Atomic decomposition of noncommutative martingales

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atomic decomposition

Atomic decomposition plays a fundamental role in the classical martingale theory or harmonic analysis. Atoms for martingales are usually defined in terms of stopping times.

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Atomic decomposition plays a fundamental role in the classical martingale theory or harmonic analysis. Atoms for martingales are usually defined in terms of stopping times.
Unfortunately, the concept of stopping times is, up to now, not well-defined in the generic noncommutative setting (there are some works on this topic, see [1] and references therein).

1. J. L. Doob, Stochastic processes, John Wiley & Sons, New York, 1953.

However, we note that atoms can be defined without help of stopping times. Let us recall this in classical martingale theory. Given a probability space $(\Omega, \mathcal{F}, \mu)$, let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing filtration of σ-subalgebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\big(\cup_n \mathcal{F}_n\big)$ and let $(\mathcal{E}_n)_{n\geq 1}$ denote the corresponding family of conditional expectations.

1. S. Attal and A. Coquio, Quantum stopping times and quasi-left continuity, Ann.I.H.Poincaré-PR **40**, 497-512(2004).

atoms

An $\mathcal{F}$-measurable function $a \in L^2$ is said to be an *atom* if there exist $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$ such that

1. (i) $\mathcal{E}_n(a) = 0$;
2. (ii) $\{a \neq 0\} \subset A$;
3. (iii) $\|a\|_2 \leq \mu(A)^{-1/2}$.

Such atoms are called *simple atoms* by Weisz [3] and are extensively studied by him (see [2] and [3]).

2. F. Weisz, Martingale Hardy Spaces for $0 < p \leq 1$, Probab. Theory Related Fields **84**, 361-376(1990).
3. F. Weisz, "Martingale Hardy Spaces and their Applications in Fourier Analysis," Lecture notes in mathematics, Vol.1568, Springer-Verlag, Berlin, 1994.

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Recall that we denote by $\mathcal{H}^1(\Omega)$ the space of martingales f with respect to $(\mathcal{F}_n)_{n\geq 1}$ such that the quadratic variation $S(f) = \Big(\sum_n |df_n|^2 \Big)^{1/2}$ belongs to $L^1(\Omega)$,

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Let us point out that atomic decomposition was first introduced in harmonic analysis by Coifman [4]. It is Herz [5] who initiated atomic decomposition for martingale theory.
Recall that we denote by $\mathcal{H}^1(\Omega)$ the space of martingales f with respect to $(\mathcal{F}_n)_{n\geq 1}$ such that the quadratic variation $S(f) = \Big(\sum_n |df_n|^2\Big)^{1/2}$ belongs to $L^1(\Omega)$, and by $\mathrm{h}^1(\Omega)$ the space of martingales f such that the conditioned quadratic variation $s(f) = \Big(\sum_n \mathcal{E}_{n-1}|df_n|^2\Big)^{1/2}$ belongs to $L^1(\Omega)$.

4. R.A. Coifman, A real variable characterization of H^p, Studia Math.**51**, 269-274(1974).
5. C. Herz, Bounded mean oscillation and regulated martingales, Trans.Amer.Math.Soc.**193**, 199-215(1974).

atomic decomposition

We say that a martingale $f = (f_n)_{n\geq 1}$ is predictable in L^1 if there exists an adapted sequence $(\lambda_n)_{n\geq 0}$ of non-decreasing, non-negative functions such that $|f_n| \leq \lambda_{n-1}$ for all $n \geq 1$ and such that $\sup_n \lambda_n \in L^1(\Omega)$. We denote by $\mathcal{P}^1(\Omega)$ the space of all predictable martingales.

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In a disguised form in the proof of Theorem A_∞ in [5], Herz establishes an atomic description of $\mathcal{P}^1(\Omega)$. Since $\mathcal{P}^1(\Omega) = \mathcal{H}^1(\Omega)$ for regular martingales, this gives an atomic decomposition of $\mathcal{H}^1(\Omega)$ in the regular case.

