Heisenberg groups *Proc. Roy. Soc. Edinburgh Sect. A* **150** (2020), 771–788.

F. Uguzzoni, A note on Yamabe-type equations on the Heisenberg group, *Hiroshima Math. J.* **30** (2000), 179–189.

In the development of the theory of partial differential equations in the Heisenberg group and, more generally, in sub–Riemanian manifolds, it is important to explore new challenging problems as well as to figure out, in this emerging variety of results, whether or not the standard methods developed in the Euclidean spaces can be adapted to this new context. This kind of analysis is the main scope of this course.

Next I review some necessary background on the Heisenberg group. Analysis on the Heisenberg group is very interesting because this space is topologically Euclidean, but analytically non-Euclidean, and so some basic tools, such as dilatations, must be developed again. One of the main differences with the Euclidean case is the appearance of the so-called homogeneous dimension $Q = 2n + 2$ in the Heisenberg group. The number $Q > 2n + 1$, which we introduce later, plays a role analogous to the topological dimension in the Euclidean context. For a complete treatment, we refer to

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J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equuations* **3** (1995), 493–512.

N. Garofalo, E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, *Ann. Inst. Fourier* **40** (1990), 313–356.

G.P. Leonardi, S. Masnou, the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$, *Ann. Mat. Pura Appl.* **184** (2005), 533–553.

A. Loiudice, Improved Sobolev inequalities on the Heisenberg group, *Nonlinear Anal.* **62** (2005), 953–962.

Definition of the Heisenberg group

The Heisenberg group $\mathbb{H}^n$ is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$,

$$\mathbb{H}^n = \{\xi = (z,t) \in \mathbb{R}^{2n+1} \mid z = (x,y) \in \mathbb{R}^{2n},\ t \in \mathbb{R}\}$$

endowed with the non–Abelian group law

$$\xi \circ \xi' = \Big(z + z', t + t' + 2\sum_{i=1}^{n}(y_i x_i' - x_i y_i')\Big)$$

for all $\xi, \xi' \in \mathbb{H}^n$ with $\xi = (z,t) = (x,y,t)$ and $\xi' = (z',t') = (x',y',t')$. Clearly, $O = (0,0)$ is the identity element of $\mathbb{H}^n$ and for any $\xi \in \mathbb{H}^n$, $\xi^{-1} = -\xi$.

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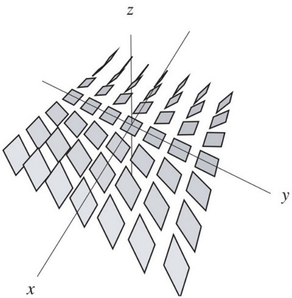
The sub–Riemannian structure

The Heisenberg group is the simplest sub–Riemannian manifold that is not Riemannian and the simplest noncommutative nilpotent Lie group.



A distribution on $\mathbb{H}^1 \cong \mathbb{R}^3$

- The constrain on admissible curves is given by a distribution of planes, that is a distribution that smoothly assigns to each point a plane (inside the 3D tangent space).
- The admissible curves are tangent to such a distribution.

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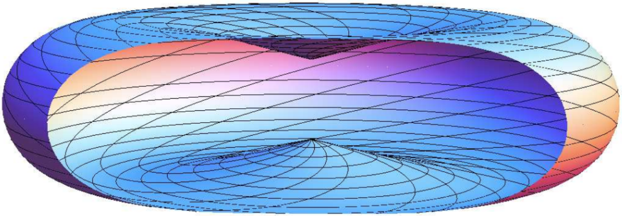
- Each pair of points of $\mathbb{H}^n$ can be connected by at least one admissible curve. From this fact one can define a distance between two points ξ and ξ' as the infimum of the length of all the admissible curves between ξ and ξ':
$$d_{CC}(\xi, \xi') = \inf\{\text{Length}(\gamma) \,|\, \gamma \text{ admissible curve between } \xi \text{ and } \xi'\}.$$
The celebrated theorem of Piotr-Zimmermann in 2015 completely describes the geodesics connecting two points.
- This metric space is not Riemannian. Indeed, the topological dimension $2n+1$ is strictly less than the homogeneous dimension $Q = 2n+2$.



The unit ball in $\mathbb{H}^1$

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An equivalent norm

- The Korányi norm is given for $\xi = (z, t) \in \mathbb{H}^n$ by

$$r(\xi) = r(z,t) = (|z|^4 + t^2)^{1/4}.$$

  and is defined via the stratification property of $\mathbb{H}^n$. Here $|\cdot|$ is the Euclidean norm of the horizontal layer. The Korányi norm is 1–homogeneous with respect to the group of dilations. From the computational viewpoint, the Korányi norm is easier to handle compared to the Carnot–Carathéodory norm. However, all homogeneous norms are equivalent in the context of Carnot groups.
- We prefer to use the Korányi distance, since it is much easier to compute than the Carnot–Carathéodory (CC) distance, even if it does not reflect the sub-Riemannian structure of the Heisenberg group. Despite this, the two metrics are closely related.

An equivalent norm

- Interestingly, it was shown by Yang in 2013 that the L-gauge $d(x)$ – sometimes also called the Korányi-Folland or Kaplan gauge – can be replaced by the CC distance, and the Hardy inequality remains valid with the same best constant $p/(Q-p)$, where $Q = 2n+2$ is the homogeneous dimension of the n Heisenberg group. Homogenoeous dimension or Hausdorff dimension Q.
- The corresponding distance, the so called Korányi distance, is

$$d_K(\xi, \xi') = r(\xi^{-1} \circ \xi')$$

for all $(\xi, \xi') \in \mathbb{H}^n \times \mathbb{H}^n$.
- This distance acts like the Euclidean distance in horizontal directions and behaves like the square root of the Euclidean distance in the missing direction.

Properties of the Korányi norm

- Translation invariance w.r.t. left translations $(\tau_\eta)_{\eta\in\mathbb{H}^n}$ given by

$$\tau_\eta : \xi \mapsto \eta \circ \xi \quad \text{for all } \xi \in \mathbb{H}^n.$$

- Homogeneity of degree 1 w.r.t. the family of dilations $(\delta_R)_{R>0}$, given by

$$\delta_R \;:\; \xi = (z,t) \mapsto (Rz, R^2 t) \qquad\qquad \text{for all } \xi \in \mathbb{H}^n,$$

since $r(\delta_R(\xi)) = r(Rz, R^2t) = (|Rz|^4 + R^4 t^2)^{1/4} = R\, r(\xi)$ for all $\xi = (z,t) \in \mathbb{H}^n$.

- The Jacobian determinant of $\delta_R : \mathbb{H}^n \to \mathbb{H}^n$ is constant and equal to R^{2n+2}. This is why the natural number $Q = 2n + 2$ is called homogeneous dimension of $\mathbb{H}^n$.

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The Lie algebra of left–invariant vector fields on $\mathbb{H}^n$ is generated by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \quad j = 1, \dots, n,$$
$$T = \frac{\partial}{\partial t}.$$

This basis satisfies the Heisenberg canonical commutation relations

$$[X_j, Y_k] = -4\delta_{jk} T, \quad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

In the span of $\{X_j, Y_j\}_{j=1}^n \simeq \mathbb{R}^{2n}$ we consider the natural inner product given by

$$\big(X, Y\big)_H = \sum_{j=1}^n \left(x^j y^j + \widetilde{x}^j \widetilde{y}^j\right)$$

for $X = \{x^j X_j + \widetilde{x}^j Y_j\}_{j=1}^n$ and $Y = \{y^j X_j + \widetilde{y}^j Y_j\}_{j=1}^n$.

The inner product $(\cdot,\cdot)_H$ produces the Hilbertian norm

$$|X|_H = \sqrt{(X,X)_H}$$

for the horizontal vector field X. The horizontal gradient of u is

$$D_H u = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u),$$

where

$$|D_H u|_H = \sqrt{\sum_{i=1}^{n} \{|X_i u|^2 + |Y_i u|^2\}}.$$

The Heisenberg group is a particular example of a wide class of nilpotent Lie groups referred to as Carnot groups in the literature.

Definition

A Carnot group is a connected and simply connected Lie group G with a Lie algebra $\mathfrak{g}$, which admits a stratification, i.e. $\mathfrak{g}$ is the direct sum of linear subspaces, $\mathfrak{g} = \bigoplus_{j=1}^{r} V_j$, such that

(i) $[V_1, V_j] = V_{j+1}$ for $j = 1, \ldots, r-1$,

(ii) $[V_1, V_r] = 0$.

The number r is called the step of the Carnot group. Clearly the Heisenberg group is a Carnot group of step 2. Most of the results that we prove in the course can be extended to this more general context. However, we do not present them in the most general framework, and we limit our treatment to the Heisenberg group context. For a detailed discussion about Carnot groups we refer to

J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equations* **3** (1995), 493–512.

S.P. Ivanov, D.N. Vassilev, *Extremals for the Sobolev inequality and the quaternionic contact Yamabe problem*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011, xviii+219 pp.

The operator Δ_H

The horizontal Laplacian in $\mathbb{H}^n$ is the operator

$$\begin{aligned}\Delta_H u &= \sum_{j=1}^{n} (X_j^2 + Y_j^2) u \\ &= \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2}.\end{aligned}$$

The operator Δ_H is subelliptic, according to Hörmander's notation.
Le us also define, for $p > 1$, the operator horizontal p–Laplacian

$$\Delta_{H,p}\varphi = \mathrm{div}_H(|D_H\varphi|_H^{p-2} D_H\varphi)$$

for any $\varphi \in C_c^\infty(\mathbb{H}^n)$.

The Haar measure and the Lebesgue spaces

We briefly recall the definition of the Haar measure of a locally compact topological group, and we specify it in the context of the Heisenberg group. For a detailed treatment about general Haar measures we refer to

- A. Bonfigli, E. Lanconelli, F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer–Verlag, Berlin Heidelberg, 2007, xxvi+802 pp.
- G.B. Folland, *Harmonic Analysis in Phase Space*, Annals of Mathematics Studies, 122. Princeton University Press, Princeton, NJ, 1989, x+277 pp.
- G.B. Folland, E.M. Stein, Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group, *Comm. Pure Appl. Math.* **27** (1974), 429–522.

and for the special case of the Heisenberg group we refer to

- J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equations* **3** (1995), 493–512.
- S. G. Krantz, *Explorations in Harmonic Analysis: With Applications to Complex Function Theory and the Heisenberg Group*, Birkhäuser B[illegible], 2009, xiv+362 pp.

Definition

Let $(G, \circ)$ be a topological group. A left Haar measure on G is a nonzero regular Borel measure μ on G such that $\mu(g \circ A) = \mu(A)$ for all $g \in G$ and all Borel measurable subsets A of G. The corresponding integral is invariant under left translations, that is

$$\int_G u(g' \circ g) d\mu = \int_G u(g) d\mu$$

for any integrable function u on G.

Similarly a right Haar measure is also defined. Moreover, by Proposition 1.3.21 in

A. Bonfigli, E. Lanconelli, F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer–Verlag, Berlin Heidelberg, 2007, xxvi+802 pp.

if μ and μ' are two left Haar measures on G then $\mu = a\mu'$ for some $a \in \mathbb{R}^+$, and so the Haar measure is unique up to a multiplicative positive constant.

