Random Covariant Quantum Channels

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Outline

Choi matrix, PPT, and EB properties

$M_d := M_d(\mathbb{C})$, $\{e_{ij}\}_{i,j=1}^d$: matrix units in M_d.

- Quantum channel = CP trace-preserving map $\Phi : M_d \to M_d$
- The Choi–Jamiołkowski matrix of $\Phi : M_d \to M_d$ is defined by

$$J(\Phi) := \sum_{i,j=1}^d e_{ij} \otimes \Phi(e_{ij}) = (\Phi(e_{ij}))_{1 \le i,j \le d} \in M_d \otimes M_d.$$

- (Choi 1975) Φ is CPTP $\iff$ $J(\Phi) \ge 0$ and $(\mathrm{id}_d \otimes \mathrm{Tr})(J(\Phi)) = I_{d^2}$.

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Definition

A quantum channel $\Phi : M_d \to M_d$ is called

1. *PPT (positive partial transpose) if $J(\Phi)^\Gamma := (\mathrm{id}_d \otimes \top)(J(\Phi)) \geq 0$,*
2. *EB (entanglement-breaking) if $J(\Phi) = \sum_i P_i \otimes Q_i$ for $P_i, Q_i \geq 0$.*

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- EB $\implies$ PPT, but PPT $\not\Longrightarrow$ EB unless $d = 2$.
- (Gurvits 2003) Determining EB is NP-hard.

Compact group symmetry

G: compact group.

Let $\begin{cases} \pi_A : G \to \mathcal{U}_{d_A} \\ \pi_B : G \to \mathcal{U}_{d_B} \end{cases}$ be unitary representations of G.

Definition

A linear map $\Phi : M_{d_A} \to M_{d_B}$ is called (π_A, π_B)-covariant if

$$\Phi(\pi_A(g) Z \pi_A(g)^*) = \pi_B(g) \Phi(Z) \pi_B(g)^* \text{ for } \forall\, Z \in B(H_A) \text{ and } g \in G.$$

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- Covariance property can be captured through Choi matrix:

  Φ is (π_A, π_B)-covariant $\iff [J(\Phi), \overline{\pi_A}(g) \otimes \pi_B(g)] = 0 \; \forall\, g \in G$.

- Decomposition of $\overline{\pi_A} \otimes \pi_B$ into irreducible representations can characterize the (π_A, π_B)-covariant maps.

Example 1: Orthogonal covariance

- Unitary covariance ($= (U, U)$-covariance)
$$\Phi(U \cdot U^*) = U\Phi(\cdot)U^*, \ \forall\, U \in \mathcal{U}_d \iff \Phi(Z) = p\frac{\mathrm{Tr}\,(Z)}{d} I_d + qZ.$$
- Conjugate unitary covariance ($= (U, \overline{U})$-covariance)
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Theorem (Vollbrecht and Werner 2001)

An orthogonal covariant quantum channel is PPT iff EB

Example 2: Diagonal orthogonal covariance

Let $\mathcal{DO}_d := \left\{ \sum_{i=1}^d s_i e_{ii} \;\middle|\; s_i \in \{\pm 1\} \right\} \cong \mathbb{Z}_2^d$ be the set of diagonal orthogonal matrices.

Definition (Singh and Nechita 2021)

A linear map $\Phi : M_d \to M_d$ is called DOC (Diagonal Orthogonal Covariant) if

$$\Phi(DZD^\top) = D\Phi(Z)D^\top, \quad D \in \mathcal{DO}_d, \quad Z \in M_d.$$

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Proposition (Singh and Nechita 2021)

Φ is DOC if and only if there exists $A, B, C \in M_d$ such that

$$\Phi(Z) = \Phi_{A,B,C}(Z) = \sum_{i,j=1}^d A_{ij} Z_{jj} e_{ii} + B \odot \mathring{Z} + C \odot (\mathring{Z})^\top,$$

where $A \odot B := (A_{ij}B_{ij})$ is the Hadamard product and $\mathring{Z} := Z - \mathrm{diag}(Z)$.

Example 2: Diagonal orthogonal covariance

- $\Phi_{A,B,C}$ is CP $\iff \begin{cases} A \in M_d(\mathbb{R}_+), \quad B \geq 0, \quad C^* = C \\ A_{ij}A_{ji} \geq |C_{ij}|^2, \ \forall\, i \neq j. \end{cases}$
- $J(\Phi_{A,B,C})^\Gamma = J(\Phi_{A,C,B})$.

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Theorem (Singh and Nechita 2021)

A DOC quantum channel $\Phi_{A,B,C}$ is

1. *PPT iff $C \geq 0$ and $A_{ij}A_{ji} \geq |B_{ij}|^2, \ \forall\, i \neq j$.*
2. *EB iff $\exists\, V, W \in M_{d\times k}$ for some k such that*
   $A = (V \odot \overline{V})(W \odot \overline{W})^*, \quad B = (V \odot W)(V \odot W)^*, \quad C = (V \odot \overline{W})(V \odot \overline{W})^*.$

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REMARK:

- There are many examples of DOC channels which are PPT not EB.
- (Dickinson and Gijben 2014) Determining whether a given DOC channel is EB or not is NP-hard.