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We say that a martingale $f = (f_n)_{n\geq 1}$ is predictable in L^1 if there exists an adapted sequence $(\lambda_n)_{n\geq 0}$ of non-decreasing, non-negative functions such that $|f_n| \leq \lambda_{n-1}$ for all $n \geq 1$ and such that $\sup_n \lambda_n \in L^1(\Omega)$. We denote by $\mathcal{P}^1(\Omega)$ the space of all predictable martingales.
In a disguised form in the proof of Theorem A_∞ in [5], Herz establishes an atomic description of $\mathcal{P}^1(\Omega)$. Since $\mathcal{P}^1(\Omega) = \mathcal{H}^1(\Omega)$ for regular martingales, this gives an atomic decomposition of $\mathcal{H}^1(\Omega)$ in the regular case.
Such a decomposition is still valid in the general case but for $\mathrm{h}^1(\Omega)$ instead of $\mathcal{H}^1(\Omega)$, as shown by Weisz [2].

atomic decomposition

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In this talk, we will present the noncommutative version of atoms and prove that atomic decomposition for the Hardy spaces of noncommutative martingales is valid for those noncommutative atoms.
Since there are two kinds of Hardy spaces, i.e., column and row Hardy spaces in the noncommutative setting, we need to define the corresponding two type atoms.
This is a main difference from the commutative case, but can be done by considering the right and left supports of martingales as being operators on Hilbert spaces.

atomic decomposition

Roughly speaking, replacing the support of atoms in the above (ii) by the right (resp. left) support we obtain the concept of noncommutative right (resp. left) atoms, which are proved to be suitable for the column (resp. row) Hardy spaces in later.

atomic decomposition

Roughly speaking, replacing the support of atoms in the above (ii) by the right (resp. left) support we obtain the concept of noncommutative right (resp. left) atoms, which are proved to be suitable for the column (resp. row) Hardy spaces in later.
On the other hand, due to the noncommutativity some basic constructions based on the stopping times for classical martingales are not valid in the noncommutative setting, our approach to the atomic decomposition for the conditioned Hardy space of noncommutative martingales is via the $\mathsf{h}^1 - \mathsf{bmo}$ duality.

atomic decomposition

Recall that the duality equality $(\mathrm{h}^1)^* = \mathrm{bmo}$ was established independently by [6] and [7]. However, this method does not give an explicit decomposition.

6. M. Junge and T. Mei, Noncommutative Riesz transforms - A probabilistic approach, Amer. J. Math. **132** (3), 611-680 (2010).
7. M. Perrin, A noncommutative Davis' decomposition for martingales, J. Lond. Math. Soc. (2) **80** (3), 627-648 (2009).

Noncommutative martingales

Let us now recall the general setup for noncommutative martingales. $\mathcal{M}$ will always denote a von Neumann algebra with a normal faithful normalized trace τ. In the sequel, we always denote by $(\mathcal{M}_n)_{n\geq 1}$ an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ whose union $\cup_{n\geq 1}\mathcal{M}_n$ generates $\mathcal{M}$ (in the w^*-topology). $(\mathcal{M}_n)_{n\geq 1}$ is called a filtration of $\mathcal{M}$.

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For $n \geq 1$, let $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ be the trace preserving conditional expectation. Note that $\mathcal{E}_n$ extends to a contractive projection from $L_p(\mathcal{M}, \tau)$ onto $L_p(\mathcal{M}_n, \tau_n)$ for all $1 \leq p \leq \infty$.

Noncommutative martingales

A sequence $x = (x_n)$ in $L^1(\mathcal{M})$ is called a *noncommutative martingale* with respect to $(\mathcal{M}_n)_{n\geq 1}$ if $\mathcal{E}_n(x_{n+1}) = x_n$ for every $n \geq 1$.

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If in addition, all x_n's are in $L^p(\mathcal{M})$ for some $1 \leq p \leq \infty$, x is called a noncommutative L^p-martingale. In this case we set

$$\|x\|_p = \sup_{n\geq 1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, then x is called a bounded L^p-martingale.

Noncommutative martingales

Let $x = (x_n)$ be a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$. Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the usual convention that $x_0 = 0$. The sequence $dx = (dx_n)$ is called the *martingale difference sequence* of x. x is called a *finite martingale* if there exists N such that $dx_n = 0$ for all $n \geq N$. In the sequel, for any operator x we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$.