In the special case of the Heisenberg group, it is easy to check that the Lebesgue measure on $\mathbb{R}^{2n+1}$ is invariant under left translations. Thus, from here on, we denote by $d\xi$ the Haar measure on $\mathbb{H}^n$ that coincides with the $(2n+1)$–Lebesgue measure.
Moreover, $|U|$ is the $(2n+1)$–dimensional Lebesgue measure of any measurable Borel set $U \subset \mathbb{H}^n$. Furthermore, the Haar measure on $\mathbb{H}^n$ is Q–homogeneous with respect to dilations δ_R. Then,

$$|\delta_R(U)| = R^Q |U|, \qquad d(\delta_R \xi) = R^Q d\xi.$$

In particular $|B_R| = |B_1| R^Q$.

The Lebesgue Spaces on $\mathbb{H}^n$

For any measurable set $U \subset \mathbb{H}^n$ and for any $1 \leq p \leq \infty$, we denote $L^p(U)$ the set of all measurable functions $u : U \to \mathbb{R}$ such that $\|u\|_{L^p(U)} < \infty$, where

$$\|u\|_{L^p(U)} = \left(\int_U |u|^p d\xi \right)^{1/p}, \quad \text{if } 1 \leq p < \infty,$$

while

$$\|u\|_{L^\infty(U)} = \operatorname*{ess\,sup}_U u = \inf\{M \, : \, |u(\xi)| \leq M \text{ for a.e. } \xi \in U\}.$$

When $U = \mathbb{H}^n$ or when there is not ambiguity about the set considered, for simplicity we denote the norm $\| \cdot \|_p$.
The symbol $\mathbb{1}_U$ denotes the characteristic function of a Lebesgue measurable subset U of any Lebesgue σ–algebra.

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Properties of $L^p(\Omega)$

Let Ω be a nonempty open set of $\mathbb{H}^n$.
The Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, are Banach spaces and $C_c(\Omega)$ is dense in $L^p(\Omega)$ when $1 \leq p < \infty$.
Moreover, the spaces $L^p(\Omega)$ are reflexive when $1 < p < \infty$, while $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive.
Indeed, for $1 \leq p < \infty$ the dual space $\big(L^p(\Omega)\big)'$ of $L^p(\Omega)$ can be identified with $L^{p'}(\Omega)$, where p' denotes the Hölder conjugate of p, that is $1/p + 1/p' = 1$, so that $\big(L^1(\Omega)\big)'$ can be identified with $L^\infty(\Omega)$.
On the other hand, the dual space of $L^\infty(\Omega)$ is identified with the space of all absolutely continuous, finitely additive set functions of bounded total variation on Ω, cf. Chapter IV.9 of

K. Yosida, *Functional analysis*, Reprint of the sixth (1980) edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995, xvi+504 pp.

The horizontal Sobolev spaces and geometric inequalities

Sharp geometric inequalities on the Heisenberg group $\mathbb{H}^n$, such as the Hardy–Sobolev inequality and the Trudinger–Moser inequality, play an important role in the study of the existence of solutions to nonlinear partial differential equations involving power nonlinearities with critical exponents and nonlinearities of exponential growth, cf.

- N. Lam, G. Lu, Sharp Moser–Trudinger inequality on the Heisenberg group at the critical case and applications, *Adv. Math.* **231** (2012), 3259–3287.
- N. Lam, G. Lu, H. Tang, On nonuniformly subelliptic equations of Q–sub–Laplacian type with critical growth in the Heisenberg group, *Adv. Nonlinear Stud.* **12** (2012), 659–681.
- N. Lam, G. Lu, H. Tang, Sharp subcritical Moser–Trudinger inequalities on Heisenberg groups and subelliptic PDEs, *Nonlinear Anal.* **95** (2014), 77–92.
- J. Li, G. Lu, M. Zhu, Concentration–compactness principle for Trudinger–Moser inequalities on Heisenberg groups and existence of ground state solutions, *Calc. Var. Partial Differential Equations* **57** (2018), 26 pp.
- G. Mingione, A. Zatorska–Goldstein, X. Zhong, Gradient regularity for elliptic equations in the Heisenberg group, *Adv. Math.* **222** (2009), 62–129.

Following

- J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equations* **3** (1995), 493–512.
- S.P. Ivanov, D.N. Vassilev, *Extremals for the Sobolev inequality and the quaternionic contact Yamabe problem*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011, xviii+219 pp.

we introduce some necessary background on the horizontal Sobolev spaces in the Heisenberg group. Then, we collect useful comments and results related to the sharp geometric inequalities on $\mathbb{H}^n$. Our presentation is partly taken from the above monograph and

- L. Capogna, D. Danielli , S.D. Pauls, J.T. Tyson, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Birkhäuser Basel, Progress in Mathematics **259**, 2007, xvi+224 pp.
- G.B. Folland, E.M. Stein, Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group, *Comm. Pure Appl. Math.* **27** (1974), 429–522.
- N. Lam, PhD Thesis, *Moser–Trudinger And Adams Type Inequalities And Their Applications*, Wayne State University Dissertations, 2014, 90 pp.
- P.–L. Lions, The concentration compactness principle in the calculus of variations. The limit case I, *Rev. Mat. Iberoamericana* **1** (1985), no. 1, 145–201.

References in more general settings

- R. Bramati, A family of sharp inequalities on real spheres *Complex Var. Elliptic Equ.* **67** (2022), 2030–2042.
- R. Bramati, P. Ciatti, J. Green, J. Wright, Oscillating spectral multipliers on groups of Heisenberg type, *Rev. Mat. Iberoam.* **38** (2022), 1529–1551.
- M. Chatzakou, M. Ruzhansky, N. Tokmagambetov, Fractional Schrödinger equations with singular potentials of higher order. II: hypoelliptic case, *Rep. Math. Phys.* **89** (2022), 59–79.
- A. Kassymov, M. Ruzhansky, D. Suragan, Hardy inequalities on metric measure spaces, III: the case $q \leq p \leq 0$ and applications *Proc. A.* **479** (2023), no. 2269, Paper No. 20220307, 16 pp.
- A. Kassymov, M. Ruzhansky, D. Suragan, Reverse Stein-Weiss, Hardy-Littlewood-Sobolev, Hardy, Sobolev and Caffarelli-Kohn-Nirenberg inequalities on homogeneous groups, *Forum Math.* **34** (2022), 1147–1158.

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- A. Kassymov, M. Ruzhansky, D. Suragan, Reverse integral Hardy inequality on metric measure spaces *Ann. Fenn. Math.* **47** (2022), 39–55.
- A. Kassymov, M. Ruzhansky, D. Suragan, Hardy-Littlewood-Sobolev and Stein-Weiss inequalities on homogeneous Lie groups *Integral Transforms Spec. Funct.* **30** (2019), 643–655.
- A. Kassymov, M. Ruzhansky, B.T. Torebek, Rayleigh-Faber-Krahn, Lyapunov and Hartmann-Wintner inequalities for fractional elliptic problems, *Mediterr. J. Math.* **20** (2023), Paper No. 119, 14 pp.
- M. Ruzhansky, N. Yessirkegenov, Critical Gagliardo-Nirenberg, Trudinger, Brezis-Gallouet-Wainger inequalities on graded groups and ground states *Commun. Contemp. Math.* 24 (2022), Paper No. 2150061, 29 pp.

The horizontal Sobolev space $HW^{1,p}(\Omega)$

Let us restrict to the special case $1 \leq p < \infty$ and to an open set Ω of $\mathbb{H}^n$.

Definition

Let $HW^{1,p}(\Omega)$ be the horizontal Sobolev space consisting of all functions $u \in L^p(\Omega)$ such that $D_H u$ exists in the sense of distributions and $|D_H u|_H \in L^p(\Omega)$, endowed with the natural norm

$$\|u\|_{HW^{1,p}(\Omega)} = \left(\int_\Omega |u|^p d\xi + \int_\Omega |D_H u|_H^p d\xi \right)^{1/p} .$$

We recall that the underlying measure in use here is the Haar measure on $\mathbb{H}^n$, which agrees with the Lebesgue measure on $\mathbb{R}^{2n+1}$. From here on,

$$\|D_H u\|_{L^p(\Omega)} = \left(\int_\Omega |D_H u|_H^p d\xi \right)^{1/p} .$$

and $\|D_H u\|_p = \|D_H u\|_{L^p(\mathbb{H}^n)}$, when $\Omega = \mathbb{H}^n$, for simplicity.

It is easy to check that the distributional horizontal gradient of a function $u \in HW^{1,p}(\Omega)$ is uniquely defined a.e. in Ω. Furthermore, if u is a smooth function, then its classical horizontal gradient is also the distributional horizontal gradient. For this reason, if u is a nonsmooth function, $D_H u$ is meant in the distributional sense.

The space $HW^{1,p}(\Omega)$ is a separable Banach space if $1 \leq p < \infty$ and a reflexive Banach space if $1 < p < \infty$.

For $1 < p < \infty$ the dual space of $HW^{1,p}(\mathbb{H}^n)$ is

$$HW^{-1,p'}(\mathbb{H}^n) = \Big\{h^0 + \sum_{j=1}^{n} \left(h_j^1 X_j + h_j^2 Y_j\right) \ : \ h^0, h_j^1, h_j^2 \in L^{p'}(\mathbb{H}^n), j = 1, \dots, n\Big\},$$

where the pairing between a function $u \in HW^{1,p}(\mathbb{H}^n)$ and a functional $h = h^0 + \sum_{j=1}^{n} \left(h_j^1 X_j + h_j^2 Y_j\right)$ is given as usual by

$$\langle h, u\rangle_{HW^{-1,p'}, HW^{1,p}} = \int_{\mathbb{H}^n} \Big\{h^0 u + \sum_{j=1}^{n} \left(h_j^1 X_j u + h_j^2 Y_j u\right)\Big\} d\xi,$$

as shown in

A. Baldi, B. Franchi, N. Tchou, M.C. Tesi, Compensated compactness for differential forms in Carnot groups and applications, *Adv. Math.* **223** (2010), 1555–1607.

The corresponding norm is

$$\|h\|_{HW^{-1,p'}} = \inf \left\{ \|h^0\|_{p'} + \sum_{j=1}^n \left(\|h_j^1\|_{p'} + \|h_j^2\|_{p'} \right) \ : \ h = h^0 + \sum_{j=1}^n \left(h_j^1 X_j + h_j^2 Y_j \right) \right\}.$$

Let us now recall some density results for the horizontal Sobolev spaces, such as the analogous of the celebrated Meyers-Serrin theorem, which can be found in

B. Franchi, R. Serapioni, F. Serra Cassano, Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields, *Houston Math. J.* **22** (1996), 859–889.

The density theorem for horizontal Sobolev spaces in the Heisenberg group is analogous to the Euclidean one, thatis, $C_c^\infty(\mathbb{H}^n)$ is dense in $HW^{1,p}(\mathbb{H}^n)$ for every p with $1 \le p < \infty$.

As shown in

B. Franchi, R. Serapioni, F. Serra Cassano, Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields, *Houston Math. J.* **22** (1996), 859–889.

N. Garofalo, D.–M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.* **49** (1996), 1081–1144.

we have a complete extension of

Theorem (Meyers–Serrin theorem)

Let Ω be an open set in $\mathbb{H}^n$ and $1 \leq p < \infty$. Then,

$$C^\infty(\Omega) \cap HW^{1,p}(\Omega) \text{ is dense in } HW^{1,p}(\Omega).$$

The Folland–Stein inequality in the Heisenberg group

The subelliptic variant of the Sobolev inequality has a form similar to the Euclidean version, but the exponent governing the transition to the supercritical case is the homogeneous dimension $Q = 2n + 2$. The Folland–Stein inequality is valid also for any Carnot group, but we state it in the context the Heisenberg group, see also Franchi, Gallot and Wheeden, Math. Ann. 1994.

