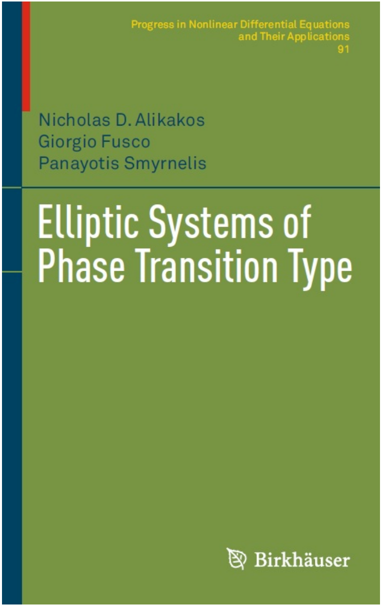
Elliptic systems of phase transition type

Panayotis Smyrnelis

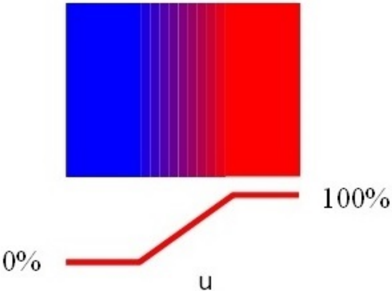
University of Athens

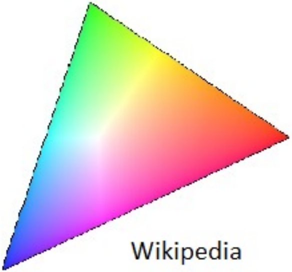
Our monograph



Introduction - Phase transition

• Phase transition models describe how different components which are called *phases* coexist in a substance. They also study the *interfaces* separating them.





• To determine the concentrations (= local mass fractions) of $m+1$ phases in a substance, we need a vector function u taking its values in $\mathbb{R}^m$ (since the mass fraction of one component is determined by the others).

Introduction - Phase transition

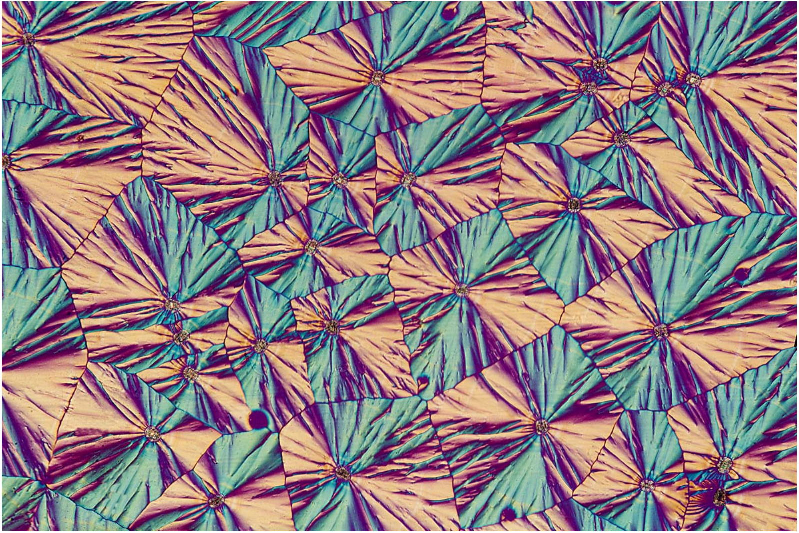


Figure: Vitamin C crystals. David Malin

The Allen-Cahn PDE

• A standard phase transition model is described by the Allen-Cahn PDE:

$$\Delta u = u^3 - u = W'(u), u : \mathbb{R}^n \to \mathbb{R}, W(u) = \frac{(u^2-1)^2}{4}, \qquad (1)$$

where the potential W has two zeros at 1 and -1.

• The zeros of the potential are called *phases*, and they are constant solutions of (1).

• The energy functional corresponding to (1) is

$$E(u,\Omega) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + W(u)\right].$$

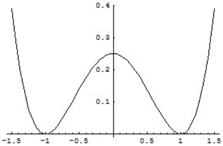
A function solves equation (1) iff it is a critical point of E:

$$\forall \xi \in C_c^1(\mathbb{R}^n) : \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} E(u+\lambda\xi, \operatorname{supp}\xi) = \int_{\mathbb{R}^n} (\nabla u \cdot \nabla \xi + W'(u)\xi) x = 0. \qquad (2)$$

The Allen-Cahn ODE: $u'' = u^3 - u$

- The solutions we are interested in are those *connecting* the phases, as the heteroclinic orbit:

$$e'' = e^3 - e = W'(e), \lim_{x \to \pm\infty} e(x) = \pm 1, e(x) = \tanh(x/\sqrt{2}). \quad (3)$$



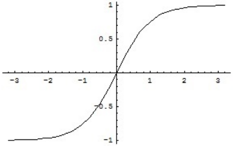
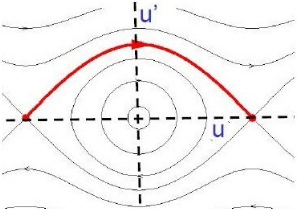


Figure: Phase plane of the ODE $u'' = u^3 - u$.



The Allen-Cahn PDE and minimal surfaces

In 1977, Modica and Mortola established that the Allen-Cahn equation is related to minimal surfaces.

Theorem: We consider $\Omega \subset \mathbb{R}^n$, and for small $\epsilon > 0$, the rescaled Allen-Cahn energy:

$$E_\epsilon(u,\Omega) = \int_\Omega \left[\frac{\epsilon}{2}|\nabla u|^2 + \frac{(1-u^2)^2}{4\epsilon}\right] = \epsilon^{n-1} E(\tilde{u}_\epsilon, \frac{\Omega}{\epsilon}), \tilde{u}_\epsilon(x) := u(\epsilon x). \tag{4}$$

Let also u_ϵ be a minimizer of $E_\epsilon(\cdot,\Omega)$ under the mass constraint:

$$\frac{1}{|\Omega|}\int_\Omega u(x)dx = m, \qquad m \in (-1,1) \tag{5}$$

Then, if $u_\epsilon \to u_0$ in $L^1(\Omega)$, we have that $u_0(x) \in \{\pm 1\}$ for a.e. $x \in \Omega$, and the boundary in Ω of the set $A := \{x \in \Omega : \ u_0(x) = 1\}$ has minimal perimeter among all subsets $B \subset \Omega$ such that $|B| = |A| = \frac{m+1}{2}|\Omega|$ ($|\cdot|$=n-dimensional Lebesgue measure).

The Allen-Cahn PDE and minimal surfaces

• In view of these results, De Giorgi stated in 1978 a famous conjecture for the Allen-Cahn PDE which is the analog of the Bernstein conjecture for minimal graphs.

• De Giorgi's conjecture: Let $n \leq 8$ and $u : \mathbb{R}^n \to \mathbb{R}$ a solution to the Allen-Cahn PDE such that $\frac{\partial u}{\partial x_n} > 0$. Then u is one dimensional in the sense that $u(x_1, \ldots, x_n) = e(x \cdot \nu + a)$ holds for a unit vector $\nu \in \mathbb{R}^n$ and some $a \in \mathbb{R}$, where $e(t) = \tanh(t/\sqrt{2})$.

• Bernstein conjecture: Up to dimension 7, the *minimal* graphs of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ are hyperplanes. Minimal means that any local perturbation of the graph has greater or equal area.

• Comments: 1) $\frac{\partial u}{\partial x_n} > 0 \Rightarrow$ the level sets of u are graphs of functions defined in $\mathbb{R}^{n-1}$ (with $n - 1 \leq 7$). 2) If these graphs are minimal, it follows from Bernstein conjecture that the level sets of u are hyperplanes, and u is $1D$.