Noncommutative martingales

For $1 \leq p < \infty$ and any finite sequence $a = (a_n)_{n\geq 1}$ in $L^p(\mathcal{M})$, we set

$$\|a\|_{L^p(\mathcal{M};\ell^2_c)} = \Big\| \Big(\sum_{k\geq 1} |a_k|^2 \Big)^{1/2} \Big\|_p, \quad \|a\|_{L^p(\mathcal{M};\ell^2_r)} = \Big\| \Big(\sum_{k\geq 1} |a_k^*|^2 \Big)^{1/2} \Big\|_p.$$

Then $\|\cdot\|_{L^p(\mathcal{M};\ell^2_c)}$ (resp. $\|\cdot\|_{L^p(\mathcal{M};\ell^2_r)}$) defines a norm on the family of finite sequences of $L^p(\mathcal{M})$. The corresponding completion is a Banach space, denoted by $L^p(\mathcal{M};\ell^2_c)$ (resp. $L^p(\mathcal{M};\ell^2_r)$).

Noncommutative martingales

Recall that $L^p(\mathcal{M};\ell^2_c)$ (resp. $L^p(\mathcal{M};\ell^2_r)$) can be identified with the closed subspace of $L^p(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell^2))$ consisting of column (resp. row) matrices with values in $L^p(\mathcal{M})$.

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For $p=\infty$, we define $L^\infty(\mathcal{M};\ell^2_c)$ (respectively $L^\infty(\mathcal{M};\ell^2_r)$) as the Banach space of the sequences in $L^\infty(\mathcal{M})$ such that $\sum_{n\geq 1} x_n^* x_n$ (respectively $\sum_{n\geq 1} x_n x_n^*$) converge for the weak operator topology.

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Thus $L^\infty(\mathcal{M};\ell_c^2)$ (resp. $L^\infty(\mathcal{M};\ell_r^2)$) is isometric to the subspace of $L^\infty(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell^2))$ consisting of column (resp. row) matrices.
Then, as observed in [8], both $L^p(\mathcal{M};\ell_c^2)$ and $L^p(\mathcal{M};\ell_r^2)$ are one-complemented subspaces of $L^p(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell^2))$.

8. G. Pisier and Q.Xu, Non-commutative martingale inequalities, *Commun. Math. Phys.* **189**, 667-698 (1997).

Noncommutative martingales

Let us now recall the definitions of the square functions and martingale Hardy spaces for noncommutative martingales. Following [8], we introduce the column and row versions of square functions relative to a (finite) martingale $x = (x_n)$:

$$S_{c,n}(x) = \Big(\sum_{k=1}^{n} |dx_k|^2 \Big)^{1/2}, \quad S_c(x) = \Big(\sum_{k=1}^{\infty} |dx_k|^2 \Big)^{1/2};$$

and

$$S_{r,n}(x) = \Big(\sum_{k=1}^{n} |dx_k^*|^2 \Big)^{1/2}, \quad S_r(x) = \Big(\sum_{k=1}^{\infty} |dx_k^*|^2 \Big)^{1/2}.$$

Noncommutative martingales

Let $1 \leq p < \infty$. Observe that for a finite L^p-martingale x, we have

$$\|dx\|_{L^p(\mathcal{M};\ell^2_c)} = \|S_c(x)\|_p \quad \text{and} \quad \|dx\|_{L^p(\mathcal{M};\ell^2_r)} = \|S_r(x)\|_p.$$

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Define $\mathcal{H}_c^p(\mathcal{M})$ (resp. $\mathcal{H}_r^p(\mathcal{M})$) as the completion of all finite L^p-martingales under the norm $\|x\|_{\mathcal{H}_c^p} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_r^p} = \|S_r(x)\|_p$).

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Define $\mathcal{H}^p_c(\mathcal{M})$ (resp. $\mathcal{H}^p_r(\mathcal{M})$) as the completion of all finite L^p-martingales under the norm $\|x\|_{\mathcal{H}^p_c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}^p_r} = \|S_r(x)\|_p$).
For $p = \infty$, we define $\mathcal{H}^\infty_c(\mathcal{M})$ (resp. $\mathcal{H}^\infty_r(\mathcal{M})$) as the Banach space of the $L^\infty(\mathcal{M})$-martingales x such that $\sum_{k\geq 1} |dx_k|^2$ (resp. $\sum_{k\geq 1} |dx_k^*|^2$) converge for the weak operator topology.

Noncommutative martingales

The Hardy space of noncommutative martingales is defined as follows: if $1 \le p < 2$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}^p_c(\mathcal{M}) + \mathcal{H}^p_r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p} = \inf \left\{ \|y\|_{\mathcal{H}^p_c} + \|z\|_{\mathcal{H}^p_r} \right\}$$

where the infimum is taken over all $y \in \mathcal{H}^p_c(\mathcal{M})$ and $z \in \mathcal{H}^p_r(\mathcal{M})$ such that $x = y + z$.

