Can we divide vectors?

Geometric calculus in Science and Engineering

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Complex and Hypercomplex Analysis

Classic Vector calculus - Algebraic View

- Iron rule for students: Never divide vectors!
- Comes from the problem of multiplication of vectors
- Known products: scalar product $\langle x, y \rangle$ and cross product $x \times y$ (in $\mathbb{R}^3$)
- Problem for scalar product: $\langle x, y \rangle = 1$ with $|x| = |y| = 1$ does only imply $y = \pm x$
- Same problem for cross product: $\langle x, y \rangle = 1$ only implies $x \perp y$
- Inverse is not well defined!

- Reflection on the unit disk:

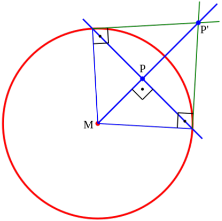


Figure: Inversion on the circle - Wikipedia

- Kelvin inverse in vector calculus: $x \to x/|x|^2$
- What is the algebraic meaning of the Kelvin inverse?

Complex numbers and complex plane

- Vector $a = (a_1.a_2) \to$ complex number $a = a_1 + ia_2$
- Geometric meaning of multiplication of complex numbers a, b:

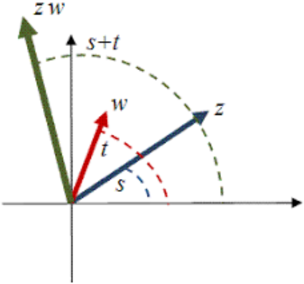


Figure: Complex Multiplication - Wikipedia

- Given $a = |a|e^{i\varphi}$ geometrically multiplication with a is a rotation by φ plus a dilation by a.
- Inverse $a^{-1} = \overline{a}/|a|^2$ corresponds to the Kelvin inverse plus a reflection on the real axis

Multiplication of vectors

- Vectors $(a_1, a_2) = a_1 e_1 + a_2 e_2$ and $(b_1, b_2) = b_1 e_1 + b_2 e_2$
- Critical point in Multiplication: What to do with $e_1 e_2$?
- Rotations in dimensions higher than 2 are not commutative $\Rightarrow e_1 e_2 = -e_2 e_1$
- What is the meaning of $e_1 e_2$?
- $e_1 e_2 = e_1$ or $e_1 e_2 = e_2$? No good geometric meaning.

Small historic remark

- R.W. Hamilton 1843: $e_1 e_2$ is a new vector:

x 

Figure: Broom Bridge - Wikipedia

$$i = e_1, j = e_2, k = e_1 e_2$$

- Rather dogmatic approach $\rightarrow$ Gibbs and others developed vector calculus in response
- German secondary school teacher Grassmann: $e_1 e_2$ is an (oriented) plane!
- W.K. Clifford 1878: New algebra (Geometric or Clifford algebra)
- A. Einstein 1913: Semi-vector calculus

Complex and Hypercomplex Analysis

Clifford Algebras

- Universal Clifford Algebra $\mathcal{C}\ell_{p,q}$
  - generated by $e_0 = 1$ and $e_1, \cdots, e_n$ satisfying to

$$e_i e_j = -e_j e_i, i \neq j,$$

$$e_i^2 = +1, i = 1, \ldots, p, \qquad e_j^2 = -1, j = p+1, \ldots, n = p+q.$$

  - Clifford number:

$$\begin{aligned} a &= a_0 + e_1 a_1 + \ldots + e_n a_n + e_1 e_2 a_{12} + \ldots + e_1 e_2 e_3 a_{123} \\ &\quad + \ldots + e_1 \ldots e_n a_{12\ldots n} \end{aligned}$$

    Then $\dim(\mathcal{C}\ell_{p,q}) = 2^n$.
  - Conjugation defined as

$$\overline{1} = 1, \ \overline{e_i} = -e_i, \ \overline{ab} = \overline{b}\,\overline{a}.$$

    Hence for a vector $x = \sum e_i x_i$ we have $x^2 = -|x|^2$ and $x^{-1} = \frac{\overline{x}}{|x|^2}$.

Complex and Hypercomplex Analysis

Geometric interpretation

- Scalars and Vectors
- Bi-vectors $\rightarrow$ oriented planes
- Tri-vectors $\rightarrow$ oriented volumes
- Clifford number = Scalar plus vector plus oriented plane plus $\cdots$ plus pseudoscalar (n-dimensional volume)

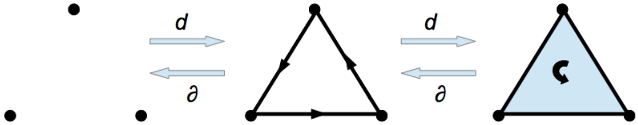


Figure: DeRham Cohomology

Rotations - Classic Approach in 3D

- Given rotation axis ω and rotation angle φ
- Standard approach: Euler angles ϕ, θ, ψ

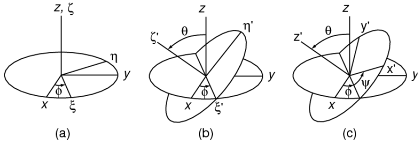


Figure: Euler Angles - Wolfram Alpha

- Example: Rotation in $e_1 e_2$-plane - Multiplication by matrix

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotations - Clifford Approach in 3D