Theorem (The Folland–Stein inequality, Comm. Pure Appl. Math. 1974)

Let $1 < p < Q$ *and set* $p^* = \dfrac{pQ}{Q-p}$. *Then, there exists a constant* $C_{p,Q}$ *such that*

$$\text{(FS)} \qquad \left(\int_{\mathbb{H}^n} |\varphi|^{p^*} d\xi \right)^{1/p^*} \le C_{p,Q} \left(\int_{\mathbb{H}^n} |D_H \varphi|^p d\xi \right)^{1/p}$$

for all $\varphi \in C_c^\infty(\mathbb{H}^n)$.

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Some remarks

- The Folland–Stein inequality is also true for functions in the space $HW_0^{1,p}(\Omega)$, where Ω is an open subset of $\mathbb{H}^n$ and $HW_0^{1,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \|D_H u\|_{L^p(\Omega)}^p\right)^{1/p}.$$

- In particular, if $\Omega = \mathbb{H}^n$, the Folland–Stein inequality holds in the horizontal Sobolev space $HW^{1,p}(\mathbb{H}^n)$ consisting of the functions $u \in L^p(\mathbb{H}^n)$ such that $D_H u$ exists in the sense of distributions and $|D_H u|_H \in L^p(\mathbb{H}^n)$, and $HW^{1,p}(\mathbb{H}^n)$ is endowed with the norm $\|u\|_{HW^{1,p}(\Omega)}$, $\Omega = \mathbb{H}^n$.

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Some remarks

If I know that there exists q such that

$$\|\varphi\|_q \leq C\|D_H\varphi\|_p$$

holds for any $\varphi \in C_c^\infty(\mathbb{H}^n)$, then $q = p^* = \dfrac{pQ}{Q-p}$. Indeed, taking $\varphi_\lambda(\xi) = \varphi(\delta_\lambda(\xi))$, $\lambda > 0$,

$$\|\varphi_\lambda\|_q \leq C\|D_H\varphi_\lambda\|_q,$$

$$\lambda^{-Q/q}\|\varphi\|_q \leq C\lambda^{1-Q/p}\|D_H\varphi\|_p,$$

so that

$$\frac{1}{q} = \frac{Q-p}{pQ}.$$

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Best constant of the Folland–Stein inequality

- Actually, much less is known about sharp constants for the Folland and Stein inequality on the Heisenberg group than for Sobolev inequality on the Euclidean space.

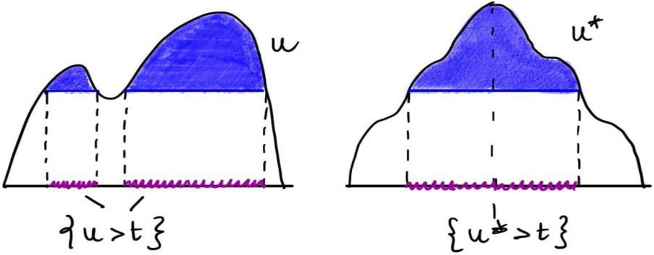
- As in most proofs of sharp constants in Euclidean spaces (note the celebrated results of Talenti and Aubin), one attempts to use the radial nonincreasing rearrangement u^* of functions u (in terms of a certain norm) on the Heisenberg group.

T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, *J. Differential Geometry* **11** (1976), 573–598.

G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* **110** (1976), 353–372.

Radial nonincreasing rearrangement

The radial nonincreasing rearrangement u^* of a function u is the unique function such that the level sets $\{x \in \mathbb{R}^n \,:\, u^*(x) > t\}$ of u^* are balls which have the same measure as the level sets $\{x \in \mathbb{R}^n \,:\, u(x) > t\}$ of u.



The radial rearrangement function u^*

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Radial nonincreasing rearrangement

Equivalently, the radial nonincreasing rearrangement of a function u is defined as

$$u^*(x) = \sup\{t \,:\, \mu(t) > \omega_n |x|^n\},$$

where ω_n denotes volume of the unit ball in $\mathbb{R}^n$ and

$$\mu(t) = meas\{x \in \mathbb{R}^n \,:\, u(x) > t\}$$

is the measure of the level sets of u.

Theorem (Pólya–Szegő inequality)

Given $u \in W^{1,p}(\mathbb{R}^n)$, *the non–increasing rearrangement* u^* *satisfies*

$$\|u^*\|_p = \|u\|_p, \qquad \|\nabla u^*\|_p \leq \|\nabla u\|_p.$$

In particular, $u^* \in W^{1,p}(\mathbb{R}^n)$.

- The Pólya–Szegő inequality is used to prove the Rayleigh–Faber–Krahn inequality, which states that among all the domains of a given fixed volume, the ball has the smallest first eigenvalue for the Laplacian with Dirichlet boundary conditions.
- The optimal constant in the Sobolev inequality in $\mathbb{R}^n$ can be obtained by combining the Pólya–Szegő inequality with some integral inequalities.

The Euclidean case

In the Euclidean case, if $1 < p < n$ the best constant of the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{1/p^*} \leq C_{p,n} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{1/p}$$

was computed by Talenti (Ann. Mat. Pura Appl., 1976). The proof is accomplished by symmetrization and consists in two main steps.

1. the ratio
$$J(u) = \frac{\|u\|_{p^*}}{\|\nabla u\|_p}$$
attains its maximum value on spherically symmetric functions.
2. The functional J has a maximum in a class of spherically symmetric functions.

More precisely, the functional $J(u) = \dfrac{\|u\|_{p^*}}{\|\nabla u\|_p}$ increases if u is replaced by u^*.

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Radial nonincreasing rearrangement in $\mathbb{H}^n$

- Of course the definition of u^* can be generalized to the Heisenberg context.
- Problem: In the Heisenberg context, it is NOT KNOWN whether or not the L^p norm of the horizontal gradient of the rearrangement of a function is dominated by the L^p norm of the horizontal gradient of the function.
- In other words, the Pólya–Szegő inequality

$$\|D_H u^*\|_p \leq \|D_H u\|_p$$

for the horizontal gradient in the Heisenberg group is NOT available.

Radial nonincreasing rearrangement in $\mathbb{H}^n$

- Manfredi and Vera De Serio (Acta Math. Sin., 2019) proved a generalization of the Pólya–Szegő inequality for $p \geq 1$ for the Heisenberg group asserting that there exists a constant $C = C(p) \geq 1$, depending only on p, such that

$$\|D_H u^*\|_p \leq C\|D_H u\|_p$$

(actually the result holds in the more general context of Carnot groups).
- However, the exact value of C is not known and so this result cannot be used to determine the sharp constant of the Folland–Stein inequality.
- The work of Jerison and Lee (J. Amer. Math. Soc., 1988) indicates that this inequality does not hold with $C = 1$ if $p = 2$.

Proof of the Pólya–Szegő inequality

The three main ingredients are:

- The coarea formula: given Ω open set in $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}$ a Lipschitz function and $g \in L^1(\mathbb{R}^n)$, then

$$\int_\Omega g(x)|\nabla u(x)|\,dx = \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} g(x)\,d\mathcal{H}_{n-1}(x) \right) dt$$

- The Hölder inequality
- The (classical) isoperimetric inequality Given $E \subset \mathbb{R}^n$ a regular set such that $\mathcal{H}^{n-1}(\partial E) < \infty$, then

$$\min\{\text{meas}(E), \text{meas}(\mathbb{R}^n \backslash E)\} \leq C_I(n) \left[\mathcal{H}^{n-1}(\partial E)\right]^{1^*}, \quad 1^* = \frac{n}{n-1},$$

and the best constant $C_I(n)$ in the previous inequality is given by

$$\text{meas}(B_1) / \left[\mathcal{H}^{n-1}(\partial B_1)\right]^{1^*} = \omega_n / [n\omega_n]^{1^*}.$$

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Main issue: Isoperimetric inequality in the Heisenberg group

- The isoperimetric problem in the Heisenberg group is one of the central questions of sub–Riemannian geometric analysis.
- The inequality (for $n = 1$)

$$\min\{\text{meas}(E), \text{meas}(\mathbb{H} \setminus E)\} \leq C_{I,H} \left[P_H(\partial E)\right]^{4/3}$$

  has been proved by Pansu (R. Acad. Sci. Paris Sér. I Math., 1982)
- Open problem: the best constant and the extremal configuration are still unknown
- Pansu conjecture: the best constant $C_{I,H}$ in the previous inequality is given by

$$\text{meas}(\mathcal{B}(O,R) / \left[P_H(\partial\mathcal{B}(O,R))\right]^{4/3} = 3/[4\sqrt{\pi}]^{4/3}$$

  and $\mathcal{B}(O,R)$ is a bubble set.

Bubble sets in $\mathbb{H}^1$

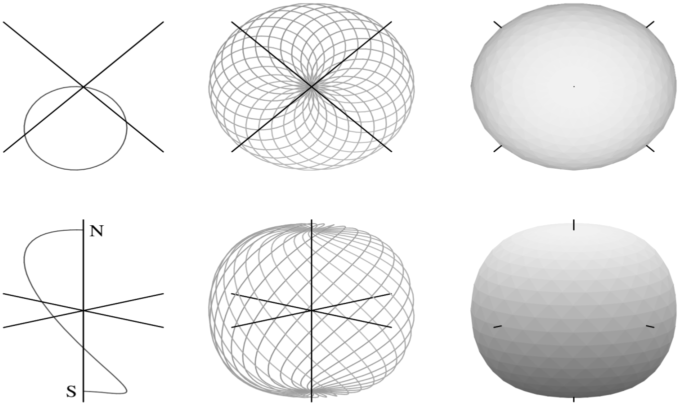

Figura: Leonardi and Masnou, *On the isoperimetric problem in the Heisenberg group* $\mathbb{H}^n$, Springer–Verlag (2005)

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The case $p = 2$

In conclusion, new ideas are needed in deriving sharp inequalities on the Heisenberg group.

- The first major breakthrough came after the works of Jerison and Lee (J. Amer. Math. Soc., 1988) on the sharp constants for the Sobolev inequality and extremal functions on the Heisenberg group, in conjunction with the solution of the CR Yamabe problem.
- In particular, the best constant for the Folland–Stein inequality on $\mathbb{H}^n$ for $p = 2$

$$\left(\int_{\mathbb{H}^n} |u|^{2^*} d\xi \right)^{1/2^*} \leq C_{2,Q} \left(\int_{\mathbb{H}^n} |D_H u|^2 d\xi \right)^{1/2}$$

was found and the extremal functions were identified.

Theorem (Jerison and Lee, J. Amer. Math. Soc., 1988)

The best constant for the Folland–Stein inequality for $p = 2$ on $\mathbb{H}^n$

$$\text{(FS)} \qquad \left(\int_{\mathbb{H}^n} |u|^{2^*} d\xi \right)^{1/2^*} \leq C_{2,Q} \left(\int_{\mathbb{H}^n} |D_H u|^2 d\xi \right)^{1/2}$$

is given by

$$C_{2,Q} = \frac{1}{4\pi n^2} \left[\Gamma(n+1) \right]^{1/(n+1)},$$

and all the extremals of (FS) are obtained by dilations and left translations of the function

$$K \left| \left(t + i(|z|^2 + 1) \right) \right|^{-n}.$$

Furthermore, the extremals in (FS) are constant multiples of images under the Cayley transform of extremals for the Yamabe functional on the sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$.

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Open problems

The work of Jerison and Lee raised two natural questions.

1. What is the best constant $C_{p,Q}$ Folland Stein inequality for all p, when $1 < p < Q$ and $p \neq 2$?
2. What about the borderline case $p = Q$?