Noncommutative martingales

For $2 \leq p \leq \infty$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}_c^p(\mathcal{M}) \cap \mathcal{H}_r^p(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p} = \max\left\{\|x\|_{\mathcal{H}_c^p}, \|x\|_{\mathcal{H}_r^p}\right\}.$$

The reason that $\mathcal{H}^p(\mathcal{M})$ is defined differently according to $1 \leq p < 2$ or $2 \leq p \leq \infty$ is presented in [8].

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The reason that $\mathcal{H}^p(\mathcal{M})$ is defined differently according to $1 \leq p < 2$ or $2 \leq p \leq \infty$ is presented in [8].
In that paper Pisier and Xu prove the noncommutative Burkholder-Gundy inequalities which imply that $\mathcal{H}^p(\mathcal{M}) = L^p(\mathcal{M})$ with equivalent norms for $1 < p < \infty$.

Noncommutative martingales

We now consider the conditioned version of square functions and Hardy spaces developed in [9, 10].

Let $1 \le p < \infty$. For a finite sequence in $\mathcal{M}$, we define (with $\mathcal{E}_0 = \mathcal{E}_1$)

$$\|a\|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)} = \Big\| \Big(\sum_{k\ge 1} \mathcal{E}_{k-1} |a_k|^2 \Big)^{1/2} \Big\|_p .$$

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It is shown in [9] that $\| \cdot \|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)}$ is a norm on the vector space of all finite sequences in $\mathcal{M}$. Then let $L^p_{\mathrm{cond}}(\mathcal{M}; \ell^2_c)$ be the corresponding completion.

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Note that $L^p_{\text{cond}}(\mathcal{M};\ell^2_c)$ is the conditioned version of $L^p(\mathcal{M};\ell^2_c)$ defined earlier. Similarly, we define the conditioned row space $L^p_{\text{cond}}(\mathcal{M};\ell^2_r)$.

9. M. Junge, Doob's inequality for non-commutative martingales, J.Reine Angew.Math.**549**, 149-190(2002).

10. M. Junge and Q. Xu, Noncommutative Burkholder/Rosenthal inequalities, Ann.Probab.**31**, 948-995(2003).

Recall that $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ and $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_r)$ can be viewed as column and row (resp.) closed subspaces of the noncommutative space $L^p(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell^2(\mathcal{N}^2)))$.

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Furthermore, for $1 < p < \infty$, $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ and $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_r)$ are complemented in $L^p(\mathcal{M};\ell^2_c(\mathcal{N}^2))$ via Stein's projection.
We refer to [9] for more details on this.

Let $x = (x_n)_{n \geq 1}$ be a finite martingale in $L^2(\mathcal{M})$; we set

$$s_{c,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1} |dx_k|^2 \Big)^{1/2}, \quad s_c(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1} |dx_k|^2 \Big)^{1/2};$$

and

$$s_{r,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1} |dx_k^*|^2 \Big)^{1/2}, \quad s_r(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1} |dx_k^*|^2 \Big)^{1/2}.$$

These will be called the column and row conditioned square functions, respectively.

Observe that for $1 \leq p < \infty$,

$$\|dx\|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)} = \|s_c(x)\|_p \quad \text{and} \quad \|dx\|_{L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_r)} = \|s_r(x)\|_p.$$

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Let $\mathsf{h}^p_c(\mathcal{M})$ (resp. $\mathsf{h}^p_r(\mathcal{M})$) denote the closure in $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_c)$ (resp. $L^p_{\mathrm{cond}}(\mathcal{M};\ell^2_r)$) of all finite martingales in $\mathcal{M}$. (As usual we have identified a martingale with its difference sequence.)

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For $p = \infty$, we define $\mathsf{h}^\infty_c(\mathcal{M})$ (resp. $\mathsf{h}^\infty_r(\mathcal{M})$) as the Banach space of the $L^\infty(\mathcal{M})$-martingales x such that $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k|^2$ (respectively $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k^*|^2$) converge for the weak operator topology.

