Restriction estimates

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The Fourier transform of $f \in L^1$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx,$$

where $x \cdot \xi = x_1 \xi_1 + \cdots + x_d \xi_d$.
The integral converges absolutely to a continuous function vanishing at infinity, that satisfies

$$\|\hat{f}\|_\infty \leqslant \|f\|_1.$$

If also $\hat{f} \in L^1$, then

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} dx.$$

For $f \in L^1$ we will denote by $\check{f}(\xi)$ the integral

$$\check{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi i x \cdot \xi} dx.$$

A C^∞ function f is Schwartz if together with all its derivatives decays at infinity more rapidly than $|x|^{-N}$ for all $N \in \mathbb{N}$, that is

$$\| |x|^N D^\alpha f \|_\infty < C_{\alpha,N} < \infty.$$

The space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ is dense in any L^p with $p < \infty$ and the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$.

If $f, g \in \mathcal{S}$ their convolution,

$$f * g(x) = \int f(x-y)g(y)dy,$$

is a Schwartz function. We can therefore compute the Fourier transform

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

We also have

$$\widehat{fg}(\xi) = \hat{f}(\xi) * \hat{g}(\xi).$$

If f is a general L^2 function the integral

$$\int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$$

does not converge absolutely. To define the Fourier transform in L^2, we observe that if g is a Schwartz function ($g \in \mathcal{S}$), then g obeys the Plancherel identity

$$\|g\|_2 = \|\hat{g}\|_2.$$

For any $f \in L^2$ there is $\{f_n\} \subset \mathcal{S}$ such that $\|f - f_n\|_2 \to 0$. Since $\{f_n\}$ is a Cauchy sequence, by

$$\|f_n - f_m\|_2 = \|\hat{f}_n - \hat{f}_m\|_2,$$

$\{\hat{f}_n\}$ is also Cauchy. We call $\hat{f}$ the L^2 limit of $\{\hat{f}_n\}$ (sometimes we will call it Fourier-Plancherel transform for clarity).

Since $L^p \subset L^1 + L^2$ for $1 \leqslant p \leqslant 2$, the Fourier transform $\hat{f}$ of a function f in $L^p(\mathbb{R}^d)$, $1 < p < 2$, can be defined, by decomposing f into a sum $f_1 + f_2$ with $f_1 \in L^1(\mathbb{R}^d)$ and $f_2 \in L^2(\mathbb{R}^d)$, as

$$\hat{f} = \hat{f}_1 + \hat{f}_2,$$

where

$$\hat{f}_1(\xi) = \int_{\mathbb{R}^d} f_1(x) e^{-2\pi i x \cdot \xi} dx,$$

and $\hat{f}_2$ is the Fourier-Plancherel transform.

The Fourier transform of a function in L^p is a function in $L^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, and satisfies the Hausdorff-Young's inequality.

Theorem (Hausdorff-Young's inequality)

Let $1 \leqslant p \leqslant 2$ and let f be a Schwartz function. Then

$$\|\hat{f}\|_{p'} \leqslant \|f\|_p.$$

Similarly to what we did for the Fourier transform in L^2, one can also define $\hat{f}$ for f in $L^p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant 2)$, showing first that the Hausdorff-Young inequality holds for Schwartz functions and then exploit the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$.

The Hausdorff-Young inequality can be proved by interpolating between

$$\|\hat{f}\|_\infty \leqslant \|f\|_1$$

and

$$\|\hat{f}\|_2 = \|f\|_2$$

with the Riesz-Thorin theorem.

Theorem (M.Riesz-O.Thorin)

Let p_0, p_1, q_0, q_1 be Lebesgue exponents and let T be a linear operator mapping $L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$. Suppose that we have

$$\|Tf\|_{q_0} \leqslant C_0 \|f\|_{p_0}, \quad \|Tf\|_{q_1} \leqslant C_1 \|f|_{p_1}.$$

Define for $1 \leqslant t \leqslant 1$,

$$\frac{1}{p} = \frac{(1-t)}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q} = \frac{(1-t)}{q_0} + \frac{t}{q_1},$$

then

$$\|Tf\|_q \leqslant C_0^{1-t} C_1^t \|f\|_p.$$

Young's convolution inequality,

$$\|f * g\|_q \leqslant \|f\|_p \|g\|_r$$

holds for

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r},$$

and can also be proved using the Riesz-Thorin theorem.

It may be proved by interpolating between

$$\|f * g\|_r \leqslant \|f\|_r \|g\|_1 \quad \text{and} \quad \|f * g\|_\infty \leqslant \|f\|_r \|g\|_{r'}.$$

Setting $q_0 = r$, $p_0 = 1$, $q_1 = \infty$, $p_1 = r'$ and $Tg = f * g$, we get

$$\|f * g\|_q = \|Tg\|_q \leqslant \|f\|_r \|g\|_p,$$

with

$$\frac{1}{q} = \frac{1-t}{r} + \frac{1}{\infty} = \frac{1-t}{r}, \quad \frac{1}{p} = \frac{1-t}{1} + \frac{t}{r'} = 1 - \frac{t}{r}.$$

Note that

$$\frac{1}{p} + \frac{1}{r} = 1 - \frac{t}{r} + \frac{1}{q} + \frac{t}{r} = 1 + \frac{1}{q}.$$

Restriction of the Fourier transform

Functions in L^1 have a Fourier transform $\hat{f}$ that is continuous and may be evaluated at every point; in particular, it may be restricted to a set of measure zero, like, for instance, the unit sphere,

$$S^{d-1} = \{x \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 = 1\}.$$

On the other hand, by the Plancherel theorem any function in L^2 is a Fourier transform. Since L^2 functions are defined only almost everywhere, their restriction to sets of measure zero does not make sense. As a consequence, for $f \in L^2$ the restriction of $\hat{f}$ to S^{d-1} is meaningless.

For $f \in L^p$, $1 < p < 2$, we have

$$\hat{f} = \hat{f}_1 + \hat{f}_2,$$

where $f_1 \in L^1$ and $f_2 \in L^2$ and

$$f = f_1 + f_2.$$

The function $\hat{f}_1$ is continuous and may be restricted to S^{d-1}. However, the function $\hat{f}_2$ is in L^2 and cannot in general be meaningfully restricted.

The first hint that things might be quite different is the observation that for certain p in $(1,2)$ if f in L^p is radial, then $\hat{f}$ is continuous away from the origin.

Proposition

Suppose $f \in L^p$ is a radial function. If $1 \leqslant p < \frac{2d}{d+1}$, then $\hat{f}$ is continuous away from the origin. In particular $\hat{f}$ may be restricted to S^{d-1}.

This proposition is a consequence of the decay of the Fourier transform of $d\sigma$: the surface measure on S^{d-1}.

Fourier transform of complex measures

In these lectures we will only consider measures $d\mu = \psi d\lambda$, where $\psi \in L^1$ and λ is a (positive) Borel measure with bounded support or dx.
Examples are: $d\mu = fdx$ with $f \in L^1$; $d\mu = d\sigma$; the Dirac mass δ_0.

Definition

The Fourier transform of μ is

$$\hat{\mu}(\xi) = \int e^{-2\pi i x\cdot\xi} d\mu(x).$$

The integral converges absolutely and $\hat{\mu}$ is a bounded function since

$$|\hat{\mu}(\xi)| \leqslant \int |\psi(x)| d\lambda(x) = \|\mu\|.$$

In general, however, $\hat{\mu}$ does not decay at infinity ($\hat{\delta}_0 = 1$).