The Allen-Cahn PDE and minimal surfaces

- The conjecture of Bernstein was solved by Bernstein, Fleming, De Giorgi, Almgren and Simons. In dimension 8 there is a counterexample of Bombieri-De Giorgi-Giusti.
- On the other hand, the De Giorgi conjecture
  - was solved in dimension $n = 2, 3$ (Ghoussoub+Gui, 1998, and Ambrosio+Cabré, 2000).
  - was solved in dimensions $4 \leq n \leq 8$ (Savin, 2009) under the additional assumption that

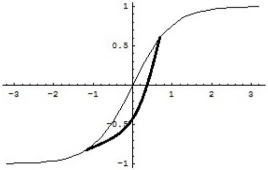
$$\lim_{x_n \to \pm\infty} u(x_1, \ldots, x_{n-1}, x_n) = \pm 1.$$

  - In dimension $n = 9$, Del Pino, Kowalczyk and Wei constructed a monotone solution which is not $1D$ (2011).

The Allen-Cahn PDE and minimal surfaces

• A problem related to the De Giorgi conjecture is the classification of minimal solutions of the Allen-Cahn PDE. By definition, a solution u is *minimal* if

$$E_{\operatorname{supp}\phi}(u) \le E_{\operatorname{supp}\phi}(u+\phi),\ \forall \phi \in C_c^1(\mathbb{R}^n;\mathbb{R}). \tag{6}$$



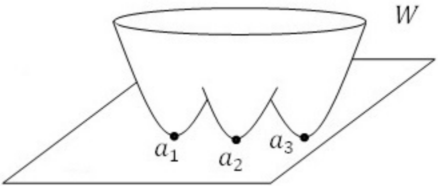
• Savin (Ann. Math., 2009) proved that when $n \le 7$, the only minimal solutions $u : \mathbb{R}^n \to \mathbb{R}$ of (1) are the constants ± 1, and (up to change of coordinates) the heteroclinic orbit e.

Elliptic systems of phase transition type

- In our book, we study the vector Allen-Cahn equation:

$$\Delta u(x) = \nabla W(u(x)), x \in \mathbb{R}^n, u : \mathbb{R}^n \to \mathbb{R}^m, W \in C^2(\mathbb{R}^m; [0, \infty)), \tag{7}$$

where the potential $W \geq 0$ has a finite number of zeros: $\{a_1, a_2, \ldots, a_N\}$. Theses zeros are called *phases*.



- We also assume that

$$\liminf_{|u|\to\infty} W(u) > 0. \tag{8}$$

Elliptic systems of phase transition type

- I will focus on three basic problems:
  - $1D$ solutions and the heteroclinic connection problem in the vector case (Chapter 2),
  - The heteroclinic double layers problem (Chapters 8 and 9),
  - The triple junction problem (Chapter 6). The triple junction is a solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ modeling the junction of three phases.

$1D$ solutions - heteroclinics

• In Chap. 2, we focus on the $1D$ problem, by considering the ODE system (Newton's equation):

$$u''(x) = \nabla W(u(x)), x \in \mathbb{R}, u : \mathbb{R} \to \mathbb{R}^m, W \in C^2(\mathbb{R}^m; [0, \infty)). \tag{9}$$

• Problem: Existence of heteroclinic orbits in the vector case. Main contributions: Rabinowitz 1989, Sternberg 1988-1991, Alikakos-Fusco 2008, Sourdis 2016, Antonopoulos-Smyrnelis 2016, Sternberg-Zuniga 2016, Fusco-Gronchi-Novaga 2017, Monteil-Santambrogio 2018, Alessio-Montecchiari-Zuniga 2019, Smyrnelis 2020.

$1D$ solutions - heteroclinics

• <u>Theorem</u> (simplest version): Let $W \in C^2(\mathbb{R}^m; [0,\infty))$, such that $\{W = 0\} = \{a^-, a^+\}$, and (8) holds. Then, there exists a minimizing heteroclinic orbit connecting a^- to a^+:

$$e \in C^2(\mathbb{R};\mathbb{R}^m),\ e''(x) = \nabla W(e(x)),\ \lim_{x\to\pm\infty} e(x) = a^{\pm}. \qquad (10)$$

• In addition the heteroclinic e is by construction a minimizer of the energy $E_{\mathbb{R}}(u) = \int_{\mathbb{R}} \left[\frac{1}{2}|u'|^2 + W(u)\right]$ in the class

$$K = \{u \in W^{1,2}_{loc}(\mathbb{R};\mathbb{R}^m) : \lim_{x\to\pm\infty} u(x) = a^{\pm}\}.$$

Thus, it solves the Euler-Lagrange equation:

$$\int_{\mathbb{R}} [e' \cdot \xi' + \nabla W(e) \cdot \xi] = 0, \forall \xi \in C^1_0(\mathbb{R};\mathbb{R}^m). \qquad (11)$$

$1D$ solutions - heteroclinics

- Extensions of the Theorem are available:
  - When the zero set of W is partitioned into two compact subsets, there exists a heteroclinic connecting these two subsets.
  - For lower semicontinuous potentials.
  - For homoclinic and periodic orbits.
  - For potentials defined in Hilbert spaces.

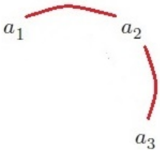
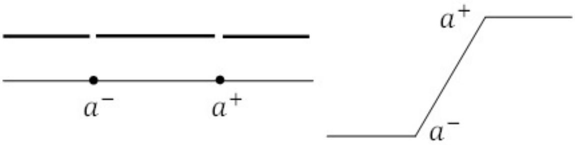


Figure: For a triple well potential:
$\{W = 0\} = \{a_1, a_2, a_3\} = \{a_1\} \cup \{a_2, a_3\} = \{a_3\} \cup \{a_1, a_2\}$, at least two heteroclinics exist. The existence of the third heteroclinic is not always ensured.

$1D$ solutions - heteroclinics

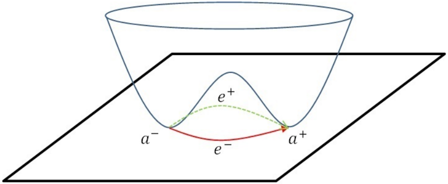
• If $W \notin C^1$, then the heteroclinic orbit does not solve the Euler-Lagrange equation, but it is defined as a minimizer.

Figure: The potential $W(u) = \chi_{\mathbb{R}\setminus\{a^-,a^+\}}(u)$ (with $\{W = 0\} = \{a^-, a^+\}$) and the corresponding heteroclinic orbit.



Heteroclinic double layers

- First constructions: Alama-Bronsard-Gui (1997, under symmetry assumptions), Schatzman (2002).
- We suppose that the potential $W : \mathbb{R}^2 \to [0, \infty)$ has two zeros $a^{\pm}$ and that the ODE system $e'' = \nabla W(e)$ has *exactly* (up to translations) two minimizing heteroclinics $e^{\pm}$.



- A solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ of $\Delta u(t,x) = \nabla W(u(t,x))$, satisfying the B.C.:

$$\lim_{x \to \pm\infty} u(t,x) = a^{\pm}. \tag{12a}$$

$$\lim_{t \to \pm\infty} u(t,x) = e^{\pm}(x), \tag{12b}$$

is called *heteroclinic double layers*.

Heteroclinic double layers

• It is an important result establishing also the existence of $2D$ *minimal solutions* in the vector case. Indeed, the heteroclinic double layers are minimal by construction, since they are obtained by minimization, starting from minimizing heteroclinics.