We also need $\ell_p(L_p(\mathcal{M}))$, the space of all sequences $a = (a_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \Big(\sum_{n\geq 1} \|a_n\|_p^p\Big)^{1/p} < \infty \quad \text{if } 1 \leq p < \infty,$$

and

$$\|a\|_{\ell_\infty(L_\infty(\mathcal{M}))} = \sup_n \|a_n\|_\infty \quad \text{if } p = \infty.$$

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Let $\mathsf{h}_p^d(\mathcal{M})$ be the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of all martingale difference sequences.
It is easy to check the following duality identity

$$\mathsf{h}_d^p(\mathcal{M})^* = \mathsf{h}_d^q(\mathcal{M})$$

if $1 \leq p < \infty$ with $q = p/(p-1)$.

Following [10], we define the conditioned version of martingale Hardy spaces as follows: If $1 \leq p < 2$,

$$\mathsf{h}^p(\mathcal{M}) = \mathsf{h}_d^p(\mathcal{M}) + \mathsf{h}_c^p(\mathcal{M}) + \mathsf{h}_r^p(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathsf{h}^p} = \inf \left\{ \|w\|_{\mathsf{h}_d^p} + \|y\|_{\mathsf{h}_c^p} + \|z\|_{\mathsf{h}_r^p} \right\},$$

where the infimum is taken over all $w \in \mathsf{h}_d^p(\mathcal{M}), y \in \mathsf{h}_c^p(\mathcal{M})$ and $z \in \mathsf{h}_r^p(\mathcal{M})$ such that $x = w + y + z$.

For $2 \le p \le \infty$,

$$\mathsf{h}^p(\mathcal{M}) = \mathsf{h}_d^p(\mathcal{M}) \cap \mathsf{h}_c^p(\mathcal{M}) \cap \mathsf{h}_r^p(\mathcal{M})$$

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$$\|x\|_{\mathsf{h}^p} = \max\left\{\|x\|_{\mathsf{h}_d^p}, \|x\|_{\mathsf{h}_c^p}, \|x\|_{\mathsf{h}_r^p}\right\}.$$

For $2 \leq p \leq \infty$,

$$\mathsf{h}^p(\mathcal{M}) = \mathsf{h}_d^p(\mathcal{M}) \cap \mathsf{h}_c^p(\mathcal{M}) \cap \mathsf{h}_r^p(\mathcal{M})$$

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$$\|x\|_{\mathsf{h}^p} = \max\big\{\|x\|_{\mathsf{h}_d^p}, \|x\|_{\mathsf{h}_c^p}, \|x\|_{\mathsf{h}_r^p}\big\}.$$

The noncommutative Burkholder inequalities proved in [10] state that $\mathsf{h}^p(\mathcal{M}) = L^p(\mathcal{M})$ with equivalent norms for all $1 < p < \infty$.

supports

Let $\mathbb{H}$ be a Hilbert space on which $\mathcal{M}$ acts. For $x \in L^p(\mathcal{M})$ there exists a unique polar decomposition $x = u|x|$, where u is a partial isometry such that $u^*u = P_{(\ker x)^\perp}$ and $uu^* = P_{\overline{\mathrm{rang} x}}$.

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If x is selfadjoint, $r(x) = l(x)$. This common projection is called the support of x and denoted by $s(x)$.
By von Neumann's double commutant theorem, one has that $r(x), l(x) \in \mathcal{M}$.

atoms

Let us now introduce the concept of noncommutative atoms.

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Definition

$a \in L^2(\mathcal{M})$ is said to be a $(1,2)_c$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that

1. $\mathcal{E}_n(a) = 0$;
2. $r(a) \leq e$;
3. $\|a\|_2 \leq \tau(e)^{-1/2}$.

Replacing (ii) by the inequality $(\mathrm{ii})'$ $l(a) \leq e$, we get the notion of a $(1,2)_r$-atom.

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Here, $(1,2)_c$-atoms and $(1,2)_r$-atoms are noncommutative analogues of $(1,2)$-atoms for classical martingales.

In a later remark we will discuss the noncommutative analogue of $(p,2)$-atoms. These atoms satisfy the following useful estimations.

Proposition

If a is a $(1,2)_c$-atom then

$$\|a\|_{\mathcal{H}_c^1} \leq 1 \quad \textit{and} \quad \|a\|_{\mathsf{h}_c^1} \leq 1.$$

The same estimations hold for $(1,2)_r$-atoms.

Now, atomic Hardy spaces are defined as follows.