- Given rotation axis ω and rotation angle θ
- Rotation in the $e_i e_j$-plane

$$s = \cos(\phi/2) + e_i e_j \sin(\phi/2) \to \bar{s} x s$$

- General rotation

$$s = \cos(\varphi/2) + \omega \sin(\varphi/2) \to \bar{s} x s$$

- 3D: Computer implementation cost one operation less
- Computer game Elite (David Braben 1984 - vector graphics)

Figure: Frontier: Elite II - Amiga 1993

Classic vector calculus with Clifford algebras

- Consider two vectors $x = x_1 e_1 + \ldots + x_n e_n$, $y = y_1 e_1 + \ldots + y_n e_n$
- $xy = \frac{xy-yx}{2} + \frac{xy+yx}{2}$
- Inner product (scalar): $\langle x, y \rangle = \frac{xy-yx}{2}$
- Outer product (bi-vector): $x \wedge y = \frac{xy+yx}{2}$
- In 3D: $x \wedge y = x \times y$ (right-hand rule)
- Does not work in other dimensions: number of basic vectors $n \neq$ number of basic planes $n(n-1)/2$.
- Commuting vectors are parallel
- Anti-commuting vectors are perpendicular

Coordinate-free working and Möbius transformations

- Basic principle: Work coordinate-free!
  H. Weyl: *The introduction of numbers as coordinates is an act of violence.*
- Good example: Möbius transformations (Vahlen 1902, Ahlfors 1982)

$$f(x) = (ax + b)(cx + d)^{-1}$$

- Maps spheres into spheres and preserves angles

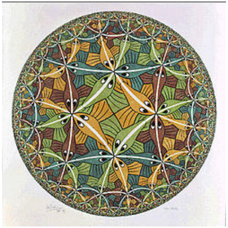


Figure: Möbius transformations of the unit disk - M.C. Escher

Monogenic signal

- Signal $f(x)$ is mapped to $F(x) = s(x)e^{\omega(x)\theta(x)}$ with $f(x) = \mathrm{Re}F(x)$
- $a(x)$- amplitude, $\theta(x)$ - phase, ω - phase angle, $\theta'(x)$ - (instantaneous) frequency
- Edge detection - detect singularities which are singular in a point and in one direction
- Find x such that $\theta'(x)$ is very large (ideally $\theta'(x) = +infty$)
- ω is perpendicular to the edge

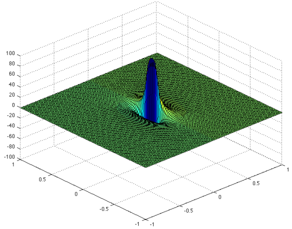
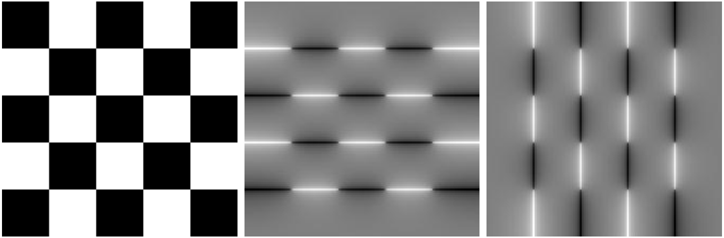


Figure: Singularity - singular in one point and one direction

Monogenic signal
Example



Monogenic signal of a checkerboard image with its directional components

- Take analyzing function (mother wavelet) ψ
- Classic Clifford wavelets: $\psi\left(\overline{s}\frac{x-b}{a}s\right)$
- F. Brackx, N. de Schepper, F. Sommen (from 2002 onwards)
- Spherical wavelets: $\psi\left(\varphi_{te_n}(\overline{s}xs)\right)$

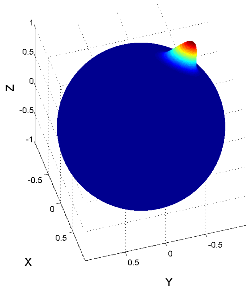


Figure: Spherical dog-wavelet

Conformal model and Robotics

- In Robotics we have two types of movements: Rotations and Translations
- Classic case: Use matrix multiplication and addition
- Embed $\mathbb{R}^n$ into $\mathbb{R}^{1,n+1}$:

$$\underline{x} \in \mathbb{R}^n \mapsto (\underline{x}, \frac{1-|\underline{x}|^2}{2}, \frac{1+|\underline{x}|^2}{2})$$

- $x \mapsto \bar{s}xs$ with $s \in \mathrm{Spin}(n+1,1)$ include now rotations and translations
- Translation by t: $s = 1 + t/2(e_+ - e_-)$
- In fact $\mathrm{Spin}(n+1,1)$ includes all Möbius transformations
- Interesting fact: Patented under U.S. Patent 6,853,964