• I will present a construction (Smyrnelis, 2020) where the heteroclinic double layers $u(t,x)$ are derived from a heteroclinic orbit $U(t) : x \mapsto u(t,x)$ taking its values in a Hilbert space of functions. This Hilbert space is defined by the B.C. (12a). Then, the initial P.D.E. is reduced to an O.D.E. problem. It is a robust method that can be applied to a large class of PDEs.

• Other approaches were also proposed recently by Alessio (2013), Alessio-Montecchiari (2017), Fusco (2017), Monteil-Santambrogio (2020).

The triple junction

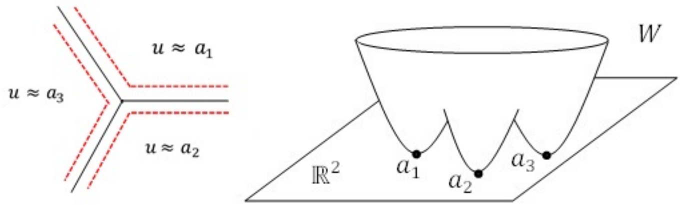
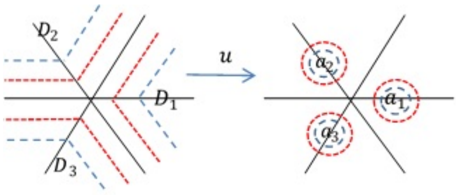


Figure: The triple junction $u : \mathbb{R}^2 \to \mathbb{R}^2$, $\Delta u(x) = \nabla W(u(x))$, and its interface (called triod), for a triple well potential $W : \mathbb{R}^2 \to [0, \infty)$.

• Without symmetry assumptions, this problem has been studied very recently, by Sandier-Sternberg, Alikakos-Geng, and Fusco. However, the desired asymptotic property of the solution described in the picture above, could not be established.

The triple junction for symmetric triple well potentials

• The first construction of a triple junction for symmetric potentials is due to Bronsard-Gui-Schatzman (1996).



• The potential $W : \mathbb{R}^2 \to [0, \infty)$ is invariant by the group of symmetry of the equilateral triangle (3 rotations and 3 reflections). The group of symmetry partitions the plane into six fundamental domains (60 degree sectors).

• The solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ is symmetric: two points x and y which are symmetric by a reflection line, have symmetric images $u(x)$ and $u(y)$ by the same reflection line.

• $x \in D_i, \; d(x, \partial D_i) \to \infty \implies u(x) \to a_i$, and $u(D_i) \subset D_i$.

Symmetric solutions

• Symmetic solutions have been constructed for general reflection groups: Alikakos - Fusco 2011, Alikakos - Smyrnelis 2012, Bates - Fusco - Smyrnelis 2017.

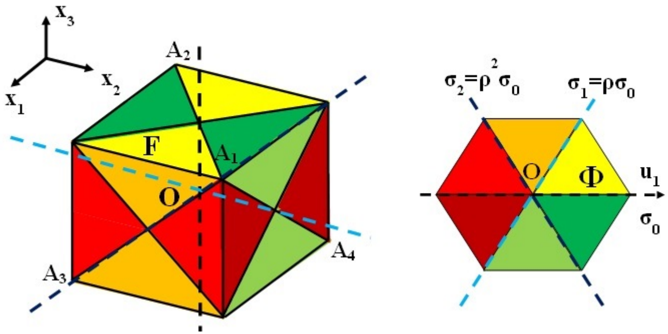
• We consider a finite or discrete reflection group G acting on $\mathbb{R}^n$, and a finite reflection group Γ acting on$\mathbb{R}^m$. We also assume the existence of a homomorphism $f : G \to \Gamma$.

• Then, a map $u : \mathbb{R}^n \to \mathbb{R}^m$ is called f-equivariant when

$$u(gx) = f(g)u(x), \forall g \in G, \forall x \in \mathbb{R}^n. \tag{13}$$

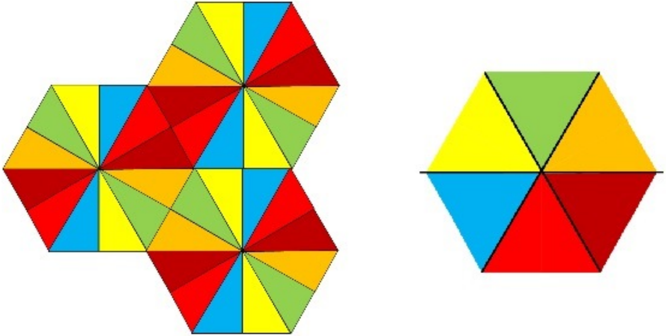
Symmetric solutions

Figure: G = symmetry group of the regular tetrahedron, Γ = symmetry group of the equilateral triangle. Every fundamental domain is mapped into the corresponding fundamental domain with the same color in the range.



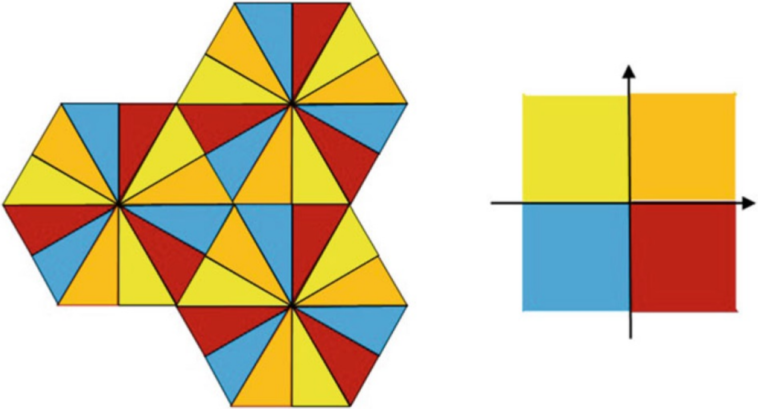
Symmetric solutions: lattices

Figure: G = discrete reflection group of the regular hexagon, $\Gamma = D_3$ reflection group of the equilateral triangle.



Symmetric solutions: lattices

Figure: G = discrete reflection group of the regular hexagon, $\Gamma = D_2$ reflection group of a line segment.

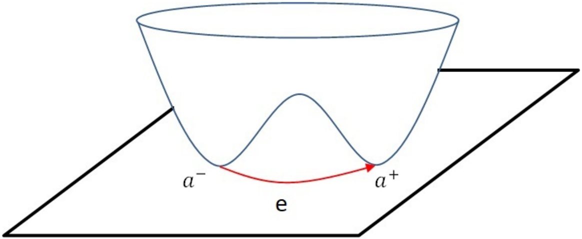


- Due to the variety of choices for the groups G and Γ, and the homomorphism f, a large class of symmetric solutions is obtained.

Existence of heteroclinics

Theorem: Let $W \in C^2(\mathbb{R}^m)$, $W \geq 0$, $\{W = 0\} = \{a^-, a^+\}$. We also assume that $\liminf_{|u|\to\infty} W(u) > 0$. Then, there exists a heteroclinic orbit connecting a^- to a^+:

$$e \in C^2(\mathbb{R}; \mathbb{R}^m), \; e''(x) = \nabla W(e(x)), \; \lim_{x\to\pm\infty} e(x) = a^\pm. \qquad (14)$$



Proof (Existence of heteroclinics)

- The idea is to minimize the energy

$$E_{\mathbb{R}}(u) = \int_{\mathbb{R}} \left[\frac{1}{2}|u'|^2 + W(u)\right]$$

in the class $K = \{u \in W^{1,2}_{loc}(\mathbb{R};\mathbb{R}^m) : \lim_{x\to\pm\infty} u(x) = a^{\pm}\}$.