Definition

We define $\mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M})$ as the Banach space of $x \in L^1(\mathcal{M})$ which admit a decomposition

$$x = \sum_k \lambda_k a_k$$

with for each k, a_k a $(1,2)_c$-atom or an element in $L^1(\mathcal{M}_1)$ of norm ≤ 1, and $\lambda_k \in \mathbb{C}$ satisfaying $\sum_k |\lambda_k| < \infty$. We equip this space with the norm

$$\|x\|_{\mathsf{h}^1_{c,\mathrm{at}}} = \inf \sum_k |\lambda_k|$$

where the infimum is taken over all decompositions of x described above.

Similarly we define $\mathsf{h}^1_{r,\mathrm{at}}(\mathcal{M})$ and $\|\cdot\|_{\mathsf{h}^1_{r,\mathrm{at}}}$.

atoms

By Proposition 3.1 we have the contractive inclusion $\mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M}) \subset \mathsf{h}^1_c(\mathcal{M})$. The following theorem shows that these two spaces coincide. That establishes the atomic decomposition of the conditioned Hardy space $\mathsf{h}^1_c(\mathcal{M})$.

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Theorem

We have

$$\mathsf{h}^1_c(\mathcal{M}) = \mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M}) \quad \textit{with equivalent norms.}$$

More precisely, if $x \in \mathsf{h}^1_c(\mathcal{M})$

$$\frac{1}{2\sqrt{2}}\|x\|_{\mathsf{h}^1_{c,\mathrm{at}}} \leq \|x\|_{\mathsf{h}^1_c} \leq \|x\|_{\mathsf{h}^1_{c,\mathrm{at}}}. \tag{1}$$

Similarly, $\mathsf{h}^1_r(\mathcal{M}) = \mathsf{h}^1_{r,\mathrm{at}}(\mathcal{M})$ *with the same equivalence constants.*

atoms

We will show the remaining inclusion $\mathsf{h}_c^1(\mathcal{M}) \subset \mathsf{h}_{c,\mathrm{at}}^1(\mathcal{M})$ by duality. Recall that the dual space of $\mathsf{h}_c^1(\mathcal{M})$ is the space $\mathsf{bmo}_c(\mathcal{M})$ defined as follows (we refer to [6] and [7] for details).

We will show the remaining inclusion $\mathsf{h}_c^1(\mathcal{M}) \subset \mathsf{h}_{c,\mathrm{at}}^1(\mathcal{M})$ by duality. Recall that the dual space of $\mathsf{h}_c^1(\mathcal{M})$ is the space $\mathsf{bmo}_c(\mathcal{M})$ defined as follows (we refer to [6] and [7] for details).
Let

$$\mathsf{bmo}_c(\mathcal{M}) = \left\{x \in L^2(\mathcal{M}) : \sup_{n\geq 1} \|\mathcal{E}_n|x - x_n|^2\|_\infty < \infty\right\}$$

and equip $\mathsf{bmo}_c(\mathcal{M})$ with the norm

$$\|x\|_{\mathsf{bmo}_c} = \max\Big(\sup_{n\geq 1} \|\mathcal{E}_n|x - x_n|^2\|_\infty^{1/2}, \|\mathcal{E}_1(x)\|_\infty\Big).$$

This is a Banach space.

atoms

Similarly, we define the row version $\mathrm{bmo}_r(\mathcal{M})$. Note that

$$\mathcal{E}_n|x - x_n|^2 = \mathcal{E}_n\Big(\sum_{k\geq n+1} |dx_k|^2\Big). \tag{2}$$

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Since $x_n = \mathcal{E}_n(x)$, we have

$$\mathcal{E}_n|x - x_n|^2 = \mathcal{E}_n|x|^2 - |x_n|^2 \leq \mathcal{E}_n|x|^2.$$

Thus the contractivity of the conditional expectation yields

$$\|x\|_{\mathrm{bmo}_c} \leq \|x\|_\infty. \tag{3}$$

atoms

We will describe the dual space of $\mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M})$ as a noncommutative Lipschitz space defined as follows. We set

$$\Lambda_c(\mathcal{M}) = \left\{x \in L^2(\mathcal{M}) : \|x\|_{\Lambda_c} < \infty\right\}$$

with

$$\|x\|_{\Lambda_c} = \max\left(\sup_{n\geq 1}\sup_{e\in\mathcal{P}_n} \tau(e)^{-1/2}\tau\big(e|x-x_n|^2\big)^{1/2}, \|\mathcal{E}_1(x)\|_\infty\right)$$

where $\mathcal{P}_n$ denote the lattice of projections of $\mathcal{M}_n$.