An application of dominated convergence shows that $\hat{\mu}$ is a continuous function. If in addition we assume that $d\mu$ has a bounded support (like $d\sigma$ or δ_0), then $\hat{\mu}$ extends to an entire function on $\mathbb{C}^d$.

The convolution of $f \in L^1$ and $d\mu$ is

$$f * \mu(x) = \int f(x-y)d\mu(y).$$

By Fubini's theorem, we have

$$\widehat{f * \mu}(\xi) = \hat{f}(\xi)\hat{\mu}(\xi).$$

The main point for us is that $\hat{\sigma}$ decays at infinity and in fact enjoys the bound

$$|\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{(d-1)}{2}}.$$

This is due the curvature of the sphere. Indeed, letting

$$\gamma = \{(x_1, 0) \in \mathbb{R}^2 : 0 \leqslant x_1 \leqslant 1\}$$

and $d\mu = dx_1$, we get

$$\hat{\mu}(\xi_1, \xi_2) = \int_\gamma e^{-2\pi i \xi \cdot x} d\mu(x) = \int_0^1 e^{-2\pi i \xi_1 x_1} dx_1,$$

so that $\hat{\gamma}(\xi_1, \xi_2) = \hat{\gamma}(\xi_1, 0)$ and does not decay in the vertical direction.

Proposition

Suppose $f \in L^p$ is a radial function ($f(x) = f(r)$, $r = |x|$). If $1 \leqslant p < \frac{2d}{d+1}$, then $\hat{f}$ is continuous away from the origin.

Proof: $f\chi_{B(0,1)} \in L^1$ has a continuous Fourier transform. Write $x = r\omega$ with $\omega \in S$ and $\rho = |\xi|$. We have

$$\begin{aligned}
\widehat{(1-\chi_B)f}(\xi) &= \int_1^\infty f(r)\left(\int_{S^{d-1}} e^{-2\pi i r\omega\cdot\xi}\, d\sigma(\omega)\right) r^{d-1}dr \\
&= \int_1^\infty f(r)\hat{\sigma}(r\rho) r^{d-1}dr \\
&= \rho^{-\frac{d-1}{2}} \int_1^\infty f(r)\rho^{\frac{d-1}{2}}\hat{\sigma}(r\rho) r^{d-1}dr = \rho^{-\frac{d-1}{2}} I(\rho).
\end{aligned}$$

By dominated convergence it is easy to see that $I(\rho)$ is continuous for $\rho > 1$, since

$$|f(r)\rho^{\frac{d-1}{2}}\hat{\sigma}(r\rho)|r^{d-1} \lesssim |f(r)|\rho^{\frac{d-1}{2}}(r\rho)^{-\frac{d-1}{2}}r^{d-1}$$
$$= |f(r)|r^{-\frac{d-1}{2}}r^{d-1} \in L^1(dr)$$

by

$$\int_1^\infty |f(r)|r^{-\frac{d-1}{2}}r^{d-1}dr = \int_1^\infty |f(r)|r^{-\frac{d-1}{2}}r^{(d-1)\left(\frac{1}{p}+\frac{1}{p'}\right)}dr$$
$$\lesssim \left(\int_1^\infty |f(r)|^p r^{d-1}dr\right)^{\frac{1}{p}}\left(\int_1^\infty r^{(d-1)\left(1-\frac{p'}{2}\right)}dr\right)^{\frac{1}{p'}}$$
$$\lesssim \|f\|_p$$

since

$$-1 > (d-1)\left(1-\frac{p'}{2}\right) \iff p' > \frac{2d}{d-1}.$$

Definition

Given a Borel set E in $\mathbb{R}^d$ and a measure $d\mu$ on E, we say that E enjoys the $R(p \to q)$-restriction property if the estimate

$$\|\hat{f}\|_{L^q(E,d\mu)} = \left(\int_E |\hat{f}(\xi)|^q d\mu(\xi)\right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

holds for all $f \in \mathcal{S}$.

The $R(1 \to \infty)$ estimate holds trivially for any Borel set $E \subset \mathbb{R}^d$. A less trivial example is given by the Hausdorff-Young inequality, which tells that $E = \mathbb{R}^d$ satisfies the $R(p \to p')$-restriction estimate (with the Lebesgue measure).

If E enjoys the $R(p \to q)$-restriction property, we may define the restriction of $\hat{f}$ to E (in the L^q sense) for all $f \in L^p$:

Since the Schwartz space is dense in L^p (for $p < \infty$), there is a sequence $\{f_n\} \subset \mathcal{S}$ converging to f in the L^p norm. The functions in $\{\hat{f}_n\}$ are Schwartz and hence continuous, so

$$E \ni \xi \mapsto Rf_n(\xi) = \hat{f}_n|_E(\xi) = \int e^{-2\pi i \xi \cdot x} f_n(x)dx$$

is well defined. Since $\{f_n\}$ is a Cauchy sequence in L^p, we deduce from $R(p \to q)$ that $\{\hat{f}_n\}$ is a Cauchy sequence in $L^q(E)$. Finally, we define $Rf = \hat{f}|_E$ as the $L^q(E)$ limit of $\{\hat{f}_n|_E\}$.

It is often more convenient to consider the adjoint of R, which for $g \in C^{\infty}_{com}(S^{d-1})$ is given by

$$R^*g(x) = \int_{S^{d-1}} e^{2\pi i\xi\cdot x} g(\xi)\, d\sigma(\xi)$$

and is called extension operator. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^d} f(\xi)\overline{R^*g(\xi)}\, d\sigma(\xi) &= \int_S Rf(\xi)\overline{g(\xi)}\, d\sigma(\xi) \\
&= \int_{S^{d-1}} \left(\int_{\mathbb{R}^d} f(x) e^{-2\pi i\xi\cdot x}\, dx \right) \overline{g(\xi)}\, d\sigma(\xi) \\
&= \int_{\mathbb{R}^d} f(x) \overline{\left(\int_{S^{d-1}} e^{2\pi i\xi\cdot x} g(\xi)\, d\sigma(\xi) \right)}\, dx.
\end{aligned}$$

By duality S^{d-1} enjoys an $R(p \to q)$ restriction estimate if and only if it satisfies the $R^*(q' \to p')$-extension inequality:

$$\|R^* g\|_{p'} = \left(\int_{\mathbb{R}^d} |R^* g(x)|^{p'} dx \right)^{\frac{1}{p'}} \lesssim \left(\int_{S^{d-1}} |g(\xi)|^{q'} d\sigma(\xi) \right)^{\frac{1}{q'}},$$

for all continuous g on M. In other words:

Lemma

The restriction operator R maps $L^p(\mathbb{R}^d)$ to $L^q(S^{d-1}, d\sigma)$ if and only if R^* maps $L^{q'}(S^{d-1}, d\sigma)$ to $L^{p'}(\mathbb{R}^d)$.

It is easy to see that when a hypersurface $M \subset \mathbb{R}^d$ (endowed with the surface measure) contains a non trivial portion of a hyperplane, then there are no non-trivial restriction estimate: for $d = 2$ consider $M = \{(\xi_1, 0) : 1 < \xi_1 < 1\}$ with $d\xi_1$, then

$$R^* g(x_1, x_2) = \int_0^1 e^{2\pi i x_1 \xi_1} g(\xi_1) d\xi_1$$

is independent of x_2 and thus does not belong to any $L^{p'}(\mathbb{R}^2)$ with $p' < \infty$.