Step 1: $\exists u_0 \in K$ such that $E_0 := E_{\mathbb{R}}(u_0) < \infty$. Let

$$K_b = \{u \in K : E_{\mathbb{R}}(u) \leq E_0\},$$

then $\inf_K E_{\mathbb{R}} = \inf_{K_b} E_{\mathbb{R}}$.

Proof (Existence of heteroclinics)

Step 2: The maps $u \in K_b$ are equicontinuous.
Indeed, for $u \in K_b$, and $-\infty < x < y < +\infty$, we have

$$|u(y) - u(x)| \leq \int_x^y |\dot{u}(t)| \mathrm{d}t \leq \left(\int_x^y |\dot{u}(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}} |y - x|^{1/2}$$
$$\leq \sqrt{2E_{\mathbb{R}}(u)} |y - x|^{1/2}$$
$$\leq \sqrt{2E_0} |y - x|^{1/2}.$$

Proof (Existence of heteroclinics)

Step 3: Let $q = \frac{|a^+ - a^-|}{4}$, we estimate the energy necessary for a map $u \in K_b$ to reach a point at a distance q from the phases $a^\pm$:

(i) $|u(t) - a^-| = q \Rightarrow E_{(-\infty,t]}(u) \geq \frac{q\sqrt{w_q^-}}{\sqrt{2}}$,

(ii) $|u(t) - a^+| = q \Rightarrow E_{[t,+\infty)}(u) \geq \frac{q\sqrt{w_q^+}}{\sqrt{2}}$,

for the positive constants $w_q^\pm := \min\{W(u) : \frac{q}{2} \leq |u - a^\pm| \leq q\}$.

- To prove (i), we notice that there exist $t_1 < t_2 \leq t$ such that
  - $|u(t_1) - a^-| = \frac{q}{2} \quad (t_1 = \max\{s < t : |u(s) - a^-| = \frac{q}{2}\})$,
  - $|u(t_2) - a^-| = q \quad (t_2 = \min\{s > t_1 : |u(s) - a^-| = q\})$,
  - $\forall s \in [t_1, t_2]$: $\frac{q}{2} \leq |u(s) - a^-| \leq q$.
- As a consequence,

$$E_{(-\infty,t]}(u) \geq E_{[t_1,t_2]}(u) = \int_{t_1}^{t_2} [\frac{1}{2}|u'|^2 + W(u)] \geq \int_{t_1}^{t_2} \sqrt{2W(u)}|u'|$$

$$\geq \sqrt{2w_q^-} \int_{t_1}^{t_2} |u'| \geq \sqrt{2w_q^-}|u(t_2) - u(t_1)| \geq \sqrt{2w_q^-}\,\frac{q}{2},$$

(since $A^2 + B^2 \geq 2AB$, $A = |u'|/\sqrt{2}$, $B = \sqrt{W(u)}$).

Proof (Existence of heteroclinics)

Step 4: • We choose a constant $\eta \in (0, q)$ such that

- $\frac{\eta^2}{2} + \max\{W(u) : |u - a^-| \leq \eta\} \leq \frac{q\sqrt{w_q^-}}{\sqrt{2}}$,
- $\frac{\eta^2}{2} + \max\{W(u) : |u - a^+| \leq \eta\} \leq \frac{q\sqrt{w_q^+}}{\sqrt{2}}$.

• Next, for every $u \in K_b$, we define the times:

- $\lambda_u^- = \max\{s \in \mathbb{R} : |u(s) - a^-| = \eta\}$,
- $\lambda_u^+ = \min\{s > \lambda^- : |u(s) - a^+| = \eta\}$.

• By definition of $\lambda_u^\pm$, we have

$$\begin{aligned} t \in [\lambda_u^-, \lambda_u^+] &\Rightarrow |u(t) - a^-| \geq \eta \text{ and } |u(t) - a^-| \geq \eta \\ &\Rightarrow W(u(t)) \geq w_0 > 0 \text{ (constant)}, \\ &\Rightarrow w_0(\lambda_u^+ - \lambda_u^-) \leq \int_{\lambda_u^-}^{\lambda_u^+} W(u(t))\mathrm{d}t \leq E_{\mathbb{R}}(u) \leq E_0. \end{aligned}$$

• Conclusion: $\forall u \in K_b$: $(\lambda_u^+ - \lambda_u^-) \leq \Lambda$ (constant).

Proof (Existence of heteroclinics)

Step 5: Replacement lemma.

- Let $u \in K_b$, then we define a competitor $\tilde{u}$ as follows:
  - $\tilde{u} = u$ on the interval $[\lambda_u^-, \lambda_u^+]$.
  - If $|u(t) - a^-| = q$ for some $t < \lambda_u^-$, then

$$\tilde{u}(x) = \begin{cases} a^- + (u(\lambda_u^-) - a^-)(x - \lambda_u^- + 1) & \text{for } x \in [\lambda_u^- - 1, \lambda_u^-] \\ a^- & \text{for } x \in (-\infty, \lambda_u^- - 1]. \end{cases} \tag{15}$$

    Otherwise, $\tilde{u} = u$ on the interval $(-\infty, \lambda_u^-]$.
  - Similarly, if $|u(t) - a^+| = q$ for some $t > \lambda_u^+$, then

$$\tilde{u}(x) = \begin{cases} u(\lambda_u^+) + (a^+ - u(\lambda_u^+))(x - \lambda_u^+) & \text{for } x \in [\lambda_u^+, \lambda_u^+ + 1] \\ a^+ & \text{for } x \in [\lambda_u^+, +\infty). \end{cases} \tag{16}$$

    Otherwise, $\tilde{u} = u$ on the interval $[\lambda_u^+, +\infty)$.

Proof (Existence of heteroclinics)

Step 6: Properties of $\tilde{u}$.

- By construction $\tilde{u} \in K$. In addition,

$$\forall x \leq \lambda_u^- : \ |\tilde{u}(x) - a^-| < q \text{ while } \forall x \geq \lambda_u^+ : \ |\tilde{u}(x) - a^+| < q.$$

- $E_{\mathbb{R}}(\tilde{u}) \leq E_{\mathbb{R}}(u) \Rightarrow \tilde{u} \in K_b$.

Indeed, if for instance $|u(t) - a^-| = q$ for some $t < \lambda_u^-$, then

$$\tilde{u}(x) = \begin{cases} a^- + (u(\lambda_u^-) - a^-)(x - \lambda_u^- + 1) & \text{for } x \in [\lambda_u^- - 1, \lambda_u^-] \\ a^- & \text{for } x \in (-\infty, \lambda_u^- - 1]. \end{cases}$$

Thus, in view of Step 4 (def. of λ_u^- and η) and Step 3, we have

$$E_{(-\infty,\lambda_u^-]}(\tilde{u}) = \int_{\lambda_u^- - 1}^{\lambda_u^-} \Big[\frac{1}{2}|\tilde{u}'|^2 + W(\tilde{u})\Big] \leq \frac{1}{2}|u(\lambda_u^-) - a^-|^2 + \sup_{B(a^-,\eta)} W$$

$$\leq \frac{\eta^2}{2} + \sup_{B(a^-,\eta)} W \leq \frac{q\sqrt{w_q^-}}{\sqrt{2}} \leq E_{(-\infty,t]}(u) \leq E_{(-\infty,\lambda_u^-]}(u).$$

Proof (Existence of heteroclinics)

Step 7: Minimizing sequence.

• Let $u_n \in K_b$ be such that $\lim_{n\to\infty} E_{\mathbb{R}}(u_n) = \inf_{K_b} E_{\mathbb{R}}$. For every n, we denote by $\lambda_n^{\pm}$ the times associated in Step 4 to the map u_n.