While the first question still seems to be open, the second question was answered in the work of Cohn and Lu (Indiana Univ. Math. J., 2001).

Existence of minimizers via concentration–compactness

When $1 < p < Q$, while the symetrization argument fails, the concentration–compactness method by Lions (Ann. Inst. H. Poincaré Anal. Non Linéaire, 1984) can be generalized to deal with the problem of finding minimizers of the variational problem

$$\mathcal{I} = \inf_{\substack{u \in C_c^\infty(\mathbb{H}^n) \\ u \neq 0}} \frac{\|D_H u\|_p}{\|u\|_{p^*}}.$$

The natural space where looking for extremals is the Folland–Stein space $S^{1,p}(\mathbb{H}^n)$, defined as the completion of $C_c^\infty(\mathbb{H}^n)$ w.r.t. the norm

$$\|u\| = \|D_H u\|_p = \left(\int_{\mathbb{H}^n} |D_H u|^p d\xi \right)^{1/p}.$$

Theorem (Ivanon and Vassilev, 2011)

Let $1 < p < Q$. Every minimizing sequence $(u_n)_n$ in $S^{1,p}(\mathbb{H}^n)$ of the variational problem

$$\text{(VP)} \qquad \mathcal{I} = \inf_{\substack{u \in C_c^\infty(\mathbb{H}^n) \\ u \neq 0}} \frac{\|D_H u\|_p}{\|u\|_{p^*}}$$

has a convergent subsequence in $S^{1,p}(\mathbb{H}^n)$ after possibly translating and dilating its elements by

$$u^\eta(\xi) = u(\tau_\eta(\xi)), \quad u_\lambda(\xi) = \lambda^{Q/p^*} u(\delta_\lambda(\xi))$$

Moreover, the infimum in (VP) is achieved by a nonnegative function $u \in S^{1,p}(\mathbb{H}^n)$ which is a weak nonnegative solution of the critical equation

$$-\Delta_{H,p} u = u^{p^*-1} \quad \text{in } \mathbb{H}^n.$$

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Some remarks

- The difficulty in finding minimizers of (VP) stems from the fact that the Sobolev embedding is not compact and there is a (noncompact) group of dilations preserving the set of extremals. In particular, starting from an extremal we can construct a sequence of extremals which converge to the zero function. Thus, an argument proving the existence of an extremal by taking a sequence of functions converging to an extremal will fail unless a more delicate analysis and modification (by scaling and translating) of the sequence is performed.
- The concentration–compactness principle of Lions can be applied to prove that the best constant in the Folland–Stein embedding is achieved.
- This method does not allow an explicit determination of the best constant or the functions for which it is achieved.

Continuous embeddings and compact embeddings

Theorem (Subelliptic Sobolev embeddings)

(i) If $1 \le p < Q$, then $HW^{1,p}(\Omega) \hookrightarrow L^{\mathfrak{q}}(\Omega)$ is continuous for any $\mathfrak{q} \in [p, p^*]$, $p^* = \dfrac{pQ}{Q-p}$;
(ii) if $p = Q$, then $HW^{1,Q}(\Omega) \hookrightarrow L^{\mathfrak{q}}(\Omega)$ is continuous for any $\mathfrak{q} \in [Q, \infty[$.
Moreover, if Ω is bounded and $1 < p < Q$, then the embedding $HW^{1,p}(\Omega) \hookrightarrow L^{\mathfrak{q}}(\Omega)$ is continuous for any $\mathfrak{q} \in [1, p^*]$, while if Ω is bounded and $p = Q$, embedding $HW^{1,Q}(\Omega) \hookrightarrow L^{\mathfrak{q}}(\Omega)$ is continuous for any $\mathfrak{q} \in [1, \infty)$.

For the proof of the Subelliptic Sobolev embedding Theorem we refer e.g. to Proposition 5.27 in

J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equations* **3** (1995), 493–512.

We conclude this discussion with some hints about the compactness of embedding $HW^{1,p}(\Omega) \hookrightarrow L^{\mathfrak{q}}(\Omega)$ on well behaved domains Ω. For a complete treatment we refer to

- J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equations* **3** (1995), 493–512.
- B. Franchi, C. Gutierrez, R.L. Wheeden, Weighted Sobolev–Poincaré inequalities for Grushin type operators, *Comm. Partial Differential Equations* **19** (1994), 523–604.
- N. Garofalo, D.–M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.* **49** (1996), 1081–1144.
- P. Hajłasz, P. Koskela, *Sobolev met Poicaré*, Mem. Amer. Math. Soc. **145**, 2000, x+101 pp.

and the monograph of Hajłasz and Koskela contains an extensive bibliography on this subject. More precisely, let us introduce the following definition as given by Garofalo and Nhieu.

Definition

An open set Ω of $\mathbb{H}^n$ is said to be a Poincaré–Sobolev domain, briefly *PS* domain, if there exists a covering $\{B\}_{B\in\mathcal{F}}$ of Ω by Carnot–Carathéodory balls B and numbers $N > 0$, $\alpha \geq 1$, and $\nu \geq 1$ such that

(i) $\sum_{B\in\mathcal{F}} \mathbb{1}_{(\alpha+1)B}(\xi) \leq N \mathbb{1}_{\Omega}(\xi)$ for every $\xi \in \mathbb{H}^n$;

(ii) there exists a ball $B_0 \in \mathcal{F}$ such that for all $B \in \mathcal{F}$ there is a finite chain $B_0, B_1, \ldots, B_{s(B)}$, with $B_i \cap B_{i+1} \neq \emptyset$ and $|B_i \cap B_{i+1}| \geq \max\{|B_i|, |B_{i+1}|\}/N$,

(iii) $B \subset \nu B_i$ for $i = 1, \ldots, s(B)$.

The Definition of a PS domain is purely metric. In the context of the Heisenberg groups, one can produce a large class of *PS* domains as explained in details inthe cited paper of Garofalo and Nhieu.

For our needs, it is important also to report a version of the Rellich–Kondrachov compact embedding, in the Heisenberg group context. In particular, the next theorem is a particular case of Theorem 1.3.1 in

S.P. Ivanov, D.N. Vassilev, *Extremals for the Sobolev inequality and the quaternionic contact Yamabe problem*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011, xviii+219 pp.

Theorem (The Rellich–Kondrachov theorem)

(i) *If Ω denotes a bounded PS domain in $\mathbb{H}^n$ and $1 \leq p < Q$, then the embedding*

$$HW^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{\mathfrak{q}}(\Omega)$$

is compact provided that $1 \leq \mathfrak{q} < p^*$, where p^* is the critical Sobolev exponent related to p.

(ii) *The Carnot–Carathéodory balls are PS domains.*

Part (i) was first proved by Garofalo and Nhieu in 1996 in the subelliptic setting, while for a proof of part (ii) we refer to the paper of Franchi, Gutierrez and Wheeden in 1994.
Combining the Rellich–Kondrachov theorem, with the fact that the Carnot–Carathéodory distance and the Korányi distance are equivalent on $\mathbb{H}^n$, we get

Corollary

Let $1 \leq p < Q$ and let $B_R(\xi_0)$ be any Korányi ball, centered at $\xi_0 \in \mathbb{H}^n$, with radius $R > 0$. Then, the embedding

$$HW^{1,p}(B_R(\xi_0)) \hookrightarrow\hookrightarrow L^{\mathfrak{q}}(B_R(\xi_0)) \tag{1}$$

is compact provided that $1 \leq \mathfrak{q} < p^* = \dfrac{pQ}{Q-p}$.

The Hardy inequality

Following

N. Garofalo, E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, *Ann. Inst. Fourier* **40** (1990), 313–356.

put $\psi(\xi) = |D_H r(\xi)|_H = \dfrac{|z|}{r(\xi)}$ for $\xi = (z,t) \neq O$, where $r(\xi) = (|z|^4 + t^2)^{1/4}$ is the Korányi norm.

Theorem (The Hardy inequality in $\mathbb{H}^n$)

If $1 < p < Q$, then

$$\int_{\mathbb{H}^n} |\varphi|^p \psi^p \frac{d\xi}{r^p} \leq \left(\frac{p}{Q-p}\right)^p \int_{\mathbb{H}^n} |D_H\varphi|_H^p d\xi$$

for all $\varphi \in C_c^\infty(\mathbb{H}^n \setminus \{O\})$.
The Hardy inequality remains valid in the Folland–Stein space $S^{1,p}(\mathbb{H}^n)$.

The Hardy inequality was obtained by Garofalo and Lanconelli in in 1990 when $p = 2$ and then extended to all $p > 1$ in

L. D'Ambrosio, Hardy–type inequalities related to degenerate elliptic differential operators, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **4** (2005), 451–486.

P. Niu, H. Zhang, Y. Wang, Hardy–type and Rellich type inequalities on the Heisenberg group, *Proc. Amer. Math. Soc.* **129** (2001), 3623–3630.

When $p = 2$, the optimality of the constant $(2/(Q-2))^2$ is shown in

J.A. Goldstein, Q.S. Zhang, On a degenerate heat equation with a singular potential, *J. Funct. Anal.* **186** (2001), 342–359.

A sharp Hardy inequality of type has been derived in general Carnot–Carathéodory spaces in the monograph

D. Danielli, N. Garofalo, N.C. Phuc, *Inequalities of Hardy–Sobolev type in Carnot–Carathéodory, Sobolev spaces in mathematics. I*, 117–151, Int. Math. Ser. (N.Y.), 8, Springer, New York, 2009.

The case $p = Q$

- As in the Euclidean case, functions in $HW^{1,Q}(\mathbb{H}^n)$ have exponential integrability. This result is refined by the Trudinger–Moser inequality.
- The Trudinger–Moser inequality was first proved, when Ω is a subset of $\mathbb{R}^n$ of finite measure, by Trudinger (J. Math. Mech., 1967). Short after, Moser (Indiana Univ. Math. J., 1970) obtained a different proof, which allows the determination of the corresponding sharp constant.
- In the entire Euclidean space $\mathbb{R}^n$, the first related inequalities have been proved by Cao (Commun. Partial Differ. Equ., 1992) for $n = 2$, and for any dimension by do Ó (Abstr. Appl. Anal., 1997) and Adachi and Tanaka (Proc. Amer. Math. Soc., 2000).

The Trudinger–Moser inequality in $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$

Theorem (Trudinger–Moser inequality)

Let Ω be a domain with finite measure in $\mathbb{R}^n$, $n \geq 2$. Then there exist a sharp constant $\alpha_n = n\left(\frac{n\pi^{n/2}}{\Gamma(n/2+1)}\right)^{1/(n-1)}$ *and a positive constant $C_0 = C_0(n)$ such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha|u|^{n'}\right)dx \leq C_0$$

for any $u \in W^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx \leq 1$ and any $\alpha \leq \alpha_n$, where $n' = n/(n-1)$.

This constant α_n is sharp in the sense that if $\alpha > \alpha_n$, then the above inequality does not hold any longer with a constant C_0 independent of u.

The Trudinger–Moser inequality in $\mathbb{R}^n$

Theorem (Trudinger–Moser inequality in $\mathbb{R}^n$)

For any $\alpha \in (0, \alpha_n)$, there exists a constant $C_\alpha > 0$ such that

$$\int_{\mathbb{R}^n} \left\{ \exp\left(\alpha |u|^{n'}\right) - \sum_{j=0}^{n-2} \frac{\alpha^j}{j!} |u|^{jn'} \right\} dx \le C_\alpha \|u\|_n^n$$

for any $u \in W^{1,n}(\mathbb{R}^n)$ with $\|\nabla u\|_n \le 1$.