We can also see that there are not $R(p \to q)$ estimates for S^{d-1} if $p' \leqslant \frac{2d}{d-1}$ or $p \geqslant \frac{2d}{d+1}$. This follows by testing the extension estimate on the constant function $g = 1$. Since

$$R^* g(x) = \int_S e^{2\pi i \omega \cdot x} d\sigma(\omega) = \hat{\sigma}(x)$$

and $|\hat{\sigma}(x)| \approx |x|^{-\frac{d-1}{2}}$ for $|x| > 1$, we have

$$\|R^* g\|_{p'}^{p'} \gtrsim \int_{|x|>1} |x|^{-p' \frac{d-1}{2}} dx = \infty,$$

if $p' \frac{d-1}{2} \leqslant d$ or $p' \leqslant \frac{2d}{d-1}$, that is $p \geqslant \frac{2d}{d+1}$.
From this computation we see that $\hat{\sigma} \in L^r$ if and only if $r > \frac{2d}{d-1}$.

Hystorically, the first non trivial restriction estimate was obtained in the sixties of the last century by E. Stein, who proved a $(p, 2)$-estimate holding on the sphere, by conjugating the decay of the Fourier transform of $d\sigma$ and the Hardy-Littlewood-Sobolev inequality.
This estimate is based on the following observation holding for $q = 2$.

Lemma

The following estimates are equivalent:

$$\|Rf\|_{L^2(d\sigma)} \leqslant C\|f\|_{L^p(\mathbb{R}^d)}, \tag{1}$$

$$\|R^*g\|_{L^{p'}(\mathbb{R}^d)} \leqslant C\|g\|_{L^2(d\sigma)}, \tag{2}$$

$$\|R^*Rf\|_{L^{p'}(\mathbb{R}^d)} \leqslant C^2\|f\|_{L^p(\mathbb{R}^d)}, \tag{3}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ *and all* $g \in C_{com}(E)$*.*

Proof.

(1) and (2) are equivalent by the previous lemma. The bound in (3) follows by composition of those in (1) and (2). Finally, assuming (3), we have

$$\begin{aligned}\|Rf\|^2_{L^2(E)} = (Rf, Rf)_{L^2(E)} = (f, R^*Rf)_{L^2(\mathbb{R}^d)} \\ \leqslant \|f\|_{L^p(\mathbb{R}^d)}\|R^*Rf\|_{L^{p'}(\mathbb{R}^d)} \leqslant C^2\|f\|^2_{L^p(\mathbb{R}^d)}.\end{aligned}$$

We prove an estimate holding for R^*R. The advantages of that are first that R^*Rf is still a function on $\mathbb{R}^d$ (like f) and second that R^*R is a convolution operator and the effects of the oscillations are all contained in its kernel.

First we obtain an expression for the operator R^*R,

$$R^*(Rf)(x) = \int_{S^{d-1}} e^{2\pi i\omega\cdot x} Rf(\omega)d\sigma(\omega) = \int_{S^{d-1}} e^{2\pi i\omega\cdot x}\hat{f}(\omega)d\sigma(\omega)$$
$$= \int_{S^{d-1}} e^{2\pi i\omega\cdot x}\left(\int_{\mathbb{R}^d} f(y)e^{-2\pi i y\cdot\omega}dy\right)d\sigma(\omega),$$

which by Fubini's theorem yields

$$R^*R\,f(x) = \int_{\mathbb{R}^d} f(y)\left(\int_{S^{d-1}} e^{-2\pi i(x-y)\cdot\omega}d\sigma(\omega)\right)dy$$
$$= \int_{\mathbb{R}^d} f(y)\hat{\sigma}(x-y)dy$$
$$= f*\hat{\sigma}(x).$$

Being

$$|\hat{\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{(d-1)}{2}},$$

$\hat{\sigma}$ lies in $L^r(\mathbb{R}^d)$ if and only if

$$r > \frac{2d}{d-1}.$$

Hence, by Young's inequality we get

$$\|R^*Rf\|_{L^{p'}(\mathbb{R}^d)} = \|f * \hat{\sigma}\|_{L^{p'}(\mathbb{R}^d)} \leqslant \|f\|_{L^p(\mathbb{R}^d)}\|\hat{\sigma}\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

if

$$1+\frac{1}{p'} = \frac{1}{p}+\frac{1}{r} < \frac{1}{p}+\frac{d-1}{2d}$$

which is equivalent to $p < \frac{4d}{3d+1}$.

Using the Hardy-Littlewood-Sobolev estimate instead of Young's convolution inequality Stein proved the estimate for $p \leqslant \frac{4d}{3d+1}$.

Stein himself and P. Tomas eventually showed that the restriction inequality holds exactly for $1 \leqslant p \leqslant \frac{2d+2}{d+3}$.

We first show that there are not L^2 restriction estimate in the range: $p > \frac{2d+2}{d+3}$. To do that we introduce the **Knapp example**.
We test the inequality

$$\|\widehat{gd\sigma}\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|g\|_{L^2(S)},$$

on the characteristic function g of a cap C_δ of radius $0 < \delta \ll 1$ centered at the north pole $e_d = (0, \cdots, 0, 1)$.
We have

$$C_\delta = S^{d-1} \cap D_\delta,$$

where D_δ is the cylinder around the x_d coordinate axis, with radius δ and top face tangent to the sphere at e_d. The bottom of the cylinder lies at hight $\sqrt{1-\delta^2}$, so

$$D_\delta = \{y = (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : |y'| \leqslant \delta, 1 - y_d \leqslant 1 - \sqrt{1-\delta^2} \approx \delta^2\}.$$

The thickness of D_δ is chosen in order to maximize the intersection with S^{d-1}.

The norm on the right hand side is given by

$$\|g\|_{L^2(S)} = |C_\delta|^{\frac{1}{2}} \approx \delta^{\frac{d-1}{2}}.$$

We need an estimate from below for $\|\widehat{gd\sigma}\|_{p'}$. So we need a bound from below for

$$|\widehat{gd\sigma}(x)| = \left| \int_{C_\delta} e^{2\pi i x \cdot \omega} d\sigma(\omega) \right|.$$

Observe that

$$\begin{aligned} |\widehat{gd\sigma}(x)| &= \left| e^{2\pi i x\cdot e_d} \int_{C_\delta} e^{2\pi i x\cdot(\omega - e_d)} d\sigma(\omega) \right| \\ &= \left| \int_{C_\delta} e^{2\pi i x\cdot(\omega - e_d)} d\sigma(\omega) \right| \\ &\geqslant \left| \mathrm{Re} \int_{C_\delta} e^{2\pi i x\cdot(\omega - e_d)} d\sigma(\omega) \right| \\ &= \left| \int_{C_\delta} \cos(2\pi x \cdot (\omega - e_d)) d\sigma(\omega) \right|. \end{aligned}$$

We look for a subset of $\mathbb{R}^d$ where

$$\cos(2\pi x \cdot (\omega - e_d)) \geqslant \frac{1}{2} \quad \text{for all } \omega \in C_\delta \subset D_\delta.$$

For $c > 0$, let

$$D^*_{\delta,c} = \{(x', x_d) : |x'| \leqslant c/\delta, \quad |x_d| \leqslant c/\delta^2\}.$$

If $x = (x', x_d) \in D^*_{\delta,c}$ and $\omega = (\omega', \omega_d) \in C_\delta \subset D_\delta$, then

$$\begin{aligned} |2\pi x \cdot (\omega - e_d)| &\leqslant 2\pi|x' \cdot \omega'| + 2\pi|x_d||\omega_d - 1| \\ &\leqslant 2\pi|x'||\omega'| + 2\pi|x_d||\omega_d - 1| \leqslant 2\pi c + 2\pi c \leqslant \frac{\pi}{3}, \end{aligned}$$

if c is sufficiently small.