• We define the sequence:

$$v_n(x) = \tilde{u}_n(x - \lambda_n^-),$$

which is also minimizing by the translation invariance of $E_{\mathbb{R}}$:

$$E_{\mathbb{R}}(v_n) = E_{\mathbb{R}}(\tilde{u}_n) \leq E_{\mathbb{R}}(u_n) \to \inf_{K_b} E_{\mathbb{R}} \text{ as } n \to \infty.$$

• In addition

- $\forall n$, $\forall x \leq 0$: $|v_n(x) - a^-| \leq q$ (cf. Step 6),
- $\forall n$, $\forall x \geq \Lambda$: $|v_n(x) - a^+| \leq q$ (cf. Steps 4 and 6),
- $\forall n$, $\forall x \in [0, \Lambda]$: $|v_n(x) - v_n(0)| \leq \sqrt{2E_0}|x - 0|^{1/2} \leq \sqrt{2E_0\Lambda}$ (cf. Step 2).

• Conclusion: the minimizing sequence v_n is equibounded and equicontinuous (cf. Step 2).

Proof (Existence of heteroclinics)

Step 8: Convergence of the minimizing sequence v_n.
• By the theorem of Ascoli, up to subsequence, v_n converges in $C_{loc}(\mathbb{R};\mathbb{R}^m)$ to a limit $e \in C(\mathbb{R};\mathbb{R}^m)$.
• On the other hand, since $\|v_n'\|^2_{L^2(\mathbb{R};\mathbb{R}^m)} \leq 2E(v_n) \leq 2E_0$, we have that up to subsequence, v_n' converges weakly in $L^2(\mathbb{R};\mathbb{R}^m)$ to a limit f. It is easy to see that $e \in W^{1,2}_{loc}(\mathbb{R};\mathbb{R}^m)$, and $e' = f$.
• Next, by Fatou's lemma and the weak lower semicontinuity of the L^2 norm, we deduce that

- $\int_{\mathbb{R}} W(e) \leq \liminf_{n\to\infty} \int_{\mathbb{R}} W(v_n)$,
- $\int_{\mathbb{R}} |e'|^2 \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |v_n'|^2$,
- $E_{\mathbb{R}}(e) \leq \liminf_{n\to\infty} E_{\mathbb{R}}(v_n) = \inf_{K_b} E_{\mathbb{R}}$.

Proof (Existence of heteroclinics)

Step 9: We prove that $e \in K_b$ and $E_{\mathbb{R}}(e) = \min_{K_b} E_{\mathbb{R}}$.
• We have to check that $\lim_{x \to \pm\infty} e(x) = a^{\pm}$. To see this we notice that

- e is bounded (cf. Steps 7 and 8) and uniformly continuous (because $e' \in L^2(\mathbb{R}; \mathbb{R}^m)$) cf. Step 2).
- Thus, $W(e)$ is also uniformly continuous, and since $\int_{\mathbb{R}} W(e) < \infty$, we have $\lim_{x \to \pm\infty} W(e(x)) = 0$.

• Finally, since $|e(x) - a^-| < q$ for $x \leq 0$, while $|e(x) - a^+| < q$ for $x \geq \Lambda$ (cf. Steps 7 and 8), we deduce that $e \in K_b$ and $E_{\mathbb{R}}(e) = \min_{K_b} E_{\mathbb{R}} = \min_K E_{\mathbb{R}}$.

Proof (Existence of heteroclinics)

Step 10: We prove that $e \in C^2(\mathbb{R}; \mathbb{R}^m)$ solves the ODE system

$$u'' = \nabla W(u).$$

• We first notice that for every $\xi \in C_c^\infty(\mathbb{R}; \mathbb{R}^m)$, we have that $e + \xi \in K$ and $E_{\mathbb{R}}(e + \xi) < \infty$.
• Thus, $E_{\mathbb{R}}(e) = \min_K E_{\mathbb{R}}$ implies that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} E_{\mathbb{R}}(e + \lambda\xi) = \int_{\mathbb{R}} (e' \cdot \xi' + \nabla W'(e) \cdot \xi) = 0.$$

• It follows that $e'' = \nabla W(e) \in C(\mathbb{R}; \mathbb{R}^m)$ in the distributional sense.
• By integrating twice, we conclude that $e \in C^2(\mathbb{R}; \mathbb{R}^m)$ solves the ODE system.

Remarks

- Every heteroclinic orbit

$$e \in C^2(\mathbb{R};\mathbb{R}^m),\ e''(x) = \nabla W(e(x)),\ \lim_{x\to\pm\infty} e(x) = a^{\pm}.$$

satisfies the equipartition relation $\frac{1}{2}|e'(x)|^2 = W(e(x))$, $\forall x \in \mathbb{R}$. Indeed, by integrating the equation

$$e''(x)\cdot e'(x) = \nabla W(e(x))\cdot e'(x),$$

we obtain that

$$\frac{1}{2}|e'(x)|^2 = W(e(x)) + Const.$$

In addition,

$$\lim_{x\to\pm\infty} e''(x) = 0 \text{ and } \lim_{x\to\pm\infty}(e(x) - a^{\pm}) = 0 \Rightarrow \lim_{x\to\pm\infty} e'(x) = 0,$$

$$\text{and } \lim_{x\to\pm\infty} W(e(x)) = 0.$$

Thus, $Const. = 0$.

Remarks

• It is sufficient to assume that $W \in C^1(\mathbb{R}^m)$ for the theorem to hold.
• If $W \in C^2(\mathbb{R}^m)$, then the heteroclinics e do not attain the phases $a^{\pm}$:

$$\forall x \in \mathbb{R} : \; e(x) \neq a^- \text{ and } e(x) \neq a^+.$$

Indeed, if we assume by contradiction that

$$e(x) = a^- \text{ or } a^+, \text{ for some } x \in \mathbb{R},$$

then by the equipartition relation we have

$$e'(x) = 0.$$

Thus the uniqueness result for O.D.E. implies that $e \equiv a^-$ or a^+, which is a contradiction.

Remarks

• If the Hessian of W at $a^{\pm}$ is positive definite (i.e. the global minima $a^{\pm}$ are *nondegenerate*), then every heteroclinic orbit converges exponentially fast to $a^{\pm}$ at $\pm\infty$.

• When $m \geq 2$, there are explicit examples of W having at least two distinct minimizing heteroclinics. When $m = 1$ the heteroclinic orbit is unique.

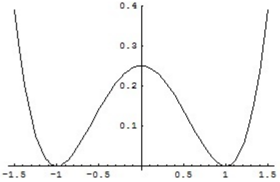
Connecting orbits in the scalar case - Heteroclinics

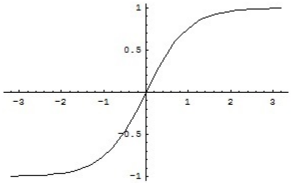
We consider the scalar ODE

$$u'' = W'(u),\ u : \mathbb{R} \to \mathbb{R},\ W \in C^2(\mathbb{R}), \tag{17}$$

for a potential such that $W > 0$ in the interval (a^-, a^+) and $W(a^\pm) = 0$. Depending on the sign of W' at the endpoints $a^\pm$, we obtain different kinds of orbits.

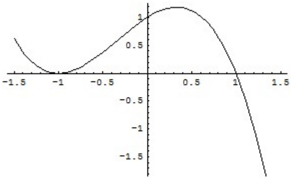
1) When $W'(a^\pm) = 0$, there exits a solution $u : \mathbb{R} \to (a^-, a^+)$ to (17) such that $\lim_{x\to\pm\infty} u(x) = a^\pm$. It is the heteroclinic connection which is unique up to translations.