This inequality is false for $\alpha \ge \alpha_n$. It can be noted that unlike the case of the bounded domains, α_n cannot be reached.

- **Cianchi, Musil, Pick, Luboš**, Int. Math. Res. Not. IMRN 2022
- **Ho, Perera**, Proc. Edinb. Math. Soc. 2022

The Trudinger–Moser inequality in the Heisenberg group

- In bounded domains of $\mathbb{H}^n$, the Trudinger–Moser inequality was first established in by Cohn and Lu (Indiana Univ. Math. J., 2001), adapting an Adams idea in deriving the Moser–Trudinger inequality for higher order derivatives in Euclidean space to avoid considering the horizontal gradient of the rearrangement function. This method requires an optimal bound on the size of a function in terms of the potential of its gradient, namely a sharp representation formula. Indeed, the main difficulty in passing to the Heisenberg context is the impossibility of using the radial nonincreasing rearrangement u^* of functions u (in terms of a certain norm) on the Heisenberg group.
- The situation is more complicated when concerning the Trudinger–Moser type inequalities for unbounded domains of $\mathbb{H}^n$, since the Adams approach does not work. However, Lam, Lu and Tang (Nonlinear Anal., 2014) obtain a sharp Trudinger–Moser inequality on the whole $\mathbb{H}^n$.

Singular Trudinger–Moser inequality in $\mathbb{H}^n$

Theorem (Lam, Lu and Tang, Nonlinear Anal. 2014)

There exists a positive constant $\alpha_Q = Q\left(\frac{2\pi^n\Gamma(1/2)\Gamma((Q-1)/2)}{\Gamma(Q/2)\Gamma(n)}\right)^{Q'-1}$ *such that for* β*, with* $0 \le \beta < Q$*, and for any* α*, with* $0 < \alpha < \alpha_Q(1-\beta/Q) = \alpha_{Q,\beta}$*, there exists a constant* $C_{\alpha,\beta} > 0$ *such that the inequality*

$$\int_{\mathbb{H}^n} \frac{1}{r(\xi)^\beta}\left\{\exp\left(\alpha|u|^{Q'}\right) - \sum_{j=0}^{Q-2}\frac{\alpha^j}{j!}|u|^{jQ'}\right\}d\xi \le C_{\alpha,\beta}\|u\|_Q^{Q-\beta}$$

holds for all $u \in HW^{1,Q}(\mathbb{H}^n)$*, with* $\|D_H u\|_Q \le 1$.

Some references

- **Capogna, Danielli, Pauls and Tyson**, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, *Birkhäuser Basel, Progress in Mathematics* (2007)
- **Garofalo and Nhieu**, *Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces*, *Comm. Pure Appl. Math.* (1996)
- **Ivanov and Vassilev**, *Extremals for the Sobolev inequality and the quaternionic contact Yamabe problem*, *World Scientific Publishing Co. Pte. Ltd.* (2011)
- **Lam**, *Moser–Trudinger And Adams Type Inequalities And Their Applications*, *Ph.D. thesis* (2014)

The fractional horizontal Sobolev spaces

There are many definitions in literature of the fractional horizontal Sobolev spaces $HW^{s,p}(\mathbb{H}^n)$ and so extremely different approaches. We refer to

G. Palatucci, M. Piccinini, Nonlocal Harnack inequalities in the Heisenberg group, *Calc. Var. Partial Differential Equations* **61** (2022), Paper No. 185.

for a detailed list of references on the subject.

Definition

Let $0 < s < 1$ and $1 < p < \infty$. The horizontal fractional Sobolev space $HW^{s,p}(\mathbb{H}^n)$ is the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|\cdot\|_{HW^{s,p}(\mathbb{H}^n)} = \left(\|\cdot\|^p_{L^p(\mathbb{H}^n)} + [\,\cdot\,]^p_{H,s,p}\right)^{1/p},$$

where

$$[\varphi]_{H,s,p} = \left(\iint_{\mathbb{H}^n\times\mathbb{H}^n} \frac{|\varphi(\xi)-\varphi(\eta)|^p}{r(\eta^{-1}\circ\xi)^{Q+sp}}d\xi d\eta\right)^{1/p}$$

along any $\varphi \in C_c^\infty(\mathbb{H}^n)$.

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For notational simplicity, the (s,p) fractional horizontal gradient of any function $u \in HW^{s,p}(\mathbb{H}^n)$ is denoted by

$$|D^s_H u|^p(\xi) = \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^p}{r(\eta^{-1}\circ\xi)^{Q+ps}} d\eta = \int_{\mathbb{H}^n} \frac{|u(\xi\circ h) - u(\xi)|^p}{r(h)^{Q+ps}} dh,$$

where we recall for all $\xi, \xi' \in \mathbb{H}^n$, with $\xi = (z,t) = (x,y,t)$ and $\xi' = (z',t') = (x',y',t')$

$$\xi \circ \xi' = \Big(z + z', t + t' + 2\sum_{i=1}^{n}(y_i x_i' - x_i y_i')\Big)$$

and $\eta^{-1} = -\eta$ for any $\eta \in \mathbb{H}^n$, while as usually $r(\xi) = r(z,t) = (|z|^4 + t^2)^{1/4}$, for $\xi = (z,t) \in \mathbb{H}^n$, is the Korányi norm.

The (s,p) horizontal gradient of a function $u \in HW^{s,p}(\mathbb{H}^n)$ is well defined a.e. in $\mathbb{H}^n$ and $|D^s_H u|^p \in L^1(\mathbb{H}^n)$ thanks to Tonelli's theorem.

Fractional continuous and compact embeddings

The fractional Sobolev embedding in the Heisenberg group is obtained in

Adimurth, A. Mallick, A Hardy type inequality on fractional order Sobolev spaces on the Heisenberg group, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **18** (2018), 917–949.

following the arguments of

E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), 521–573.

Theorem

If $0 < s < 1$, $1 < p < \infty$ *and* $sp < Q$, *then there exists a positive constant* $C_{p_s^*} = C_{p_s^*}(p, Q, s)$ *such that*

$$\|\varphi\|_{p_s^*}^p \leq C_{p_s^*}[\varphi]_{H,s,p}^p, \quad p_s^* = \frac{pQ}{Q - sp},$$

for all $\varphi \in C_c^\infty(\mathbb{H}^n)$.

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Theorem

For every sequence $(u_k)_k$ *bounded in* $HW^{s,p}(\mathbb{H}^n)$ *there exist* $u \in HW^{s,p}(\mathbb{H}^n)$ *and a subsequence* $(u_{k_j})_j \subset (u_k)_k$ *such that for all* $\xi_0 \in \mathbb{H}^n$ *and* $R > 0$

$$u_{k_j} \to u \quad \text{in } L^p(B_R(\xi_0)) \quad \text{as } j \to \infty.$$

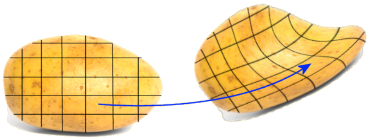
A Lie group version of the Fréchet–Kolmogorov theorem yields the existence of a function $u \in L^p(\mathbb{H}^n)$ and a subsequence of $(u_k)_k$, still denoted $(u_k)_k$, such that $u_k \to u$ a.e. in $\mathbb{H}^n$ and $u_k \to u$ in $L^p(B_R(\xi_0))$ for all $\xi_0 \in \mathbb{H}^n$ and $R > 0$. The proof of the fact that $u \in HW^{s,p}(\mathbb{H}^n)$ follows from an application of the Fatou lemma.

(p, q) problems in the Heisenberg group The above inequalities are the fundamental tool to study existence, multiplicity and qualitative properties of solutions for a class of problems with the following features:

1. We consider subelliptic problems, dealing first with the model case of the Heisenberg group.
2. We consider problems involving operators with non–standard growth conditions, also known as (p, q) operators, see Zhikov (Izv. Akad. Nauk SSSR Ser. Mat., 1986), Marcellini (J. Differential Equations, 1991) and many others.
3. We deal with entire solutions, that is solutions defined in the whole space, and we consider different types of critical nonlinearities. The combined presence of these two factors causes a "double lack of compactness" which produces new, interesting complications.

A Model arising in nonlinear elasticity

Let us first describe the model considered by Marcellini (Ann. Inst. H. Poincaré Anal. Non Linéaire 1986) and originally studied by Ball (Phil. Trans. R. Soc. Lond. 1982).



- Consider the deformation of an elastic body that occupies a bounded domain $\Omega \subset \mathbb{R}^N$.
- Let $u : \Omega \to \mathbb{R}^N$ be the displacement and Du the $N \times N$ matrix of the deformation gradient, then the total energy is

$$I(u) = \int_\Omega f(x, Du(x))dx$$

for some function f.

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The (p, q) growth condition

A natural choice for f is given by

$$f(x, \xi) = g(x, \xi) + h(\det \xi)$$

with g and h satisfying

$$c_1|\xi|^p \leq g(x, \xi) \leq c_2(1 + |\xi|^p), \quad c_3|t| \leq h(t) \leq c_4(1 + |t|)$$

with $1 < p \leq N$. Therefore, since $|\det \xi| \leq c_5(1 + |\xi|^N)$, the function f satisfies the condition

$$c_1|\xi|^p \leq f(x, \xi) \leq c_6(1 + |\xi|^N).$$

It is then clear the interest in considering general functions $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm} \to \mathbb{R}$ satisfying the (p, q) growth condition

$$C_1|\xi|^p \leq f(x, \xi) \leq C_2(1 + |\xi|^q), \quad 1 < p \leq q,$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^{Nm}$.

Model (p, q) problem in $\mathbb{H}^n$

The model problem is

$$-\Delta_{H,p}u - \Delta_{H,q}u + |u|^{p-2}u + |u|^{q-2}u = f(x, u), \quad \text{in } \mathbb{H}^n.$$

- $\Delta_{H,\wp}$, is the horizontal $\wp$–Laplacian operator, which is defined for $\wp > 1$ as

$$\Delta_{H,\wp}\varphi = \mathrm{div}_H(|D_H\varphi|_H^{\wp-2}D_H\varphi) \text{ for any } \varphi \in C_c^\infty(\mathbb{H}^n);$$

- the exponents p and q are such that $1 < p < q \leq Q$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$;
- f contains a critical term;
- We consider two different cases: $q < Q$ and $q = Q$.

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A more general class of (p, q) systems

In P.P. and Temperini (Adv. Nonlinear Anal., 2020), we consider the system in $\mathbb{H}^n$

$$(\mathcal{S})\begin{cases} -\mathrm{div}_H\big(A(|D_H u|_H)D_H u\big) + B(|u|)u = \lambda F_u(u,v) + \dfrac{\alpha}{q^*}|v|^\beta |u|^{\alpha-2}u, \\ -\mathrm{div}_H\big(A(|D_H v|_H)D_H v\big) + B(|v|)v = \lambda F_v(u,v) + \dfrac{\beta}{q^*}|u|^\alpha |v|^{\beta-2}v, \end{cases}$$

- $1 < p < q < Q$ where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$;
- $\lambda > 0$ is a parameter;
- $\alpha, \beta > 1$ with $\alpha + \beta = q^*$, where $q^* = \dfrac{qQ}{Q - q}$.