This yields the bound

$$|\widehat{gd\sigma}(x)| \geqslant \frac{1}{2}\int_{C_\delta} d\sigma(\omega) \gtrsim |C_\delta| \approx \delta^{d-1}$$

for $x \in D^*_{\delta,c}$.
Since $|D^*_{\delta,c}| \approx \delta^{-(d-1)}\delta^{-2} = \delta^{-(d+1)}$, we obtain

$$\|\widehat{gd\sigma}\|_{L^{p'}(\mathbb{R}^d)} \geqslant \|\widehat{gd\sigma}\|_{L^{p'}(D^*_{\delta,c})} \gtrsim \delta^{d-1}|D^*_{\delta,c}|^{\frac{1}{p'}} \gtrsim \delta^{d-1}\delta^{-\frac{d+1}{p'}}.$$

Assuming

$$\|\widehat{gd\sigma}\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|g\|_{L^2(S)},$$

we get

$$\delta^{d-1}\delta^{-\frac{d+1}{p'}} \lesssim \delta^{\frac{d-1}{2}} \quad \text{or} \quad \delta^{\frac{d-1}{2}-\frac{d+1}{p'}} \lesssim 1,$$

for all $0 < \delta \ll 1$. This implies

$$\frac{d-1}{2} - \frac{d+1}{p'} \geqslant 0, \quad \frac{d-1}{2} \geqslant \frac{d+1}{p'},$$

or

$$p' \geqslant 2\frac{d+1}{d-1} \quad \text{or} \quad p \leqslant 2\frac{d+1}{d+3}$$

proving the claim.

The same example applied to

$$\|\widehat{gd\sigma}\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|g\|_{L^{q'}(S^{d-1})},$$

leads to the condition ($\|g\|_{L^{q'}(S^{d-1})} \approx \delta^{\frac{d-1}{q'}}$)

$$\delta^{d-1}\delta^{-\frac{d+1}{p'}} \lesssim \delta^{\frac{d-1}{q'}},$$

that is

$$1 \gtrsim \delta^{(d-1)\left(1-\frac{1}{q'}\right)}\delta^{-\frac{d+1}{p'}} = \delta^{\frac{d-1}{q}-\frac{d+1}{p'}},$$

which implies

$$\frac{d-1}{q} \geqslant \frac{d+1}{p'} \quad \text{or} \quad p' \geqslant q\frac{d+1}{d-1}.$$

The Restriction Conjecture

This lead to Stein to conjecture that the estimate

$$\|\hat{f}\|_{L^q(S^{d-1})} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

should be true for

$$1 \leqslant p < \frac{2d}{d+1} \quad \text{and} \quad p' \geqslant q\frac{d+1}{d-1}.$$

Known for $d = 2$ ($1 \leqslant p < 4/3$ and $p' \leqslant 3q$) (due to C. Fefferman-Stein and A. Zygmund).

THEOREM (P. TOMAS)

The inequality

$$\|R^*Rf\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

holds for $1 \leqslant p < \frac{2d+2}{d+3} = p^*$.

The proof is based on an interpolation theorem due to R. Hunt.

Theorem (R. Hunt)

Let (X,μ) and (Y,ν) be measure spaces. Let T be a linear operator mapping measurable functions on X to measurable functions on Y. Let p_0, p_1, q_0, q_1 be Lebesgue exponents, with $p_0 < p_1$ and $q_0 < q_1$. Suppose that for any measurable set $E \subset X$ we have

$$\|T\chi_E\|_{L^{q_0}(Y)} \lesssim |E|^{\frac{1}{p_0}}, \quad \|T\chi_E\|_{L^{q_1}(Y)} \lesssim |E|^{\frac{1}{p_1}}.$$

Define for $0 < t < 1$,

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1},$$

then

$$\|Tf\|_{L^q(Y)} \lesssim \|f\|_{L^p(X)}$$

for all measurable functions f.

Proof of the Tomas theorem (A. Carbery):

Let E be a Borel set in $\mathbb{R}^d$, we will show that

$$\|R\chi_E\|_{L^2(S^{d-1})} \lesssim |E|^{\frac{d+3}{2d+2}},$$

then the theorem will follow from Hunt's theorem interpolating with the trivial L^1 to L^∞ estimate,

$$\|R\chi_E\|_{L^\infty(S^{d-1})} \lesssim |E|.$$

The argument is based on two geometric properties of the measure σ: the dimensional estimate

$$\sigma(B(x,R)) \lesssim R^{d-1}$$

and the decay of the Fourier transform

$$|\hat{\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{d-1}{2}},$$

which is due to the curvature of S^{d-1}.

We will use the T^*T method, writing

$$\|R\chi_E\|^2_{L^2(S^{d-1})} = \langle R\chi_E, R\chi_E\rangle_{L^2(S^{d-1})} = \langle \chi_E, R^*R\chi_E\rangle_{L^2(\mathbb{R}^d)}.$$

We split $R^*Rf = f * \hat{\sigma}$ in two operators corresponding to the low and the high frequency parts of σ:
Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\chi_{B(0,1)} \leqslant \phi \leqslant \chi_{B(0,2)}$ and let $\lambda > 0$. Define

$$\hat{\sigma}_{low}(\xi) = \hat{\sigma}(\xi)\phi(\xi/\lambda)$$

and

$$\hat{\sigma}_{high}(\xi) = \hat{\sigma}(\xi)(1 - \phi(\xi/\lambda))$$

and

$$T_{low}f = f * \hat{\sigma}_{low}, \quad T_{high}f = f * \hat{\sigma}_{high},$$

so that

$$R^*Rf = T_{low}f + T_{high}f.$$

Then

$$\|R\chi_E\|_{L^2(S^{d-1})}^2 = \langle \chi_E, R^* R\chi_E \rangle = \langle \chi_E, T_{low}\chi_E \rangle + \langle \chi_E, T_{high}\chi_E \rangle.$$

We will bound the two terms on the right hand side separately.

We start with the high frequency part,

$$|\langle \chi_E, T_{high}\chi_E\rangle| \leqslant \|\chi_E\|_1 \|T_{high}\chi_E\|_\infty = |E| \|T_{high}\chi_E\|_\infty.$$

We have

$$\|T_{high}\chi_E\|_\infty = \|\chi_E * \hat{\sigma}_{high}\|_\infty \leqslant |E| \|\hat{\sigma}_{high}\|_\infty \lesssim |E|\lambda^{-\frac{d-1}{2}}$$

by the decay rate of $\hat{\sigma}$. Hence, we obtain

$$|\langle \chi_E, T_{high}\chi_E\rangle| \lesssim |E|^2\lambda^{-\frac{d-1}{2}}.$$

For the low frequency part we first use Cauchy-Schwarz,

$$|\langle \chi_E, T_{low}\chi_E\rangle| \leqslant \|\chi_E\|_2 \|T_{low}\chi_E\|_2 = |E|^{\frac{1}{2}} \|T_{low}\chi_E\|_2.$$

The Plancherel identity yields

$$\begin{aligned} \|T_{low}\chi_E\|_2 &= \|\chi_E * \hat{\sigma}_{low}\|_2 = \|\hat{\chi}_E \sigma_{low}\|_2 \\ &\leqslant \|\hat{\chi}_E\|_2 \|\sigma_{low}\|_\infty = \|\chi_E\|_2 \|\sigma_{low}\|_\infty = |E|^{\frac{1}{2}} \|\sigma_{low}\|_\infty. \end{aligned}$$