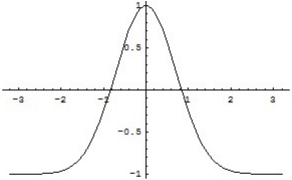




Connecting orbits in the scalar case - Homoclinics

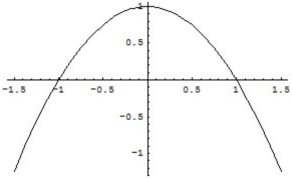
2) When $W'(a^-) = 0$ and $W'(a^+) \neq 0$, there exits a unique even solution $u : \mathbb{R} \to (a^-, a^+]$ to (17) such that $\lim_{x \to \pm\infty} u(x) = a^-$ and $u(0) = a^+$. It is the homoclinic connection.

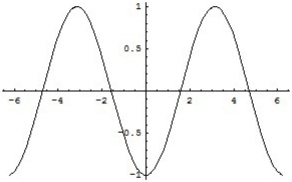




Connecting orbits in the scalar case - Periodic orbits

3) When $W'(a^-) \neq 0$ and $W'(a^+) \neq 0$, there exits a periodic solution $u : \mathbb{R} \to [a^-, a^+]$ to (17) such that $u(0) = a^-$, $u(T/2) = a^+$ and $\forall x \in \mathbb{R}$: $u(x+T) = u(x)$, $u(x + T/2) = u(-x + T/2)$, for some $T > 0$.





Connecting orbits in the vector case - Heteroclinics

<u>Theorem</u> (Antonopoulos-Sm, 2016): We consider a potential $W \in C^2(\mathbb{R}^m)$ and a connected component Ω of the set $\{W > 0\}$. We assume that

- H_1 $\partial\Omega$ is partitioned into two compact subsets A^- and A^+.
- H_2 $\liminf_{u\in\Omega,\, |u|\to+\infty} W(u) > 0$, if Ω is not bounded,

Next, we impose a uniform condition on $A^\pm$.

1) If $\nabla W(u) = 0$ holds on A^- and A^+, then there exists a heteroclinic orbit e:

$$e :\in C^2(\mathbb{R};\Omega),\ \ e''(x) = \nabla W(e(x)),\ \ \lim_{x\to\pm\infty} d(e(x), A^\pm) = 0. \tag{18}$$

Connecting orbits in the vector case - Heteroclinics

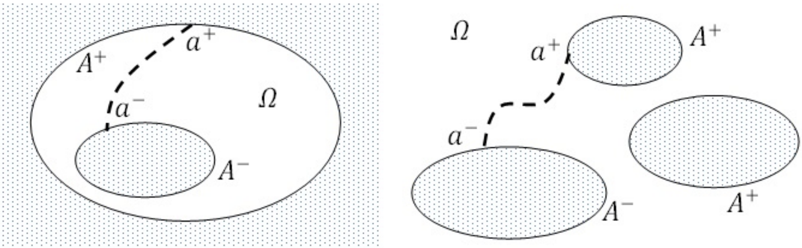


Figure: The sets Ω, $A^{\pm}$, and the orbit of the heteroclinic e. For the sake of simplicity, we assumed that the limits of e exist at $\pm\infty$.

- To ensure the existence of the limits of e at $\pm\infty$, a nondegeneracy condition is needed:

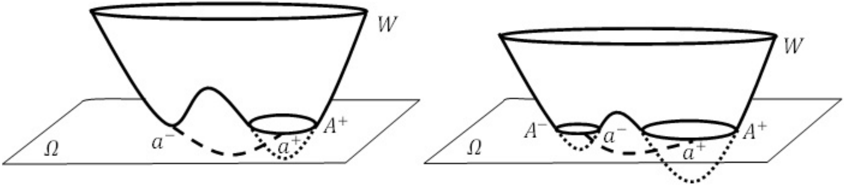
$$\liminf_{d(u,A^{\pm})\to 0} \frac{W(u)}{(d(u,A^{\pm}))^2} > 0. \tag{19}$$

Connecting orbits in the vector case - Homoclinics

2) If $\nabla W(u) = 0$ holds on A^- and $\nabla W(u) \neq 0$ holds on A^+, then there exists an *even* homoclinic orbit e,

$$e \in C^2(\mathbb{R}; \overline{\Omega}),\ e''(x) = \nabla W(e(x)),\ \lim_{x \to \pm\infty} d(e(x), A^-) = 0,$$
$$e(x) \in A^+ \Leftrightarrow x = 0,\quad e(x) \in \Omega,\ \forall x \neq 0. \tag{20}$$

Figure: On the left: a homoclinic orbit in the case where $A^- = \{a^-\}$. On the right: a periodic orbit.



Connecting orbits in the vector case - Periodic orbits

3) If $\nabla W(u) \neq 0$ holds on A^- and A^+, then there exists a periodic solution $e \in C^2(\mathbb{R}; \overline{\Omega})$ of period T, connecting A^- and A^+:

$$e\Big(x + T\Big) = e(x),$$

$$e\Big(x + \frac{T}{2}\Big) = e\Big(-x + \frac{T}{2}\Big)$$

$$u(x) \in A^- \Leftrightarrow x \in T\mathbb{Z},$$

$$e(x) \in A^+ \Leftrightarrow x + \frac{T}{2} \in T\mathbb{Z}.$$

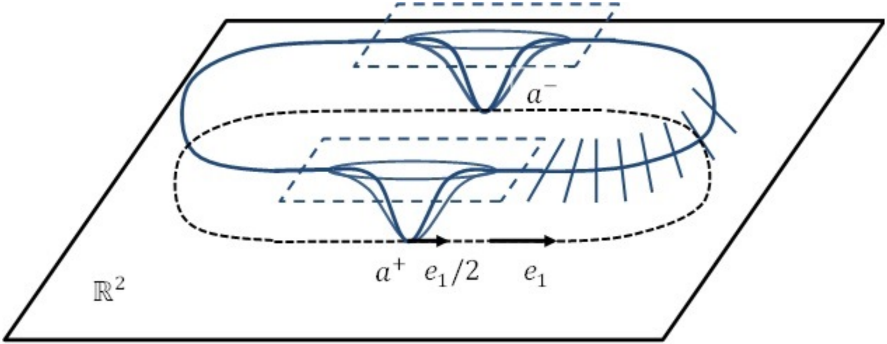
A new kind of periodic orbit in the vector case connecting two critical points of a potential

One can construct[1] a double well potential $W \in C^2(\mathbb{R}^2)$, and a solution $u \in C^\infty(\mathbb{R};\mathbb{R}^2)$ of the O.D.E. $u'' = \nabla W(u)$ with the following properties:

- $W(a^\pm) = 0$ and $W(u) > 0$ for $u \neq a^\pm$,
- $D^2 W(a^\pm)$ is a positive definite matrix,
- $\forall x \in \mathbb{R},\ u(x+T) = u(x)$ for some $T > 0$ (i.e. u is periodic),
- $u(0) = a^+$ and $u(T/2) = a^-$ (u connects the minima of W),
- the derivative of u at $x = 0$ or $x = T/2$ does not vanish.