Outline

Random Stinespring quantum channels

- (Stinespring representation) Any quantum channel $\Phi : M_d \to M_d$ can be represented by
$$\Phi(Z) = (\mathrm{id}_d \otimes \mathrm{Tr}_s)(VZV^*) \quad (*)$$
for some isometry $V : \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^s$.

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Definition

A random Stinespring quantum channel $\Phi = \Phi_V$ is defined by $(*)$, where V is a Haar-random isometry (i.e. unitarily invariant random isometry).

The probability distribution of Φ_V depends on the parameters (d, s).

Threshold phenomanon

QUESTION: What are the behaviors of $\begin{cases} P(\Phi_V \text{ is PPT}) \\ P(\Phi_V \text{ is EB}) \end{cases}$ for asymptotic regimes $d \to \infty$ and $s = s(d)$?

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Theorem (Aubrun 2012)

For $\forall \varepsilon > 0$, *we have*

1. $s \leq (4-\varepsilon)d^2 \implies P(\Phi_V \textit{ is PPT}) \to 0 \textit{ as } d \to \infty$,
2. $s \geq (4+\varepsilon)d^2 \implies P(\Phi_V \textit{ is PPT}) \to 1 \textit{ as } d \to \infty$.

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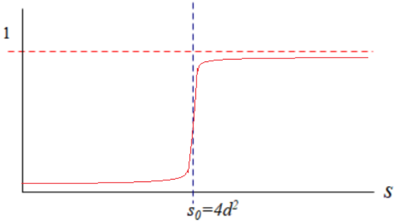

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Theorem (Aubrun, Szarek, and Ye 2014)

There exist universal constants $C_1, C_2 > 0$ *such that* $\forall\, \varepsilon > 0$,

1. $s \leq (1-\varepsilon)C_1 d^3 \implies P(\Phi_V \textit{ is EB}) \to 0 \textit{ as } d \to \infty$,
2. $s \geq (1+\varepsilon)C_2 d^3 (\log d)^2 \implies P(\Phi_V \textit{ is EB}) \to 1 \textit{ as } d \to \infty$.

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Twirling operation

QUESTION: Can we define a natural probability on the set of (π_A, π_B)-covariant quantum channels for unitary representations π_A and π_B of a compact group G?

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Proposition

For a linear map $\Phi : M_{d_A} \to M_{d_B}$, define $\mathcal{T}_{\pi_A,\pi_B}\Phi : M_{d_A} \to M_{d_B}$ by

$$(\mathcal{T}_{\pi_A,\pi_B}\Phi)(Z) := \int_G \pi_B(g)^*\Phi(\pi_A(g)Z\pi_A(g)^*)\pi_B(g)dg,$$

where dg is a (normalized) Haar measure on G. Then

1. $\mathcal{T}_{\pi_A,\pi_B}\Phi$ *is (π_A, π_B)-covariant.*
2. Φ *is (π_A, π_B)-covariant $\iff \mathcal{T}_{\pi_A,\pi_B}\Phi = \Phi$.*

In particular, we can regard $\mathcal{T}_{\pi_A,\pi_B}$ as a projection onto the set of (π_A, π_B)-covariant maps.

Random covariant quantum channel

RECALL: $\Phi_V := (\mathrm{id}_d \otimes \mathrm{Tr}_s)(V \cdot V^*)$ where $V : \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^s$ is a Haar-random isometry.

Definition (Nechita and P., in progress)

We define random (π_A, π_B)-covariant quantum channels (of parameters (d, s)) by

$$\Phi^{(d,s)}_{\pi_A,\pi_B} := \mathcal{T}_{\pi_A,\pi_B}\Phi_V = \int_G \pi_B(g)^*\Phi_V\Big(\pi_A(g) \cdot \pi_A(g)^*\Big)\pi_B(g)\, dg.$$

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QUESTION: What is the limit

$$\lim_{d\to\infty} P(\Phi^{(d,s)}_{\pi_A,\pi_B} \text{ is PPT / EB})$$

for choices of (π_A, π_B) and regimes $s = s(d)$?