Recall that

$$\hat{\sigma}_{low}(\xi) = \hat{\sigma}(\xi)\phi(\xi/\lambda) = \hat{\sigma}(\xi)\phi_\lambda(\xi),$$

so that

$$\sigma_{low}(x) = \sigma * \widehat{\phi_\lambda}(x) = \lambda^d \int_{S^{d-1}} \hat{\phi}(\lambda(x-\omega))d\sigma(\omega).$$

Since ϕ has compact support, by the uncertainty principle the support of $\hat{\phi}$ cannot be bounded. However, $\hat{\phi}$ is Schwartz, so we can assume that $\hat{\phi}$ is 1 on $B(0,1)$ and decay rapidly outside of it,

$$\begin{aligned} |\sigma_{low}(x)| &\leqslant \lambda^d \int_{S^{d-1}} |\hat{\phi}(\lambda(x-\omega))|d\sigma(\omega) \\ &\lesssim \lambda^d \int_{S^{d-1}} \chi_{B(0,2)}(\lambda(x-\omega))d\sigma(\omega) \\ &= \lambda^d \sigma(B(x,2/\lambda) \cap S^{d-1}). \end{aligned}$$

$\sigma\big(B(x,1/\lambda)\cap S^{d-1}\big)$ attains the supremum for $x\in S^{d-1}$ and this is $\sigma\big(B(x,1/\lambda)\big)\approx\lambda^{-(d-1)}$, so that

$$|\sigma_{low}(x)|\lesssim\lambda^d\lambda^{-(d-1)}=\lambda.$$

Hence,

$$\|T_{high}\chi_E\|_2\leqslant|E|^{\frac{1}{2}}\|\sigma_{low}\|_\infty\lesssim\lambda|E|^{\frac{1}{2}},$$

which yields

$$|\langle\chi_E,T_{low}\chi_E\rangle|\leqslant|E|^{\frac{1}{2}}\|T_{high}\chi_E\|_2\lesssim\lambda|E|.$$

Therefore,

$$\|R\chi_E\|^2_{L^2(S^{d-1})} = \langle \chi_E, T_{low}\chi_E \rangle + \langle \chi_E, T_{high}\chi_E \rangle$$
$$\lesssim |E|\lambda + |E|^2\lambda^{-\frac{d-1}{2}}.$$

By optimizing in λ, we get the claim:

$$\|R\chi_E\|_{L^2(S^{d-1})} \lesssim |E|^{\frac{d+3}{2d+2}}.$$

Why do we have $|\hat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}$ for $|\xi| \gg 1$?

We want to estimate $\hat{\sigma}(\xi)$ for $R/2 < |\xi| < R$ for some $R \gg 1$. This is the same as understanding

$$\hat{\sigma}_R(\xi) = \hat{\sigma}(\xi)\phi_R(\xi)$$

with ϕ_R a smooth bump function localized at $|\xi| \approx R$. We take $g \in \mathcal{S}(\mathbb{R})$ essentially supported at $t = 1$, that is, we suppose that $g(1) = 1$ and that $|g(t)| \ll 1$ for $t < \frac{1}{2}$ and $t > \frac{3}{2}$. We also assume that $\hat{g}(0) = 1$, $\operatorname{supp} \hat{g} \subset (-1, 1)$.
Define

$$\phi_R(\xi) = g(|\xi|/R).$$

Since

$$\widehat{\phi_R}(\xi) = R^d \hat{\phi}(R\xi),$$

taking the Fourier transform of $\hat{\sigma}_R = \hat{\sigma}\phi_R$ we get

$$\sigma_R = \sigma * \widehat{\phi_R},$$

a function of total mass 1 supported in a neighborhood of S^{d-1} of radius $1/R$.

Indeed,

$$\begin{aligned}\int_{\mathbb{R}^d} \sigma * (R^d \check{\phi}(R|\cdot|))(x)dx &= \int_{\mathbb{R}^d}\int_{S^{d-1}} R^d \check{\phi}(R|x-\omega|)d\sigma(\omega)dx \\ &= \int_{S^{d-1}}\int_{\mathbb{R}^d} R^d \check{\phi}(R|x-\omega|)dxd\sigma(\omega) \\ &= \int_{S^{d-1}}\int_{\mathbb{R}^d} R^d \check{\phi}(R|x|)dxd\sigma(\omega) \\ &= \int_{\mathbb{R}^d} R^d \check{\phi}(R|x|)dx \int_{S^{d-1}} d\sigma(\omega) \\ &= |S^{d-1}| \int_{\mathbb{R}^d} \check{\phi}(|x|)dx \\ &\approx 1.\end{aligned}$$

So far we showed that

$$\sigma_R = \sigma * \widehat{\phi_R}$$

is a function of total mass 1 supported in a neighborhood of S^{d-1} of radius $1/R$. Therefore σ_R can be written as

$$\sigma_R(x) = \sigma * \widehat{\phi_R}(x) = R\Psi(R(|x| - 1))$$

with $\Psi \in \mathcal{S}(\mathbb{R})$ such that $\Psi(0) = 1$ and $\operatorname{supp} \Psi \subset (-1/2, 1/2)$.

Given a $1/\sqrt{R}$ separated set $\Lambda = \{\omega\}$ in S^{d-1}, which is maximal with respect to this property, the caps $C_{\omega,2/\sqrt{R}}$ yield a covering of S^{d-1}. Hence, introducing a partition of unity, we decompose $\Psi(R(|x|-1))$ into bump functions $\eta_{\omega,R^{-1/2}} \in \mathcal{S}(\mathbb{R}^d)$ adapted to the sets $D_{\omega,R^{-1/2}}$, where $D_{\omega,R^{-1/2}}$ is a cylinder centered at ω with axis ω, radius $1/\sqrt{R}$ and thickness $1/R$:

$$\Psi(R(|x|-1)) = \sum_{\omega\in\Lambda} \eta_{\omega,R^{-\frac{1}{2}}}(x).$$

So we get

$$\sigma_R(x) = R\Psi(R(|x|-1)) = R\sum_{\omega\in\Lambda}\eta_{\omega,R^{-\frac{1}{2}}}(x),$$

so that

$$\begin{aligned}|\hat{\sigma}(\xi)|\chi_{\{R/2<|\xi|<3R/2\}}(\xi) &\approx |\hat{\sigma}(\xi)\phi_R(\xi)| \\ &= |\hat{\sigma}_R(\xi)| = R\Big|\sum_{\omega\in\Lambda}\widehat{\eta_{\omega,R^{-\frac{1}{2}}}}(\xi)\Big|.\end{aligned}$$

By the Knapp example we know that for ξ in the dual cylinder D_ω^* (with length R and radius $\sqrt{R}$), we have

$$\left|\widehat{\eta_{\omega,R^{-\frac{1}{2}}}}(\xi)\right| \approx |D_\omega| \approx (R^{-\frac{1}{2}})^{d-1}R^{-1} = R^{-(d+1)/2}.$$

Hence,

$$R\left|\widehat{\eta_{\omega,R^{-\frac{1}{2}}}}(\xi)\right| \approx RR^{-(d+1)/2} = R^{-(d-1)/2}.$$

Note that if $|\xi| \approx R$, then ξ essentially lies in a unique cylinder D_ω^* (since the directions of the cylinders are $1/\sqrt{R}$ separated) hence

$$\begin{aligned}|\hat{\sigma}(\xi)|\chi_{\{R/2<|\xi|<R\}}(\xi) &\approx R\Big|\sum_{\omega\in\Lambda}\widehat{\eta_{\omega,R^{-\frac{1}{2}}}}(\xi)\Big| \\ &\leqslant R\sum_{\omega\in\Lambda}\left|\widehat{\eta_{\omega,R^{-\frac{1}{2}}}}(\xi)\right| \approx R^{-(d-1)/2},\end{aligned}$$

which proves the claim.