Similarly we define

$$\Lambda_r(\mathcal{M}) = \left\{x \in L^2(\mathcal{M}) : x^* \in \Lambda_c(\mathcal{M})\right\}$$

equipped with the norm

$$\|x\|_{\Lambda_r} = \|x^*\|_{\Lambda_c}.$$

The relation between Lipschitz space and bmo space can be stated as follows.

atoms

Proposition

We have $\mathrm{bmo}_c(\mathcal{M}) = \Lambda_c(\mathcal{M})$ *and* $\mathrm{bmo}_r(\mathcal{M}) = \Lambda_r(\mathcal{M})$ *isometrically.*

We now turn to the duality between the conditioned atomic space $\mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M})$ and the Lipschitz space $\Lambda_c(\mathcal{M})$.

Theorem

We have $h^1_{c,\mathrm{at}}(\mathcal{M})^* = \Lambda_c(\mathcal{M})$ *isometrically. More precisely,*

1. *Every* $x \in \Lambda_c(\mathcal{M})$ *defines a continuous linear functional on* $\mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M})$ *by*

$$\varphi_x(y) = \tau(x^* y), \quad \forall y \in L^2(\mathcal{M}). \tag{4}$$

2. *Conversely, each* $\varphi \in \mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M})^*$ *is given as (8) by some* $x \in \Lambda_c(\mathcal{M})$.

Similarly, $\mathsf{h}^1_{r,\mathrm{at}}(\mathcal{M})^* = \Lambda_r(\mathcal{M})$ *isometrically.*

Remark

Remark that we have defined the duality bracket (4) *for operators in* $L^2(\mathcal{M})$*. This is sufficient for* $L^2(\mathcal{M})$ *is dense in* $\mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M})$*. To see this we write* $L^2(\mathcal{M}) = L^2_0(\mathcal{M}) \oplus L^2(\mathcal{M}_1)$*, where* $L^2_0(\mathcal{M}) = \{x \in L^2(\mathcal{M}) : \mathcal{E}_1(x) = 0\}$.

atoms

We can generalize this decomposition to the whole space $\mathsf{h}^1(\mathcal{M})$. To this end we need the following definition.

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Definition

We set

$$\mathsf{h}^1_{\mathrm{at}}(\mathcal{M}) = \mathsf{h}^1_d(\mathcal{M}) + \mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M}) + \mathsf{h}^1_{r,\mathrm{at}}(\mathcal{M}),$$

equipped with the sum norm

$$\|x\|_{\mathsf{h}^1_{\mathrm{at}}} = \inf\left\{\|w\|_{\mathsf{h}^1_d} + \|y\|_{\mathsf{h}^1_{c,\mathrm{at}}} + \|z\|_{\mathsf{h}^1_{r,\mathrm{at}}}\right\},$$

where the infimum is taken over all $w \in \mathsf{h}^1_d(\mathcal{M}), y \in \mathsf{h}^1_{c,\mathrm{at}}(\mathcal{M})$, and $z \in \mathsf{h}^1_{r,\mathrm{at}}(\mathcal{M})$ such that $x = w + y + z$.

Thus Theorem 3 clearly implies the following result.

Theorem

We have

$$\mathsf{h}^1(\mathcal{M}) = \mathsf{h}^1_{\mathrm{at}}(\mathcal{M}) \quad \textit{with equivalent norms.}$$

More precisely, if $x \in \mathsf{h}^1(\mathcal{M})$

$$\frac{1}{\sqrt{2}}\|x\|_{\mathsf{h}^1_{\mathrm{at}}} \leq \|x\|_{\mathsf{h}^1} \leq \|x\|_{\mathsf{h}^1_{\mathrm{at}}}.$$

The noncommutative Davis' decomposition presented in [7] state that $\mathcal{H}^1(\mathcal{M}) = \mathsf{h}^1(\mathcal{M})$. Thus Theorem 6 yields that $\mathcal{H}^1(\mathcal{M}) = \mathsf{h}^1_{\mathrm{at}}(\mathcal{M})$, which means that we can decompose any martingale in $\mathcal{H}^1(\mathcal{M})$ in an atomic part and a diagonal part. This is the atomic decomposition for the Hardy space of noncommutative martingales.

recent results

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They obtain a weak form of the atomic decomposition of hp for all $0 < p < 1$, and provide a constructive proof of the atomic decomposition for $p = 1$.

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Thank you!