Result: Random DOC channels

For a random stinespring channel Φ_V, take DOC-twirling

$$\Phi_{\mathrm{DOC}} := \mathcal{T}_{\mathrm{DOC}}\Phi_V = \frac{1}{|\mathcal{DO}_d|}\sum_{D\in\mathcal{DO}_d} D^\top \Phi_V(D\cdot D^\top)D.$$

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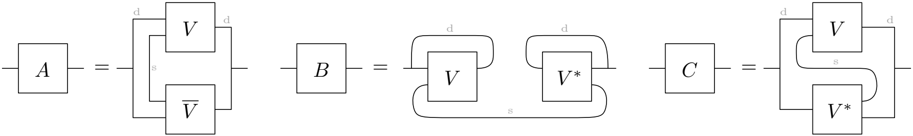
RECALL: $\Phi_{\mathrm{DOC}} = \Phi_{A,B,C}$ is PPT $\iff \begin{cases} C \geq 0, \\ A_{ij}A_{ji} \geq |B_{ij}|^2 \; \forall\, i \neq j, \end{cases}$ where

$$A_{ij} := J(\Phi_V)_{ij,ij}, \quad B_{ij} := J(\Phi_V)_{ii,jj}, \quad C_{ij} := J(\Phi_V)_{ij,ji}, \quad 1 \leq i,j \leq d.$$



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Proposition (Nechita and P., in progress)

1. *If $s \sim cd$ for some $c > 0$, then almost surely, we have,*

$$\forall\, p \geq 1, \quad \tfrac{1}{d}\mathrm{Tr}\left((sC)^p\right) \to \int x^p dSC_{c,c}(x).$$

i.e., the empirical eigenvalue distribution

$$\tfrac{1}{d}\textstyle\sum_{i=1}^{d} \delta_{\lambda_i(sC)}$$

of sC converges to $SC_{c,c}$ in moments.

Here SC_{m,σ^2} is the semicircle distribution of mean m and variance σ^2:

$$d\, SC_{m,\sigma}(x) = \tfrac{1}{2\pi\sigma}\sqrt{4\sigma^2 - (x-m)^2}\,\mathbf{1}_{[m-2\sigma, m+2\sigma]}(x)dx.$$

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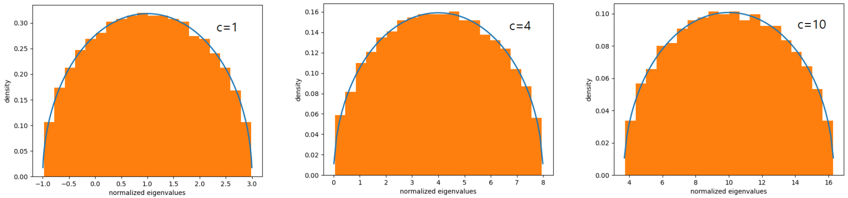

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1. *If* $s \sim cd$ *for some* $c > 0$, *then* $sC \to SC_{c,c}$ *in moments.*
2. *If* $s \lesssim d^{1-\varepsilon}$ *for some* ε, $(ds)^{1/2}C \to SC_{0,1}$ *in moments.*
3. *If* $s \gtrsim d^{1+\varepsilon}$ *for some* $\varepsilon > 0$, *then* $dC \to 1$ *in moments.*
4. *If* $s \gtrsim d^{\varepsilon}$ *for some* $\varepsilon > 0$, *we have*

$$\min_{i \neq j} d^2 \left(A_{ij}A_{ji} - |B_{ij}|^2 \right) \to 1 \text{ almost surely as } d \to \infty.$$

Here SC_{m,σ^2} is the semicircle distribution of mean m and variance σ^2:

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Note that $\mathrm{supp}(SC_{c,c}) = [c - 2\sqrt{c}, c + 2\sqrt{c}] \subset [0, \infty)$ iff $c \geq 4$.

Theorem (Nechita and P., in progress)

For any $\varepsilon > 0$, we have as $d \to \infty$,

$$P(\Phi_{\mathrm{DOC}} \text{ is } PPT) \to \begin{cases} 0 & \text{if } s \lesssim d^{1-\varepsilon} \text{ or } s \sim (4-\varepsilon)d, \\ 1 & \text{if } s \gtrsim d^{1+\varepsilon} \text{ or } s \sim (4+\varepsilon)d. \end{cases}$$

In particular, threshold occurs when $s_0 = 4d$.

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In particular, threshold occurs when $s_0 = 4d$.

Theorem (Nechita and P., in progress)

If $s \gtrsim d^{2+\varepsilon}$ for some $\varepsilon > 0$, We have

$$P(\Phi_{\mathrm{DOC}} \textit{ is } EB) \to 1 \textit{ as } d \to \infty.$$

Results: Thresholds for other types of covariance

Group symmetry	PPT property	EB property	Not EB property
(U, U)-cov.	Always	Always	Never
$(U, \overline{U})$-cov.	$s_0 \sim$ const.	$s_0 \sim$ const.	$s_0 \sim$ const.
(O, O)-cov.	$s_0 \sim$ const.	$s_0 \sim$ const.	$s_0 \sim$ const.
DOC	$s_0 \sim 4d$	$s \gtrsim d^{2+\epsilon}$	$s \leq (4-\epsilon)d$
No symmetry (Stinespring)	$s_0 \sim 4d^2$ [Aub 12]	$s \gtrsim d^3 \log^2 d$ [ASY14]	$s \lesssim d^3$ [ASY14]

Further works:

- EB properties of DOC channels in the regimes $(4+\varepsilon)d \leq s \lesssim d^2$
- Other applications of random DOC channels (e.g. additivity violation of classical capacities).

Thank you for your attention!