Restriction estimates for sub-Laplacians

joint work with V. Casarino

Gent, September 2023

If f is a Schwartz function, the Fourier inversion formula in polar coordinates yields

$$f(x) = \int_0^\infty \left(\int_{S^{d-1}} \hat{f}(r\omega) e^{2\pi i r\omega \cdot x} d\sigma(\omega) \right) r^{d-1} dr,$$

where $d\sigma$ is the measure on the unit sphere and

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx.$$

If Δ is the Laplacian, then

$$\Delta \int_{S^{d-1}} \hat{f}(r\omega) e^{2\pi i r\omega\cdot x} d\sigma(\omega) = -4\pi^2 r^2 \int_{S^{d-1}} \hat{f}(r\omega) e^{2\pi i r\omega\cdot x} d\sigma(\omega).$$

So

$$f(x) = \int_0^\infty \int_{S^{d-1}} \hat{f}(r\omega) e^{2\pi i r\omega\cdot x} d\sigma(\omega) r^{d-1} dr,$$

may be thought of as the spectral decomposition of f.

Replacing $4\pi^2 r^2$ with λ in the Fourier inversion formula, we obtain

$$
\begin{aligned}
f(x) &= \int_0^\infty \int_{S^{d-1}} \hat{f}(r\omega) e^{2\pi i r\omega\cdot x} d\sigma(\omega) r^{d-1} dr \\
&= \int_0^\infty \int_{S^{d-1}} \hat{f}(\sqrt{\lambda/2\pi}\omega) e^{i\sqrt{\lambda}\omega\cdot x} d\sigma(\omega) \lambda^{\frac{d-2}{2}} d\lambda \\
&= \int_0^\infty \mathcal{P}_\lambda f(x) d\lambda,
\end{aligned}
$$

where

$$
\mathcal{P}_\lambda f(x) = \int_{S^{d-1}} \hat{f}\left(\sqrt{\lambda/2\pi}\omega\right) e^{i\sqrt{\lambda}\omega\cdot x} d\sigma(\omega)
$$

are the spectral components of f with respect to Δ.

For $\lambda = 2\pi$ we have

$$\mathcal{P}_{2\pi} f(x) = \int_{S^{d-1}} \hat{f}(\omega) e^{i\omega \cdot x} d\sigma(\omega).$$

If g is a continuous function on S^{d-1} the extension operator is defined by

$$R^* g(x) = \int_{S^{d-1}} g(\omega) e^{2\pi i \omega \cdot x} d\sigma(\omega).$$

R^* is bounded from $L^2(S^{d-1}) \to L^{p'}(\mathbb{R}^d)$ if $1 \leqslant p \leqslant 2\frac{d+1}{d+3}$.

RESTRICTION PROBLEM:

The bounds

$$\|\hat{f}\|_{L^q(S^{d-1})} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

hold if and only if $1 \leqslant p < \frac{2d}{d+1}$ and $q \geqslant \frac{d-1}{d+1}p'$.

Settled for $d = 2$:

THEOREM (C. FEFFERMAN, E. STEIN–A. ZYGMUND)

The estimate

$$\|\hat{f}\|_{L^q(S^1)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

holds if and only if $1 \leqslant p < \frac{4}{3}$ *and* $q \geqslant \frac{p'}{3}$.

The proof of the Stein-Tomas theorem is based on the observation that $R : L^p(\mathbb{R}^d) \to L^2(S^{d-1})$ is equivalent to $R^* \circ R : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$. The operator $R^* \circ R$ is given by

$$R^*(Rf)(x) = f * \hat{\sigma}(x).$$

Indeed,

$$\begin{aligned}
R^*Rf(x) &= \int_{S^{d-1}} Rf(\omega)e^{2\pi i\omega\cdot x}d\sigma(\omega) \\
&= \int_{S^{d-1}} \hat{f}(\omega)e^{2\pi i\omega\cdot x}d\sigma(\omega) \\
&= \int_{S^{d-1}} \int_{\mathbb{R}^d} f(y)e^{-2\pi i\omega\cdot y}dye^{2\pi i\omega\cdot x}d\sigma(\omega) \\
&= \int_{\mathbb{R}^d} f(y)\left(\int_{S^{d-1}} e^{-2\pi i\omega\cdot(x-y)}d\sigma(\omega)\right)dy \\
&= \int_{\mathbb{R}^d} f(y)\hat{\sigma}(x-y)dy \\
&= \int_{\mathbb{R}^d} f(x-y)\hat{\sigma}(y)dy \\
&= f * \hat{\sigma}(x).
\end{aligned}$$

Since the spectral components of f with respect to Δ are given by

$$\mathcal{P}_{2\pi} f(x) = f * \hat{\sigma}(x) = R^* R f(x),$$

the Stein-Tomas theorem is the assertion that
$\mathcal{P}_{2\pi} : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$.

In $\mathbb{R}^3$ with coordinates (t, x, y), define

$$X = \partial_x - \frac{y}{2}\partial_t, \quad Y = \partial_y + \frac{x}{2}\partial_t, \quad T = \partial_t,$$

then $[X, Y] = T$. $\mathbb{R}^3$ equipped with this bracket gives rise to the Heisenberg Lie algebra $\mathfrak{h}$.

The sublaplacian is the operator

$$L = -X^2 - Y^2 = -\partial_x^2 - \partial_y^2 + \frac{1}{2}(x\partial_y - y\partial_x)\partial_t - \frac{1}{4}(x^2 + y^2)\partial_t^2.$$

L has a unique positive self adjoint extension to $L^2(\mathbb{R}^3)$.

Let

$$f(x,y,t) = \int_0^\infty \mathcal{P}_\lambda f(x,y,t)\,d\lambda$$

be the spectral decomposition of a Schwartz function f with respect to L,

$$L\mathcal{P}_\lambda f = \lambda \mathcal{P}_\lambda f.$$

The task is studying the mapping properties between Lebesgue spaces of the operators $\mathcal{P}_\lambda$.

The first positive result is due to D. Müller in 1990 (*A restriction theorem for the Heisenberg group, Ann. of Math., 1990*). It is expressed in terms mixed lebesgue norms:

$$\|f\|_{L_t^r L_{x,y}^p} = \left(\int_{\mathbb{R}^2} \left(\int_{-\infty}^{\infty} |f(t,x,y)|^r \, dt \right)^{\frac{p}{r}} dxdy \right)^{\frac{1}{p}}.$$

Warning: The estimate

$$\|\mathcal{P}_\lambda f\|_{L_t^s L_{x,y}^q} \lesssim_\lambda \|f\|_{L_t^r L_{x,y}^p}$$

is false for $(r,s) \neq (1,\infty)$.

THEOREM (D. MÜLLER)

The estimate

$$\|\mathcal{P}_\lambda f\|_{L_t^\infty L_{x,y}^{p'}} \lesssim_\lambda \|f\|_{L_t^1 L_{x,y}^p}$$

holds for $1 \leqslant p < 2$.