[1] P. Smyrnelis, Gradient estimates for semilinear elliptic systems and other related results. Proceedings of the Royal Society of Edinburgh, Section A. (2014)

A new kind of periodic orbit in the vector case connecting two critical points of a potential



Heteroclinic orbits in a Hilbert space $\mathcal{H}$

- Now we consider $\mathcal{W} : \mathcal{H} \to [0, +\infty]$, a weakly lower semicontinuous function such that

$$\mathcal{W} \text{ has } \textit{exactly } 2 \text{ zeros } e^- \text{ and } e^+, \text{ and } \liminf_{\|v\|\to\infty} \mathcal{W}(v) > 0. \quad (21)$$

- Let $\mathcal{K} = \{V \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; \mathcal{H}) : V(t) \rightharpoonup e^{\pm}, \text{ as } t \to \pm\infty\}$ and

$$\mathcal{J}_{\mathbb{R}}(V) := \int_{\mathbb{R}} \Big[\frac{1}{2}\|V'(t)\|^2 + \mathcal{W}(V(t))\Big]\mathrm{d}t. \quad (22)$$

Theorem (Sm. 2019): Assume that $\mathcal{W}$ satisfies (21) and $\inf_{\mathcal{K}} \mathcal{J}_{\mathbb{R}} < +\infty$, then $\mathcal{J}_{\mathbb{R}}$ admits a minimizer $U \in \mathcal{K}$ i.e. $\mathcal{J}_{\mathbb{R}}(U) = \min_{V\in\mathcal{K}} \mathcal{J}_{\mathbb{R}}(V)$. In addition, if $\mathcal{W} \in C^1(\mathcal{H}; \mathbb{R})$, then $U \in C^2(\mathbb{R}; \mathcal{H})$ is a classical solution of

$$U''(t) = \nabla\mathcal{W}(U(t)), \forall t \in \mathbb{R}, \quad (23)$$

where $\nabla\mathcal{W}(u)$ is the element of $\mathcal{H}$ corresponding to $D\mathcal{W}(u) \in \mathcal{H}'$ by identifying $\mathcal{H}$ with $\mathcal{H}'$.

Applications to P.D.E.

• To apply the previous theorem to P.D.E., the idea is to view a solution $\mathbb{R}^2 \ni (t, x) \mapsto u(t, x)$ of a P.D.E., as a map $t \mapsto [U(t) : x \mapsto [U(t)](x) := u(t, x)]$ taking its values in a space of functions $\mathcal{H}$.

• It is easier to define the space of functions $\mathcal{H}$ when the B. C. of the problem are uniform in t.

• Next, one has to reduce the initial P.D.E. to an O.D.E. problem for U.

• This approach is classical for evolution equation, but it seems also promising in the context of nonlinear elliptic PDEs.

Schatzman's theorem

• Let $W \in C^2(\mathbb{R}^m, \mathbb{R})$, $W \geq 0$, vanishing at $\{a^+, a^-\}$, which are nondegenerate zeros and satisfying the asymptotic condition:

$$\exists \rho > 0 \text{ such that } W(su) \geq W(u) \text{ for } s \geq 1 \text{ and } |u| = \rho. \tag{24}$$

• We also assume that the O.D.E. system (9) has (up to translations) exactly two minimizing heteroclinics $e^{\pm}$ which are nondegenerate[2].

• Then, the elliptic system

$$\Delta u(t,x) = \nabla W(u(t,x)),\ u : \mathbb{R}^2 \to \mathbb{R}^m\ (m \geq 2),\ (t,x) \in \mathbb{R}^2, \tag{25}$$

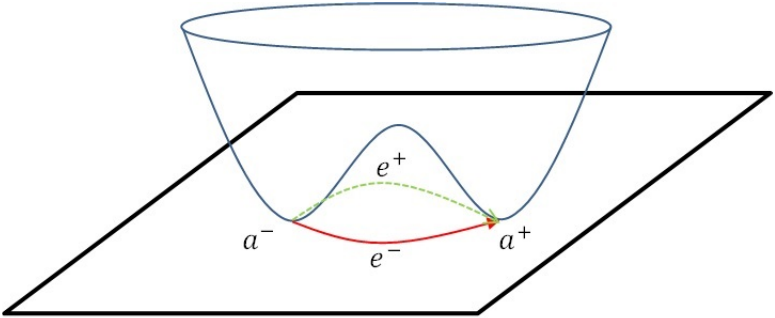
has a classical minimal solution satisfying the boundary conditions:

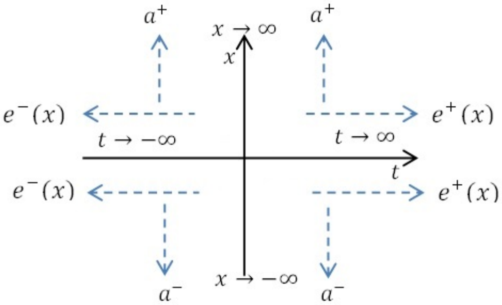
$$\lim_{x \to \pm\infty} u(t,x) = a^{\pm}, \forall t \in \mathbb{R}. \tag{26a}$$

$$\lim_{t \to \pm\infty} u(t,x) = e^{\pm}(x - m^{\pm}), \text{ for some constants } m^{\pm} \in \mathbb{R}. \tag{26b}$$

---

[2] $e^{\pm}$ are nondegenerate in the sense that 0 is a simple eigenvalue of the linearized operators $T : W^{2,2}(\mathbb{R}; \mathbb{R}^m) \to L^2(\mathbb{R}; \mathbb{R}^m)$, $T\varphi = -\varphi'' + D^2W(e^{\pm})\varphi$.


e^+
a^-
a^+
e^-


a^+
$x \to \infty$
a^+
x
$e^-(x)$
$e^+(x)$
$t \to -\infty$
$t \to \infty$
t
$e^-(x)$
$e^+(x)$
$x \to -\infty$
a^-
a^-

A new proof of Schatzman's theorem: reduction to an O.D.E. problem

• The boundary conditions (26a) suggest to set

$$e_0(x) = \begin{cases} a^-, & \text{for } x \leq -1, \\ a^- + (a^+ - a^-)\frac{x+1}{2}, & \text{for } -1 \leq x \leq 1, \\ a^+, & \text{for } x \geq 1. \end{cases} \tag{27}$$

and work in the affine subspace $\mathcal{H} := e_0 + L^2(\mathbb{R}; \mathbb{R}^m)$ which has the structure of a Hilbert space, if we identify the origin with e_0.

• More generally that in Schatzman, we assume that the set of minimizing heteroclinics $F \subset \mathcal{H}$ is partitioned into two subsets F^+ and F^-, and that $d_{\min} := d_{\mathcal{H}}(F^-, F^+) > 0$.

A new proof of Schatzman's theorem: reduction to an O.D.E. problem

- Next, we define in $\mathcal{H}$ the *effective* potential $\mathcal{W} : \mathcal{H} \to [0, +\infty]$ by

$$\mathcal{W}(u) = \begin{cases} E_{\mathbb{R}}(u) - E_{\min}, & \text{when } u' \in L^2(\mathbb{R}; \mathbb{R}^m), \\ +\infty, & \text{otherwise,} \end{cases} \tag{28}$$

where $E_{\min} := E_{\mathbb{R}}(e)$, $\forall e \in F$,

- and the functional

$$\mathcal{J}_{[\alpha,\beta]}(U) := \int_\alpha^\beta \Big[\frac{1}{2}\|U'(t)\|^2_{L^2(\mathbb{R};\mathbb{R}^m)} + \mathcal{W}(U(t))\Big] \mathrm{d}t, \tag{29}$$

for $U \in W^{1,2}([\alpha, \beta]; \mathcal{H})$.

A new proof of Schatzman's theorem: reduction to an O.D.E. problem

- Also note that setting $u(t,x) := [U(t)](x)$, we have that

$$u(t,x) - e_0(x) \in L^2([\alpha,\beta] \times \mathbb{R}; \mathbb{R}^m),$$

and

$$u_t(t,x) = [U'(t)](x) \in L^2([\alpha,\beta] \times \mathbb{R}; \mathbb{R}^m).$$

- Finally, using difference quotients we have that

$$\mathcal{J}_{[\alpha,\beta]}(U) < \infty \Rightarrow u_x(t,x) \in L^2([\alpha,\beta] \times \mathbb{R}; \mathbb{R}^m).$$

Thus, $u \in W^{1,2}_{\mathrm{loc}}((\alpha,\beta) \times \mathbb{R}; \mathbb{R}^m)$.