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Structural assumptions

Following Fiscella and P.P. (Nonlinear Anal., 2018) we assume that:

(A) *A is a positive, strictly increasing function of class $C^1(\mathbb{R}^+)$,*

(B) *$B \in C(\mathbb{R}^+)$ is a positive function and $t \mapsto tB(t)$ is strictly increasing in $\mathbb{R}^+$, with $tB(t) \to 0$ as $t \to 0^+$.*

We introduce the potentials $\mathcal{A}$ and $\mathcal{B}$, which are 0 in 0 and which are obtained by integration from $\mathcal{A}'(t) = tA(t)$,], $\mathcal{B}'(t) = tB(t)$ for all $t \in \mathbb{R}_0^+$.

(C_1) *There exist strictly positive constants a_0, $\mathfrak{a}_0$, b_0, $\mathfrak{b}_0$, a_1, $\mathfrak{a}_1$, b_1 ,$\mathfrak{b}_1$ with $a_0 \leq 1$, and there exist exponents p and q, with $1 < p < q < Q$, such that for all $t \in \mathbb{R}_0^+$*

$$a_0 t^{p-1} + a_1 t^{q-1} \leq \mathcal{A}'(t) \leq \mathfrak{a}_0 t^{p-1} + \mathfrak{a}_1 t^{q-1},$$

$$b_0 t^{p-1} + b_1 t^{q-1} \leq \mathcal{B}'(t) \leq \mathfrak{b}_0 t^{p-1} + \mathfrak{b}_1 t^{q-1}.$$

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(C_2) *There exist constants* θ *and* ϑ, *with* $p \leq \min\{\theta, \vartheta\} < q^*$, *such that*

$$\theta\mathcal{A}(t) \geq t\mathcal{A}'(t), \quad \vartheta\mathcal{B}(t) \geq t\mathcal{B}'(t) \quad \textit{for all } t \in \mathbb{R}_0^+;$$

F_u, F_v are partial derivatives of *a function of class* $C^1(\mathbb{R}^2)$, such that

(F_1) $F(u,v) \geq 0$ *for all* $(u,v) \in \mathbb{R}^2$, $F_u(u,v) = 0$ *for all* $u \leq 0$ *and* $v \in \mathbb{R}$, *while* $F_v(u,v) = 0$ *for all* $u \in \mathbb{R}$ *and* $v \leq 0$ *and* $F(u,v) > 0$ *for* $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$. *There exist* $\mathfrak{m}$, m *such that* $q < \mathfrak{m} < m < q^*$, *and for every* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$|\nabla F(u,v)| \leq \mathfrak{m}\varepsilon|(u,v)|^{\mathfrak{m}-1} + mC_\varepsilon|(u,v)|^{m-1} \quad \textit{for any } (u,v) \in \mathbb{R}^2;$$

(F_2) *there exists* σ, *with* $\max\{\theta, \vartheta\} < \sigma < q^*$, *such that*

$$0 \leq \sigma F(u,v) \leq \nabla F(u,v) \cdot (u,v) \quad \textit{for all } (u,v) \in \mathbb{R}^2.$$

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Some examples Let $1 < p < q < Q$.

- If $\mathcal{A}(t) = \mathcal{B}(t) = t^p/p + t^q/q$, $t \in \mathbb{R}_0^+$, then $a_0 = \mathfrak{a}_0 = a_1 = \mathfrak{a}_1 = b_0 = \mathfrak{b}_0 = b_1 = \mathfrak{b}_1 = 1$, $\theta = \vartheta = q$ and

$$-\Delta_{H,p}u - \Delta_{H,q}u + |u|^{p-2}u + |u|^{q-2}u;$$

- if $\mathcal{A}(t) = \frac{1}{2}(1+t^p)^{2/p} + t^q/q$ and $\mathcal{B}(t) = t^p/p + c\,t^q/q$, $t \in \mathbb{R}_0^+$, $c \geq 0$, and if $2 \leq p < q$, then $a_0 = b_0 = \mathfrak{b}_0 = \mathfrak{a}_0 = a_1 = \mathfrak{a}_1 = 1$, $\mathfrak{b}_1 = c$, $\theta = p$, $\vartheta = p$ if $c = 0$, while $\vartheta = q$ if $c > 0$ and

$$-\mathrm{div}_H\left(\frac{|D_H u|_H^{p-2} D_H u}{\left(1 + |D_H u|_H^p\right)^{1-2/p}}\right) - \Delta_{H,q}u + |u|^{p-2}u + c\,|u|^{q-2}u.$$

In the particular case in which $2 = p < q = 4 < Q$ we have

- if $\mathcal{A}(t) = \sqrt{1+t^2} - 1 + t^4/4$ and $\mathcal{B}(t) = t^2/2 + t^4/4$, $t \in \mathbb{R}_0^+$, then $a_0 = b_0 = b_1 = \mathfrak{a}_0 = \mathfrak{a}_1 = \mathfrak{b}_0 = \mathfrak{b}_1 = 1$, $a_1 = 1/2$, $\theta = \vartheta = 4$, $\alpha + \beta = q^* = 4^* = 4Q/(Q-4)$ and

$$-\mathrm{div}_H \left(\frac{D_H u}{\sqrt{1 + |D_H u|_H^2}} \right) - \Delta_{H,4} u + u + u^3;$$

- if $\mathcal{A}(t) = t \arctan t - \log \sqrt{1+t^2} + t^4/4$ and $\mathcal{B}(t) = t^2/2 + t^4/4$, $t \in \mathbb{R}_0^+$, then $a_0 = b_0 = b_1 = \mathfrak{a}_0 = \mathfrak{a}_1 = \mathfrak{b}_0 = \mathfrak{b}_1 = 1$, $a_1 = 2/3$, $\theta = \vartheta = 4$, $\alpha + \beta = q^* = 4^*$ and

$$-\mathrm{div}_H \left(\frac{\arctan |D_H u|_H}{|D_H u|_H} D_H u \right) - \Delta_{H,4} u + u + u^3.$$

Theorem (P.P. and Temperini – Adv. Nonlinear Anal., 2020)

Assume that the structural assumptions (A), (B), (C_1),(C_2), (F_1) *and* (F_2) *hold. Then, there exists* $\lambda^* > 0$ *such that for all* $\lambda \geq \lambda^*$ *the system in* $\mathbb{H}^n$

$$(\mathcal{S}) \begin{cases} -div_H\big(A(|D_H u|_H)D_H u\big) + B(|u|)u = \lambda F_u(u,v) + \dfrac{\alpha}{q^*}|v|^{\beta}|u|^{\alpha-2}u, \\ -div_H\big(A(|D_H v|_H)D_H v\big) + B(|v|)v = \lambda F_v(u,v) + \dfrac{\beta}{q^*}|u|^{\alpha}|v|^{\beta-2}v, \end{cases}$$

admits at least one solution (u_λ, v_λ) *in* W. *Moreover, each component of* (u_λ, v_λ) *is non trivial and*

$$\lim_{\lambda\to\infty} \|(u_\lambda, v_\lambda)\| = 0.$$

The proof of the existence theorem is divided into several steps.

- First, we introduce the variational setting of the problem and we prove that the underlying functional I has the geometry of Mountain pass, where

$$I(u,v) = \int_{\mathbb{H}^n} \left[\mathcal{A}(|D_H u|_H) + \mathcal{A}(|D_H v|_H)\right] d\xi + \int_{\mathbb{H}^n} \left[\mathcal{B}(|u|) + \mathcal{B}(|v|)\right] d\xi \\ - \lambda \int_{\mathbb{H}^n} F(u,v)\, d\xi - \frac{1}{q^*} \int_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi \quad \text{for all } (u,v) \in W.$$

- Then, we obtain the existence of a Palais–Smale sequence $\{(u_k, v_k)\}_k \subset W$ for I at the special level c_λ, where

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0,$$

with $\Gamma = \{\gamma \in C([0,1], W) \, : \, \gamma(0) = (0,0), \; I(\gamma(1)) < 0\}$. Furthermore, the set of critical levels $\{c_\lambda\}_\lambda$ satisfies the asymptotic condition

$$\lim_{\lambda \to \infty} c_\lambda = 0.$$

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- the key step is to prove that, up to a subsequence,

$$(u_k, v_k) \rightharpoonup (u_\lambda, v_\lambda) \text{ in } W \text{ as } k \to \infty,$$

and that there exists a threshold $\lambda^* > 0$ such that

(u_λ, v_λ) is a weak solution for $\lambda \geq \lambda^*$.

For this we need a new concentration–compactness result in the space

$$S = S^{1,\wp}(\mathbb{H}^n) \times S^{1,\wp}(\mathbb{H}^n)$$

where $S^{1,\wp}(\mathbb{H}^n)$, $1 < \wp < Q$, is the Folland–Stein space.
- Finally, the fact that the constructed solution is nontrivial is obtained via a theorem of alternatives *à la* Lions.

In order to handle the critical potential, we first study the exact behavior of the weakly convergent sequences of

$$S = S^{1,q}(\mathbb{H}^n) \times S^{1,q}(\mathbb{H}^n)$$

in the space of measures, in the spirit of Lions.
The result is based on the optimal constant

$$\mathcal{I} = \inf_{\substack{(u,v)\in S\\ u\neq 0 \wedge v\neq 0}} \frac{\|D_H u\|_q^q + \|D_H v\|_q^q}{\left(\int_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi\right)^{q/q^*}},$$

which is well defined thanks to the Folland–Stein inequality.

Exponential (p, Q) equations in $\mathbb{H}^n$

In P.P. and Temperini (Adv. Calc. Var. 15 (2022), 601–617) we consider the equation in $\mathbb{H}^n$

$$(\mathcal{E}) \qquad -\Delta_{H,p}u - \Delta_{H,Q}u + |u|^{p-2}u + |u|^{Q-2}u = \frac{f(\xi, u)}{r(\xi)^\beta} + h(\xi),$$

- $1 < p < Q$, $0 \leq \beta < Q$ where $Q = 2n + 2$,
- h is a nontrivial nonnegative functional of $HW^{-1,Q'}(\mathbb{H}^n)$, where $HW^{-1,Q'}(\mathbb{H}^n)$ is the dual space of $HW^{1,Q}(\mathbb{H}^n)$,
- $r(\xi) = r(z, t) = (|z|^4 + t^2)^{1/4}$ is the *Korányi norm* in $\mathbb{H}^n$, with $\xi = (z, t) \in \mathbb{H}^n$, $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \in \mathbb{R}$, $|z|$ the Euclidean norm of $z \in \mathbb{R}^{2n}$.