Improving on Müller's result,

$$\|\mathcal{P}_\lambda f\|_{L_t^\infty {L^{p'}}_{x,y}} \lesssim_\lambda \|f\|_{L_t^1 L_{x,y}^p},$$

means obtaining estimates in which the exponent p' on the left is replaced by $q < p'$

$$\|\mathcal{P}_\lambda f\|_{L_t^\infty {L^q}_{x,y}} \lesssim_\lambda \|f\|_{L_t^1 L_{x,y}^p}.$$

We got $q = 2$.

THEOREM (V.CASARINO, C., ADV.MATH.2013)

The estimate

$$\|\mathcal{P}_\lambda f\|_{L^\infty_t L^2_{x,y}} \lesssim_\lambda \|f\|_{L^1_t L^p_{x,y}}$$

holds for $1 \leqslant p < 2$.

Analogous results holds on more general groups ($\mathbb{H}_n$, groups of Heisenberg type).

Using classical tools it is not difficult to show that $\mathcal{P}_\lambda$ is bounded from $L^p_{x,y}L^1_t$ to $L^q_{x,y}L^\infty_t$ in the range $1 \leqslant p \leqslant \frac{6}{5}$ and $\frac{4}{3} < q$ in $\mathbb{H}_1$.

More generally, on $\mathbb{H}_n$ the range of exponents for which is known that the estimates hold is given by $1 \leqslant p \leqslant 2\frac{2n+1}{2n+3}$ and $\frac{4n}{2n+1} < q$.

By invariance under translation there cannot be estimates with $q < p$. Moreover, we have an example showing that there are no estimates of the type

$$\|\mathcal{P}_\lambda f\|_{L^q_{x,y}L^\infty_t(\mathbb{H}_n)} \lesssim_\lambda \|f\|_{L^p_{x,y}L^1_t(\mathbb{H}_n)}$$

for $q \leqslant \frac{4n}{2n+1} = \tilde{p}$. .

The Riesz diagram showing what we know is the following.

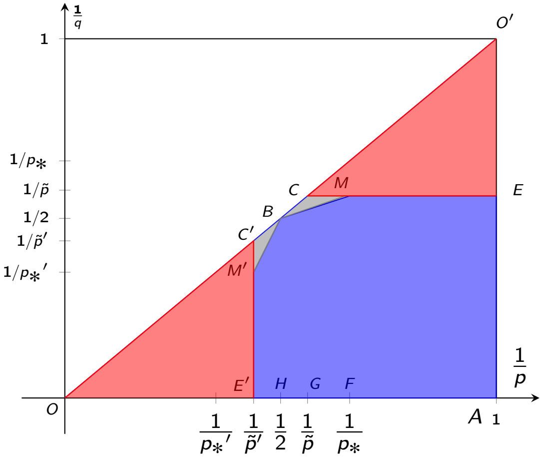


Figure 1. *In this picture* $\tilde{p} = \frac{4n}{2n+1}$, $p_* = 2\frac{2n+1}{2n+3}$.

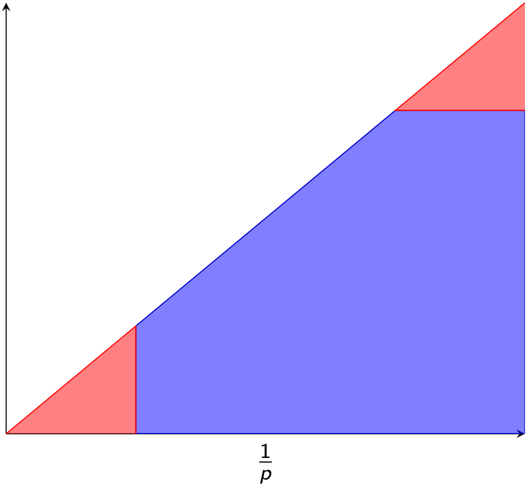
On $\mathbb{H}_1$ the problem is settled:

THEOREM (V.CASARINO, C.)

The estimate

$$\|\mathcal{P}_\lambda f\|_{L^\infty_t L^q_{x,y}} \lesssim \lambda^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^1_t L^p_{x,y}}$$

holds if and only if $1 \leqslant p < \frac{4}{3}$ *and* $q \geqslant \frac{p'}{3}$.


$\frac{1}{p}$

The spectral projectors are given by

$$\mathcal{P}_\lambda f(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(e^{i\lambda_k t} \Pi_k^{\lambda_k} f^{(\lambda_k)}(x,y) + e^{-i\lambda_k t} \Pi_k^{-\lambda_k} f^{(-\lambda_k)}(x,y) \right)$$

$\lambda_k = \frac{\lambda}{2k+1}$.
Here :

- $$f^{(\lambda)}(x,y) = \int_{-\infty}^{\infty} f(x,y,t) e^{i\lambda t} dt,$$
- Π_k^λ is the spectral projection of the twisted-laplacian $\Delta^{(\lambda)}$ corresponding to the eigenvalue $|\lambda|(2k+1)$.

The twisted laplacian is defined by

$$(Lf)^{(\lambda)}(x,y) = \Delta^{(\lambda)} f^{(\lambda)}(x,y)$$

and is given by

$$\Delta^{(\lambda)} = -\partial_x^2 - \partial_y^2 + \frac{i}{2}\lambda(x\partial_y - y\partial_x) + \frac{1}{4}\lambda^2(x^2+y^2).$$

For $\lambda \neq 0$ the operator $\Delta^{(\lambda)}$ has a pure point spectrum: the eigenvalues are given by $|\lambda|(2k+1)$, $k = 0, 1, 2, \ldots$.

Recall that the spectral projections are

$$\mathcal{P}_\lambda f(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(e^{i\lambda_k t} \Pi_k^{\lambda_k} f^{(\lambda_k)}(x,y) + e^{-i\lambda_k t} \Pi_k^{-\lambda_k} f^{(-\lambda_k)}(x,y) \right)$$

Applying the triangle inequality we get

$$|\mathcal{P}_\lambda f(x,y,t)| \leqslant \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(|\Pi_k^{\lambda_k} f^{(\lambda_k)}(x,y)| + |\Pi_k^{-\lambda_k} f^{(-\lambda_k)}(x,y)| \right),$$

and hence

$$\left(\int_{\mathbb{R}^2} |\mathcal{P}_\lambda f(x,y,t)|^q dxdy \right)^{\frac{1}{q}} \leqslant \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\|\Pi_k^{\lambda_k} f^{(\lambda_k)}\|_{L^q_{x,y}} + \|\Pi_k^{-\lambda_k} f^{(-\lambda_k)}\|_{L^q_{x,y}} \right).$$

This formula yields

$$
\begin{aligned}
\sup_t \left(\int_{\mathbb{R}^2} |\mathcal{P}_\lambda f(x,y,t)|^q dxdy \right)^{\frac{1}{q}}
&\leqslant \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\|\Pi_k^{\lambda_k} f^{(\lambda_k)}\|_{L^q_{x,y}} + \|\Pi_k^{-\lambda_k} f^{(-\lambda_k)}\|_{L^q_{x,y}} \right) \\
&\leqslant 2 \sum_{k=0}^{\infty} \frac{1}{2k+1} \|\Pi_k^{\lambda_k}\|_{L^p_{x,y} \to L^q_{x,y}} \|f^{(\lambda_k)}\|_{L^p_{x,y}} \\
&= \sum_{k=0}^{\infty} \frac{\lambda_k^{\frac{1}{p}-\frac{1}{q}}}{2k+1} \|\Pi_k\|_{L^p_{x,y} \to L^q_{x,y}} \|f\|_{L^1_t L^p_{x,y}} \\
&\lesssim \lambda^{\frac{1}{p}-\frac{1}{q}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{1+\frac{1}{p}-\frac{1}{q}}} \|\Pi_k\|_{L^p_{x,y} \to L^q_{x,y}} \|f\|_{L^1_t L^p_{x,y}},
\end{aligned}
$$

where $\Pi_k = \Pi_k^1$.