Outlook 1 - Manifolds and integral geometry

- Classic approach to manifolds: Atlas and charts $\Rightarrow$ local coordinates
- Connections, structure equations usually are given in terms of coordinates
- Work coordinate-free!
- Particular helpful when integrating over a manifold
- One application: Tomography - determining informations on the interior by integrating over rays

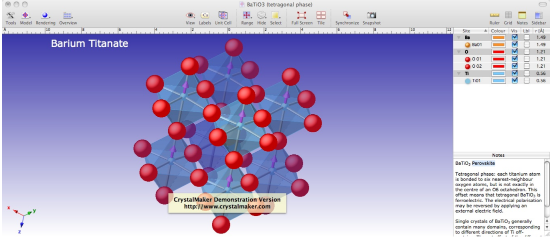
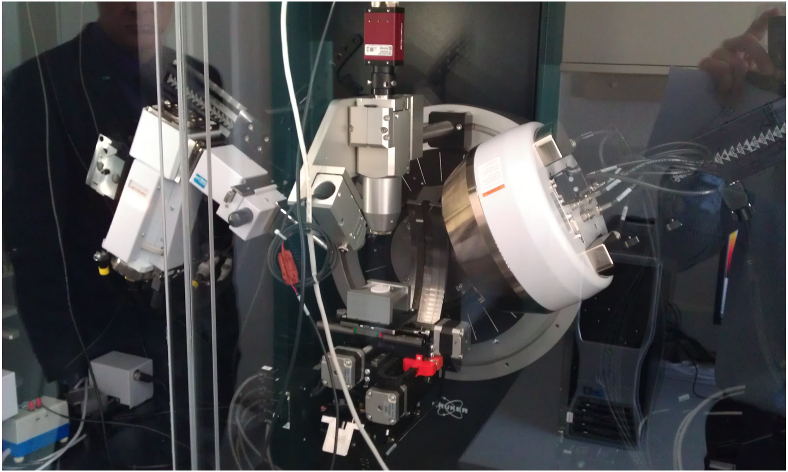


Figure: Diffraction tomography - Barium titanate

Outlook 1 - Manifolds and integral geometry

Example of diffraction tomography



Complex and Hypercomplex Analysis

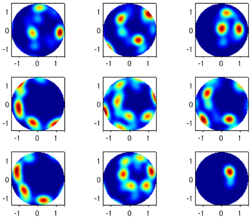


Figure: Diffraction tomography - pole figure

- Integration over all rotations which send angular vector x into angular vector y
- Such rotations are great circles in $\mathbb{S}^3$ generated by orthogonal quaternions

$$\frac{1 - yx}{|1 - yx|} \text{ and } \frac{y + x}{|y + x|}$$

- Extension of Calculus of natural numbers where geometric objects are seen as generalized natural numbers
- Examples: natural numbers $1, 2, 3, \ldots$, set $\mathbb{N}$, real line $\mathbb{R}$, cartesian spaces, projective spaces, spheres S^{n-1}, $\mathbb{C}S^{n-1}$, groups, Grassmann manifolds, homogeneous spaces
- Addition: Disjoint union and glueing whenever possible
- Subtraction $t_1 - t_2$: means that t_2 is deleted from object t_1, e.g, $\mathbb{R} - 1$ means to delete a point from a line.
- Multiplication $v \cdot w$: every point of v is replace by a copy of w and all those copies are glued together (like cartesian product or fibre bundle $E = M \cdot F$ with base space M and fibre F)
- Decomposition of real line: $\mathbb{R} = 2\mathbb{R}_+ + 1$
- $(2\mathbb{R}_+ + 1)^2 = 4\mathbb{R}_+^2 + 4\mathbb{R}_+ + 1$

- Grassmann's division problem:

  Can one work out the polynomial division $\frac{S^{n-1} \cdot S^{n-2} \ldots S^{n-k}}{S^{k-1} \ldots S^0}$, and does it result in an integral (seens as polynomial in "$\mathbb{R}$" with natural number coefficients)?

- Morphologically we have

$$\frac{S^{n-1} \cdot S^{n-2} \cdots S^{n-k}}{S^{k-1} \cdots S^0} = \frac{(\mathbb{R}^n - 1) \cdots (\mathbb{R}^n - \mathbb{R}^{k-1})}{(\mathbb{R}^k - 1) \cdots (\mathbb{R}^k - \mathbb{R}^{k-1})}$$

- Solution:

$$G_{n,k}(\mathbb{R}) = \mathbb{R}^d + c_1 \mathbb{R}^{d-1} + \ldots + c_d,$$

  c_j - number of Schubert cells of dimension $d - j$.

Examples of Grassmann manifolds:

$$
\begin{aligned}
G_{2n,2}(\mathbb{R}) &= \mathbb{CP}^{n-1} \cdot \mathbb{RP}^{2n-2} \\
G_{2n+1,2}(\mathbb{R}) &= \mathbb{RP}^{2n} \cdot \mathbb{CP}^{n-1} \\
G^{\alpha}_{6,3}(\mathbb{R}) &= \mathbb{RP}^{4} \cdot \mathbb{S}^{3} \cdot \mathbb{S}^{2}
\end{aligned}
$$

Aknowledgements

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