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Structural assumptions

(f_1) *f is a Carathéodory function, with $f(\cdot,u)=0$ for all $u\le 0$, and such that there exists $\alpha_0>0$ with the property that for all $\varepsilon>0$ there exists $\kappa_\varepsilon>0$ such that*

$$f(\xi,u)\le \varepsilon\, u^{Q-1}+\kappa_\varepsilon\left(e^{\alpha_0 u^{Q'}}-S_{Q-2}(\alpha_0,u)\right)$$

for a.e. $\xi\in\mathbb{H}^n$ and all $u\in\mathbb{R}^+_0$, where $\mathbb{R}^+_0=[0,\infty)$,

$$Q'=\frac{Q}{Q-1}\quad\text{and}\quad S_{Q-2}(\alpha_0,u)=\sum_{j=0}^{Q-2}\frac{\alpha_0^j u^{jQ'}}{j!};$$

(f_2) *there exists a number $\nu>Q$ such that $0<\nu F(\xi,u)\le uf(\xi,u)$ for a.e $\xi\in\mathbb{H}^n$ and any $u\in\mathbb{R}^+$, $\mathbb{R}^+=(0,\infty)$, where $F(\xi,u)=\int_0^u f(\xi,v)dv$ for a.e $\xi\in\mathbb{H}^n$ and all $u\in\mathbb{R}$.*

Theorem (P.P. and Temperini, Adv. Calc. Var. 15 (2022), 601–617)

Assume that the structural assumptions (f_1), (f_2) *hold. Then, there exists a constant* $\sigma > 0$ *such that the equation*

$$(\mathcal{E}) \qquad -\Delta_{H,p} u - \Delta_{H,Q} u + |u|^{p-2}u + |u|^{Q-2}u = \frac{f(\xi, u)}{r(\xi)^\beta} + h(\xi)$$

admits at least a nontrivial nonnegative solution u_h *in* W*, provided that* $0 < \|h\|_{HW^{-1,Q'}} < \sigma$. *Moreover,*

$$\lim_{h \to 0} \|u_h\| = 0.$$

Future developments and open problems

- Complete the previous theorems giving existence/nonexistence results for small λ and large h
- Consider the presence of other critical potentials, such as Hardy potentials
- Consider (p, q) non local problems, involving the fractional Laplace operator on the Heisenberg group, defined as

$$(-\Delta_H)^s_\wp u(\xi) = C_{Q,s,\wp} PV \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^{\wp-2}(u(\xi) - u(\eta))}{r(\eta^{-1} \circ \xi)^{Q+s\wp}} d\eta$$

where $Q = 2n + 2$, $C_{Q,s,\wp}$ is a positive constant, and PV is the Cauchy principal value, see Fiscella, P.P. (Fract. Calc. Appl. Anal., 2020), De Filippis, Palatucci (J. Differential Equations, 2019), Kumar and Sreenadh (Commun. Contemp. Math., 2020). Goel, Sreenadh, Radulescu, *Variational framework and Lewy-Stampacchia type estimates for nonlocal operators on Heisenberg group*, Ann. Fenn. Math. 2022.

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Critical equations in $\mathbb{H}^n$

In P.P. and Temperini, Opuscula Math., Special Issue Advances in Nonlinear Partial Differential Equations 2022 we consider the equation in $\mathbb{H}^n$

$$(\mathcal{E}) \qquad -\Delta_{H,p} u = \lambda w(\xi)|u|^{q-2}u + K(\xi)|u|^{p^*-2}u \quad \text{in } \mathbb{H}^n,$$

with $1 < p < Q$, where $Q = 2n+2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$; furthermore, $p \leq q < p^*$ and

$$p^* = \frac{pQ}{Q-p}$$

is the critical exponent associated to p.

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For $(\mathcal{E}) \quad -\Delta_{H,p} u = \lambda w(\xi)|u|^{q-2}u + K(\xi)|u|^{p^*-2}u$ in $\mathbb{H}^n$ we distinguish two different situations:

1. $1 < p < q < p^*$;
2. $1 < p = q < p^*$.

In the first case, we assume that

$(w_1) \quad w \geq 0,\ w \in L^1_{\text{loc}}(\mathbb{H}^n)$ *and* w *is such that the embedding* $S^{1,p}(\mathbb{H}^n) \hookrightarrow\hookrightarrow L^q(\mathbb{H}^n, w d\xi)$ *is compact;*

$(K_1) \quad K > 0$ *a.e. in* $\mathbb{H}^n$, $K \in L^\infty(\mathbb{H}^n)$ *and*

$$\lim_{r(\xi)\to\infty} K(\xi) = K_\infty \in \mathbb{R}^+_0,$$

where $\mathbb{R}^+_0 = [0, \infty)$.

Theorem (P.P., Temperini, Opuscula Math., S.I. Advances in Nonlinear Partial Differential Equations 2022)

Let $1 < p < Q$ *and* $p < q < p^*$. *Assume that* (w_1) *and* (H_1) *are satisfied. Then, there exists* $\lambda^* > 0$ *such that for all* $\lambda \geq \lambda^*$ *the equation*

$$(\mathcal{E}) \qquad -\Delta_{H,p} u = \lambda w(\xi)|u|^{q-2}u + K(\xi)|u|^{p^*-2}u \quad \text{in } \mathbb{H}^n,$$

admits at least a nontrivial solution.

The Theorem extends Theorem 1.1 of

[BFP] S. Bordoni, R. Filippucci, P. P., *Existence problems on Heisenberg groups involving Hardy and critical terms*, J. Geom. Anal. **30** (2020), 1887–1917.

Our Theorem is obtained via an application of the concentration–compactness results given in

- **P. P., L. Temperini**, *Existence for singular critical exponential (p, Q) equations in the Heisenberg group*, Adv. Calc. Var. 15 (2022), 601–617.
- **P. P., L. Temperini**, *On the concentration–compactness principle for Folland–Stein spaces and for fractional horizontal Sobolev spaces*, Math. Eng. 5 (2023), Special Issue: *The interplay between local and nonlocal equations - dedicated to the memory of Professor Ireneo Peral*, Paper no. 007, 21 pp.

The second case, namely when $p = q$, is more challenging and is not treated in [BFP]. Following somehow

Bonder, Saintier, Silva, *The concentration-compactness principle for fractional order Sobolev spaces in unbounded domains and applications to the generalized fractional Brézis–Nirenberg problem*, Nonlinear Differential Equations Appl. **25** (2018), 52:25.

we assume that $K(\xi) \equiv 1$, so that (K_1) is trivially satisfied, and that w verifies (w_1) and the additional request

(w_2) $\quad w \in L^\infty(\mathbb{H}^n)$ *and there exists* $\xi_0 \in \mathbb{H}^n$ *such that* w *is continuous at* ξ_0 *and* $w(\xi_0) > 0$.

We are then able to prove the following

Theorem (P.P., Temperini, Opuscula Math., S.I. Advances in Nonlinear Partial Differential Equations 2022)

Let $p > 1$ be such that $p^2 < Q$. Assume that the function w satisfies (w_1) with $p = q$ and (w_2) and that $K \equiv 1$. Then, equation

$$(\mathcal{E}) \qquad -\Delta_{H,p} u = \lambda w(\xi)|u|^{p-2}u + |u|^{p^*-2}u \quad \text{in } \mathbb{H}^n,$$

admits at least a nontrivial solution for any $\lambda \in (0, \lambda_1)$ where

$$\lambda_1 = \lambda_1(w) = \inf_{\substack{v \in S^{1,p}(\mathbb{H}^n) \\ v \neq 0}} \frac{\|D_H v\|_p^p}{\int_{\mathbb{H}^n} w(\xi)|v|^p d\xi}.$$

The idea behind the construction of the solution in the theorem above goes back to the seminal paper by

[BN] H. Brézis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

The main difficulty is the unavailability of an explicit form of the extremals for the Folland–Stein embedding.

If $1 < p < Q$, then there exists an extremal $U \in S^{1,p}(\mathbb{H}^n)$ for the Folland–Stein embedding and this estimate holds:

$$U(\xi) \sim r(\xi)^{\frac{p-Q}{p-1}} \quad \text{as } r(\xi) \to \infty.$$

The knowledge of the exact asymptotic behavior at infinity of Sobolev extremals turns out to be crucial in order to obtain existence results for the Brézis–Nirenberg type problems, whenever the explicit form of minimizers is not known.

Finally, assumption $p^2 < Q$, together with the estimate for U above, ensures that $U \in L^p(\mathbb{H}^n)$ since otherwise, as we already noted, functions in $S^{1,p}(\mathbb{H}^n)$ may not belong to the Lebesgue space $L^p(\mathbb{H}^n)$.

Few references for the p–Laplacian on the Euclidean setting

- **Bonanno, Jebelean, Şerban**, Proc. Roy. Soc. Edinburgh Sect. A 2017
- **Bonanno, Livrea, Radulescu**, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 2021
- **Bonanno, D'Aguì, Livrea**, Nonlinear Anal. 2020
- **Brasco, Prinari, Zagati**, Nonlinear Anal. 2022
- **Candito, Gasinski, Livrea, Santos**, Adv. Nonlinear Anal. 2022
- **Ciraolo, Figalli, Roncoroni**, Geom. Funct. Anal. 2020

Few references for the p–Laplacian on the Euclidean setting

- **Fusco, Mukherjee, Zhang, Yi**, Proc. Lond. Math. Soc. 2019
- **Mawhin, Skrzypek, Szymaǹska–Dębowska**, Entropy 2021
- **Mercuri, Perera**, J. Funct. Anal. 2022
- **Papageorgiou, Scapellato**, J. Differential Equations 2021

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(p,q) critical equations with Hardy terms

In [P.P., Temperini, Rend. Circ. Mat. Palermo 71 (2022), 1049–1077] we consider the critical equation with Hardy terms in $\mathbb{H}^n$

$$(\mathcal{E}) \qquad \begin{aligned} -\Delta_{H,p}u - \Delta_{H,q}u + |u|^{p-2}u + |u|^{q-2}u - \sigma\psi^q \frac{|u|^{q-2}u}{r^q} \\ = \lambda f(\xi,u) + |u|^{q^*-2}u, \end{aligned}$$

where σ and $\lambda > 0$ are real parameters. The exponents p and q are such that $1 < p < q < Q$, where $q^* = qQ/(Q-q)$ is the critical exponent related to q. As usual, $\Delta_{H,\wp}$, with $\wp \in \{p,q\}$, is the horizontal $\wp$–Laplacian defined by $\Delta_{H,\wp}\varphi = \operatorname{div}_H(|D_H\varphi|_H^{\wp-2}D_H\varphi)$ for all $\varphi \in C_c^\infty(\mathbb{H}^n)$, $r(\xi) = r(z,t) = (|z|^4 + t^2)^{1/4}$, $\xi = (z,t) \in \mathbb{H}^n$, is the Korányi norm and ψ is the weight function that appears in the Hardy inequality

$$\int_{\mathbb{H}^n} |\varphi|^p \psi^p \frac{d\xi}{r^p} \leq \left(\frac{p}{Q-p}\right)^p \int_{\mathbb{H}^n} |D_H\varphi|_H^p d\xi,$$

that is $\psi = |D_H r|_H$ in $\mathbb{H}^n \setminus \{O\}$.

On f in $(\mathcal{E})$ throughout the chapter we assume the following condition

$(\mathcal{F})$ *f is a Carathéodory function, with $f(\cdot,u)=0$ for all $u\leq 0$ and $f(\cdot,u)>0$ for all $u>0$, satisfying the two properties*

(f_1) *there exist $\mathfrak{m}$ and m, with $p<\mathfrak{m}<m<q^*$, such that for every $\varepsilon>0$ there exists $C_\varepsilon>0$ for which the inequality*

$$|f(\xi,u)|\leq \mathfrak{m}\varepsilon|u|^{\mathfrak{m}-1}+mC_\varepsilon|u|^{m-1}\quad \textit{for any } u\in\mathbb{R}$$

holds for a.e. $\xi\in\mathbb{H}^n$;

(f_2) *there exists θ, with $q<\theta<q^*$, such that the inequality*

$$0\leq\theta F(\xi,u)\leq f(\xi,u)u\quad \textit{for all } u\in\mathbb{R}$$

holds for a.e. $\xi\in\mathbb{H}^n$, where $F(\xi,u)=\displaystyle\int_0^u f(\xi,v)dv$ for a.e $\xi\in\mathbb{H}^n$ and all $u\in\mathbb{R}$.