To show that $\mathcal{P}_\lambda : L^1_t L^p_{x,y} \to L^\infty_t L^q_{x,y}$, we are thus reduced to prove

$$\sum_{k=1}^{\infty} k^{-1-\left(\frac{1}{p}-\frac{1}{q}\right)} \|\Pi_k\|_{L^p_{x,y}\to L^q_{x,y}} \lesssim_{p,q} 1.$$

In particular,

$$\|\mathcal{P}_\lambda f\|_{L^\infty_t L^q_{x,y}} \lesssim \lambda^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^1_t L^p_{x,y}},$$

for $1 \leqslant p < \frac{4}{3}$ and $q \geqslant \frac{p'}{3}$, is a consequence of (using some interpolation)

$$\|\Pi_k f\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \lesssim (\log k)^{\frac{1}{4}} \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}.$$

Theorem

For $\frac{4}{3} \leqslant q \leqslant 2$ we have

$$\|\Pi_k f\|_{L^q(\mathbb{R}^2)} \lesssim (\log k)^{\frac{1}{2}-\frac{1}{q}} \|f\|_{L^q(\mathbb{R}^2)}$$

and for $1 \leqslant q < \frac{4}{3}$

$$\|\Pi_k f\|_{L^q(\mathbb{R}^2)} \lesssim \left(\frac{q'}{q'-4}\right)^{\frac{1}{4}} k^{\frac{1}{2}-\frac{2}{q'}} \|f\|_{L^q(\mathbb{R}^2)}.$$

These bounds are sharp.

By interpolation with $\|\Pi_k f\|_{L^2(\mathbb{R}^2)} \leqslant \|f\|_{L^2(\mathbb{R}^2)}$, it suffices to prove

$$\|\Pi_k f\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \lesssim (\log k)^{\frac{1}{4}} \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^2)},$$

or equivalently

$$\|\Pi_k f\|_{L^4(\mathbb{R}^2)} \lesssim (\log k)^{\frac{1}{4}} \|f\|_{L^4(\mathbb{R}^2)}.$$

We also have:

THEOREM

Let $1 \leqslant p < 4$ *and* $q = 3p'$. *Then*

$$\|\Pi_k\|_{L^p(\mathbb{R}^2)\to L^q(\mathbb{R}^2)} = \|\Pi_k\|_{L^{q'}(\mathbb{R}^2)\to L^{p'}(\mathbb{R}^2)} \lesssim \begin{cases} k^{\frac{2}{q}-\frac{1}{2}} & 4 < q < 12 \\ k^{-\frac{4}{q}} & q > 12. \end{cases}$$

Note that $q = 12$ is excluded from the previous result. In this case there is a logarithmic loss and, so far, we have only been able to prove

$$\|\Pi_k\|_{L^4(\mathbb{R}^2)\to L^{12}(\mathbb{R}^2)} \lesssim_\epsilon k^{-\frac{1}{3}+\epsilon}$$

for all $\epsilon > 0$.
The sharp estimate should be

$$\|\Pi_k\|_{L^4(\mathbb{R}^2)\to L^{12}(\mathbb{R}^2)} \lesssim k^{-\frac{1}{3}}(\log k)^{\frac{1}{4}}.$$

The goal is

$$\|\Pi_k f\|_{L^4(\mathbb{R}^2)} \lesssim (\log k)^{\frac{1}{4}} \|f\|_{L^4(\mathbb{R}^2)}.$$

The spectral projections are given by

$$\Pi_k f(x,y) = f \times \varphi_k(x,y) = \int_{\mathbb{R}^2} f(x-u, y-v)\varphi_k(u,v) e^{i(xv-yu)} dudv,$$

where $\varphi_k(x,y) = e^{-(x^2+y^2)} L_k(x^2+y^2)$, with

$$L_k(t) = \frac{1}{\pi^{\frac{1}{2}}} \frac{e^t}{k!} \frac{d^k}{dt^k}\left(e^{-t} t^k\right)$$

a Laguerre polynomial of degree k and type 0.

The oscillations in the kernel of Π_k,

$$\Pi_k f(z) = \int_{\mathbb{R}^2} f(x-u, y-v)\varphi_k(u,v)e^{i(xv-yu)}dudv,$$

are due to $e^{i(xv-yu)}$ and to φ_k.

We know that the zeroes and hence the oscillations of are contained in the annulus in $\frac{1}{\sqrt{k}} \leqslant (u^2+v^2)^{\frac{1}{2}} \leqslant 8\sqrt{k}$.

After some standard reductions one is reduced to prove

$$\left(\int_{\mathbb{R}^2} |\tilde{\Pi}_k f(x,y)|^4 dxdy\right)^{\frac{1}{4}} \lesssim (\log k)^{\frac{1}{4}} \|f\|_{L^4(\mathbb{R}^2)},$$

for all $\epsilon > 0$, where

$$\tilde{\Pi}_k f(x,y) = \int_{\mathbb{R}^2} f(x-u, y-v)\eta_k(u,v)\varphi_k(u,v)e^{i(xv-yu)}dudv,$$

here η_k is a smooth cutoff function supported in the annulus $\frac{1}{2\sqrt{k}} < r = (x^2+y^2)^{\frac{1}{2}} \leqslant 10\sqrt{k}$ (which contains the zeroes of φ_k).

The function φ_k behaves like a Bessel function in $\frac{1}{2\sqrt{k}} \leqslant r = (u^2+v^2)^{\frac{1}{2}} \leqslant 5\sqrt{k}$ and like an Airy function in $4\sqrt{k} \leqslant r \leqslant 10\sqrt{k}$. Correspondingly $\tilde{\Pi}_k f$ must be decomposed into the sum of two terms.

The term with kernel supported in $r \leqslant 5\sqrt{k}$ is given by

$$\int_{\mathbb{R}^2} f(x-u, y-v) e^{i(xv-yu)} \tilde{\eta}_k(u,v) \hat{\sigma}\left(\sqrt{k}(u^2+v^2)^{\frac{1}{2}}\right) dudv,$$

where $\hat{\sigma}$ is the Fourier transform of $d\sigma$ and $\tilde{\eta}_k$ is another cutoff function supported in $\frac{1}{2\sqrt{k}} < r \leqslant 5\sqrt{k}$.
It is well known that $\hat{\sigma}$ is a radial function and that

$$\hat{\sigma}(r) \approx \frac{1}{\sqrt{r}} \cos(\sqrt{k} r - \pi/4) \quad \text{for} \quad r > \frac{1}{2\sqrt{k}}.$$

Hence, the integral under investigation essentially becomes the sum of two terms. This is the first

$$k^{-\frac{1}{4}} \int_{\mathbb{R}^2} f(x-u, y-v) e^{i(xv-yu)+i\sqrt{k}(u^2+v^2)^{\frac{1}{2}}} \tilde{\eta}_k(u,v) dudv,$$

which after a dyadic decomposition may be treated using a classical result of L. Carleson and P. Sjölin.