- In addition, one can see that

$$E(u, (\alpha,\beta) \times \mathbb{R}) = J_{[\alpha,\beta]}(U) + J_{\min}(\beta - \alpha).$$

A new proof of Schatzman's theorem: reformulation of the theorem

Theorem (Sm, 2019): • Under the previous assumptions on W and F, $\mathcal{J}_{\mathbb{R}}$ admits a minimizer U in the class

$$\mathcal{K} := \{V \in W^{1,2}_{\mathrm{loc}}(\mathbb{R};\mathcal{H}) : d_{\mathcal{H}}(V(t), F^{\pm}) \to 0, \text{ as } t \to \pm\infty\}.$$

• Setting $u(t,x) := [U(t)](x)$, then $u \in C^2(\mathbb{R}^2;\mathbb{R}^m)$ is a *minimal* solution of (25) satisfying

$$\lim_{x\to\pm\infty} u(t,x) = a^{\pm}, \text{ uniformly when } t \text{ remains bounded.} \tag{30}$$

• In addition, if $\mathcal{W}$ satisfies the nondegeneracy condition

$$\liminf_{d_{\mathcal{H}}(u,F)\to 0} \frac{\mathcal{W}(u)}{(d_{\mathcal{H}}(u,F))^2} > 0, \tag{31}$$

then there exist $e^{\pm} \in F^{\pm}$, such that $\lim_{t\to\pm\infty} \|U(t) - e^{\pm}\|_{H^1(\mathbb{R};\mathbb{R}^m)} = 0$, and the convergence in (30) is uniform for $t \in \mathbb{R}$.

A new proof of Schatzman's theorem: sketch of the proof

• One has to adjust the arguments in the theorem for a double well potential, since the set F is unbounded.

• However, $\mathcal{W}$ and F the following have nice properties

(i) The potential $\mathcal{W}$ is sequentially weakly lower semicontinuous.

(ii) $\mathcal{W}(u) \to 0 \Rightarrow d_{H^1(\mathbb{R};\mathbb{R}^m)}(u, F) \to 0$. This property combined with the next one are essential to address the lack of compactness issue.

(iii) Let $\{e_k\} \subset F$ be bounded in $\mathcal{H}$, then there exists $e \in F$, such that up to subsequence $\lim_{k\to\infty} \|e_k - e\|_{H^1(\mathbb{R};\mathbb{R}^m)} = 0$.

(iv) There exists a constant $\gamma > 0$, such that for every $e \in F$, we can find $T \in \mathbb{R}$ such that setting $e^T(x) = e(x - T)$, we have $\|e^T - e_0\|_{H^1(\mathbb{R};\mathbb{R}^m)} \leq \gamma$.

• To obtain the minimizer U, we consider appropriate translations of a minimizing sequence $\{U_n\}$ with respect to both variables t and x. Property (iv) is used to find the appropriate translation with respect to x.

Other applications

• We constructed a minimizing heteroclinic orbit U connecting at $\pm\infty$ the subsets $F^{\pm}$ in the Hilbert space $\mathcal{H}$.
• Question: what kind of solution is obtained if instead of $\mathcal{H} = e_0 + L^2(\mathbb{R};\mathbb{R}^m)$, we consider another space, for instance $\tilde{\mathcal{H}} = e_0 + H^1(\mathbb{R};\mathbb{R}^m)$?
• Assuming that W is as previously, and that F is partitioned into two subsets F^+ and F^- such that $d_{\tilde{\mathcal{H}}}(F^-, F^+) > 0$, we can similarly construct a minimizing heteroclinic $\tilde{U}$ connecting at $\pm\infty$ the subsets $F^{\pm}$ in $\tilde{\mathcal{H}}$. It is a minimizer of the functional

$$\tilde{\mathcal{J}}_{\mathbb{R}}(V) := \int_{\mathbb{R}} \Big[\frac{1}{2}\|V'(t)\|^2_{H^1(\mathbb{R};\mathbb{R}^m)} + \mathcal{W}(V(t))\Big]\mathrm{d}t. \qquad (32)$$

in the class
$\tilde{\mathcal{K}} := \{V \in W^{1,2}_{\mathrm{loc}}(\mathbb{R};\tilde{\mathcal{H}}) : d_{\tilde{\mathcal{H}}}(V(t), F^{\pm}) \to 0, \text{ as } t \to \pm\infty\}$.

Other applications

Theorem (Sm, 2019): Under the previous assumptions on W and F, $\tilde{\mathcal{J}}_{\mathbb{R}}$ admits a minimizer $\tilde{U} \in \tilde{\mathcal{K}}$ which is a classical solution of system $\tilde{U}''(t) = \nabla \mathcal{W}(\tilde{U}(t))$. Setting $\tilde{u}(t,x) := [\tilde{U}(t)](x)$, $\tilde{u}$ is a weak *minimal* solution of system (33):

$$\tilde{u}_{ttxx}(t,x) = \Delta \tilde{u}(t,x) - \nabla W(\tilde{u}(t,x)),\ \tilde{u} : \mathbb{R}^2 \to \mathbb{R}^m. \tag{33}$$

satisfying the boundary conditions

$$\lim_{t \to \pm\infty} d_{\tilde{\mathcal{H}}}(\tilde{U}(t), F^{\pm}) = 0, \tag{34a}$$

$$\lim_{x \to \pm\infty} \tilde{u}(t,x) = a^{\pm}, \text{ uniformly when } t \text{ remains bounded.} \tag{34b}$$

Remark: $\mathcal{W} \in C^1(\tilde{\mathcal{H}}; [0,\infty))$, and
$D\mathcal{W}(u)h = \int_{\mathbb{R}} [u' \cdot h' + \nabla W(u) \cdot h]$, $\forall u \in \tilde{\mathcal{H}}$, $\forall h \in H^1(\mathbb{R}; \mathbb{R}^m)$.
This explains why $\tilde{U}''(t) = \nabla \mathcal{W}(\tilde{U}(t))$ holds.

Other applications

• The method also applies to construct heteroclinic double layers for the Fisher-Kolmogorov P.D.E. (Sm, 2021):

$$\Delta^2 u - \beta \Delta u + \nabla W(u) = 0,\ u : \mathbb{R}^2 \to \mathbb{R}^m,\ \beta \geq 0,\ W : \mathbb{R}^m \to [0, \infty), \tag{35}$$

with W a bistable potential.

• Finally, due to the variety of choices for the space $\mathcal{H}$, several types of boundary conditions may be considered in the applications of the theorem.

A possible De Giorgi conjecture in the vector case

• For system $\Delta u = \nabla W(u)$, $u : \mathbb{R}^n \to \mathbb{R}^m$, with W a double well potential, we have seen that the existence of two minimal heteroclinics implies the existence of a two dimensional minimal solution.

• One may ask if there is a condition implying the reduction of variables for the solutions $u : \mathbb{R}^n \to \mathbb{R}^m$ of $\Delta u = \nabla W(u)$.

• Question: If the minimal heteroclinic e is unique, is it true that when n is sufficiently small, any minimal solution is either constant or equal to e up to change of coordinates?

• Question: Similarly, does the existence of exactly two minimal heteroclinics implies that any minimal solution is at most two dimensional, when $n \geq 2$ is small enough?