Due to the unbalanced growth of the (p,q) operator, the natural space where finding solutions of $(\mathcal{E})$ is

$$W = HW^{1,p}(\mathbb{H}^n) \cap HW^{1,q}(\mathbb{H}^n),$$

endowed with the norm

$$\|u\| = \|u\|_{HW^{1,p}} + \|u\|_{HW^{1,q}}$$

for all $u \in W$, where $HW^{1,\wp}(\mathbb{H}^n)$, $\wp \in \{p,q\}$, is the horizontal Sobolev space consisting of all functions $u \in L^{\wp}(\mathbb{H}^n)$ such that $D_H u$ exists in the sense of distributions and $|D_H u|_H \in L^{\wp}(\mathbb{H}^n)$, endowed with the natural norm

$$\|u\|_{HW^{1,\wp}(\mathbb{H}^n)} = \left(\int_{\mathbb{H}^n} |u|^{\wp} d\xi + \int_{\mathbb{H}^n} |D_H u|_H^{\wp} d\xi \right)^{1/\wp}.$$

The number $\mathcal{H}_q$ denotes the best Hardy constant introduced

Theorem

For any $\sigma \in (-\infty, \mathcal{H}_q)$*, there exists* $\lambda_* = \lambda_*(\sigma, Q, q, \theta) > 0$ *such that equation*

$$(\mathcal{E}) \qquad \begin{aligned} -\Delta_{H,p} u - \Delta_{H,q} u + |u|^{p-2}u + |u|^{q-2}u - \sigma \psi^q \frac{|u|^{q-2}u}{r^q} \\ = \lambda f(\xi, u) + |u|^{q^*-2}u, \end{aligned}$$

admits at least one nontrivial solution $u = u_{\sigma,\lambda}$ *in* W *for all* $\lambda \geq \lambda_*$*. Moreover,*

$$\lim_{\lambda \to \infty} \|u_{\sigma,\lambda}\| = 0.$$

The above theorem extends and complements in several directions previous results, such as the theorems contained in

- S. Bordoni, R. Filippucci, P.P., Existence problems on Heisenberg groups involving Hardy and critical terms, *J. Geom. Anal.* **30** (2020), 1887–1917.
- G.M. Figueiredo, Existence of positive solutions for a class of $p\&q$ elliptic problems with critical growth on $\mathbb{R}^N$, *J. Math. Anal. Appl.* **378** (2011), 507–518.
- A. Fiscella, P. Pucci, (p,q) systems with critical terms in $\mathbb{R}^N$, Special Issue on Nonlinear PDEs and Geometric Function Theory, in honor of Carlo Sbordone on his 70th birthday, *Nonlinear Anal.* **177** (2018), Part B, 454–479.

Existence is obtained via the mountain pass lemma of Ambrosetti and Rabinowitz and follows somehow the ideas of the last cited paper.

Moreover, the *triple loss of compactness* in $(\mathcal{E})$, caused by the simultaneous presence of the Hardy and the critical terms in the whole Heisenberg group $\mathbb{H}^n$, forces to study the exact behavior of the $(PS)_c$ sequences at special levels c, in the spirit of Lions. This analysis is deeply connected with the concentration phenomena taking place and strongly relies on the results of On the concentration-compactness principle for Folland-Stein spaces and for fractional horizontal Sobolev spaces

P.P., L. Temperini, On the concentration–compactness principle for Folland–Stein spaces and for fractional horizontal Sobolev spaces, *Math. Eng.* 5 (2023), Paper No. 007, 21 pp.

Multiplicity results

In S. Liang and P. P., *Multiple solutions for critical Kirchhoff–Poisson systems in the Heisenberg group*, Appl. Math. Lett. 127 (2022), Paper No. 107846 we study existence of multiple solutions of the following critical Kirchhoff–Poisson system in the Heisenberg group

$$(\mathcal{K}\mathscr{P}) \quad \begin{cases} -M\left(\int_\Omega |D_H u|^2 d\xi\right)\Delta_H u + \phi|u|^{q-2}u = h(\xi,u) + \lambda|u|^2 u, & \text{in } \Omega, \\ -\Delta_H \phi = |u|^q, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $1 < q < 2$, λ is a positive real parameter, and the Kirchhoff function M and the nonlinear term h satisfy the following assumptions

$(\mathscr{M})$ (M_1) $M \in C(\mathbb{R}_0^+, \mathbb{R}+)$ and there exists $m_0 > 0$ such that $M(t) \geq m_0 > 0$ for all $t \geq 0$;
(M_2) there exists $t_0 \geq 0$ such that $\mathcal{M}(t) \geq M(t)t$ for all $t \geq t_0$, where $\mathcal{M}(t) = \int_0^t M(s)ds$.

$(\mathcal{H})$ (h_1) $h \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ and there exist a constant $C > 0$ and an exponent r, with $2 < r < 4$, such that

$$|h(\xi, t)| \leq C(1 + |t|^{r-1})$$

for all $\xi \in \overline{\Omega}$, $t \geq 0$;
(h_2) $h(\xi, t) = o(|t|)$ as $t \to 0$ uniformly in $\xi \in \Omega$;
(h_3) there exist $\theta \in (2q, 4)$ and $T > 0$ such that

$$0 < \theta H(\xi, t) \leq h(\xi, t)t$$

for all $\xi \in \Omega$ and t, with $|t| \geq T$, where $H(\xi, t) = \int_0^t h(\xi, s)ds$;
(h_4) $h(\xi, -t) = -h(\xi, t)$ for all $(\xi, t) \in \Omega \times \mathbb{R}$.

In the Euclidean case, there are recent interesting papers devoted to the study of the Schrödinger-Poisson systems. For example,

Z. Wang, H. Zhou, *Positive solution for a nonlinear stationary Schrödinger-Poisson system in* $\mathbb{R}^3$, Discrete Contin. Dyn. Syst. 18 (2007) 809–816.

A. Azzollini, A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. 345 (2008) 90–108.

J. Zhang, J. M. Do Ó, M. Squassina, *Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity*, Adv. Nonlinear Stud. 16 (2016) 15–30.

To the best of our knowledge, we cannot find any result in the literature that can be directly applied to obtain the existence and multiplicity of solutions of $(\mathcal{KP})$.

However, the existence results of solutions for the critical Schrödinger–Poisson systems in the Heisenberg group are very few. In this setting, let us mention the paper

- **Y.C. An, H. Liu**, *The Schrödinger-Poisson type system involving a critical nonlinearity on the first Heisenberg group*, Israel J. Math. **235** (2020) 385–411.

in which existence of at least two positive solutions and a positive ground state solution is proved for Schrödinger–Poisson systems in the Heisenberg group, via the critical point theory. As far as we are aware, there are no results in the literature that can be directly applied to obtain the existence and multiplicity of solutions to the critical Kirchhoff–Poisson system $(\mathcal{KP})$ in the Heisenberg group, even in the Euclidean case.

Although some properties are similar between the Kohn Laplacian Δ_H and the classical Laplacian Δ, the similarities may be misleading, see

N. Garofalo, E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*, Ann. Inst. Fourier **40** (1990) 313–356.

In addition, the critical exponent $Q^* = 4$ in $\mathbb{H}^1$, while $2^* = 6$ in $\mathbb{R}^3$. This causes us some obstacles in proving compactness. In order to overcome these difficulties, we use the concentration compactness principles in the Heisenberg group. The main result of the paper is

Theorem (S. Liang, P.P., Appl. Math. Lett. 2022)

Assume that $(\mathscr{M})$ *and* $(\mathcal{H})$ *hold. Then, for any* $k \in \mathbb{N}$, *there exists* $\lambda_k^* > 0$ *such that problem*

$$(\mathcal{K}\mathscr{P}) \quad \begin{cases} -M\left(\int_\Omega |D_H u|^2 d\xi\right)\Delta_H u + \phi|u|^{q-2}u = h(\xi,u) + \lambda|u|^2 u, & \text{in } \Omega, \\ -\Delta_H \phi = |u|^q, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

admits k pairs of nonzero solutions for any $\lambda \in (0, \lambda_k^*)$.

In [S. Liang, P.P., Appl. Math. Lett. 2022], we only consider the one–dimensional case $\mathbb{H}^1$, but the method employed here is also applicable to other dimensions for this kind of problem.

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The paper P. P. and Y. Ye, *Existence for critical Kirchhoff–Poisson systems in the Heisenberg group*, Adv. Nonlinear Stud. 22 (2022), 361–371, is devoted to the study the combined effects of logarithmic and critical nonlinearities for the Kirchhoff–Poisson system

$$\begin{cases} -M\left(\int_\Omega |\nabla_H u|^2 d\xi\right)\Delta_H u + \mu\phi u = \lambda|u|^{q-2}u\ln|u|^2 + |u|^2 u & \text{in } \Omega,\\ -\Delta_H\phi = u^2 & \text{in } \Omega,\\ u = \phi = 0 & \text{on } \partial\Omega,\end{cases}$$

where Δ_H is the Kohn–Laplacian operator in the first Heisenberg group $\mathbb{H}^1$, Ω is a smooth bounded domain of $\mathbb{H}^1$, $q \in (2\theta, 4)$, $\mu \in \mathbb{R}$ and $\lambda > 0$ are some real parameters. Under suitable assumptions on the Kirchhoff function M, which cover the degenerate case, we prove the existence of nontrivial solutions for the above problem when $\lambda > 0$ is sufficiently large. Moreover, our results are new even in the Euclidean case.

As in the whole lecture, let us set for simplicity $\mathbb{R}_0^+ = [0,\infty)$ and $\mathbb{R}^+ = (0,\infty)$. Concerning the Kirchhoff term M, we assume that $M \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ satisfies:

(M_1) *For any $\tau > 0$, there exists $m_0 = m_0(\tau) > 0$ such that $M(t) \geq m_0$ for $t \geq \tau$.*

(M_2) *There exists $\theta \in [1,2)$ such that $\theta \widehat{M}(t) \geq M(t)t$ for all $t \geq 0$, where $\widehat{M}(t) = \int_0^t M(s)ds$.*

(M_3) *There exists $m_1 > 0$ such that $M(t) \geq m_1 t^{\theta-1}$ for all $t \in \mathbb{R}^+$ and $M(0) = 0$.*

A typical example is given by

$$M(t) = a + bt^{\theta-1}, \quad a,b \geq 0, \ a+b>0, \ \theta \geq 1.$$

When M is of this type, the problem is called non–degenerate if $a > 0$, while it is said to be degenerate if $a = 0$.

Theorem (P. P., Y. Ye, Adv. Nonlinear Stud. 2022)

Assume that (M_1)–(M_3) *are satisfied and* $\mu < S|\Omega|^{-\frac{1}{2}}$, *where*

$$S = \inf_{\substack{u \in S_0^1(\mathbb{H}^1) \\ u \neq 0}} \frac{\int_{\mathbb{H}^1} |\nabla_H u|^2 d\xi}{\left(\int_{\mathbb{H}^1} |u|^4 d\xi\right)^{\frac{1}{2}}}.$$

Then there exists $\lambda^* > 0$ *such that problem*

$$\begin{cases} -M\left(\int_\Omega |\nabla_H u|^2 d\xi\right) \Delta_H u + \mu\phi u = \lambda |u|^{q-2} u \ln |u|^2 + |u|^2 u & \text{in } \Omega, \\ -\Delta_H \phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

has a nontrivial solution for any $\lambda > \lambda^*$.

The degenerate case is rather appealing, not only from a mathematical point of view, but also in applications. From a physical point of view the fact that $M(0) = 0$ means that the base tension of the string is zero, a very realistic model. It is treated in well–known famous papers in Kirchhoff theory, see

P. D'Ancona,S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math. 108 (1992) 247-262.

The features of Theorem are

(i) the presence of the logarithmic term and of the critical nonlinearity, which contributes to the lack of compactness;

(ii) the fact that the result includes the degenerate case, which corresponds to the Kirchhoff function M vanishing at 0.

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