International Conference on Noncommutative Geometry, Analysis on Groups, and Mathematical Physics

Ghent University, Belgium

February 26-27, 2024

Zero-trace commutators of measurable operators

A. M. Bikchentaev

Kazan Federal University

26.02.2024

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$, $\mathfrak{S}_p$ $(0 < p < +\infty)$ be the Shatten – von Neumann ideal in $\mathcal{B}(\mathcal{H})$.
An operator $X \in \mathcal{B}(\mathcal{H})$ is called a *commutator*, if $X = [A, B] = AB - BA$ for some $A, B \in \mathcal{B}(\mathcal{H})$.

If $\dim \mathcal{H} < +\infty$ then for $X \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent:

(i) X is zero-diagonal in some basis in $\mathcal{H}$;

(ii) $\mathrm{tr}(X) = 0$;

(iii) X is a commutator.

Introduction

If $\mathcal{H}$ is separable and $\dim \mathcal{H} = +\infty$, then $X \in \mathcal{B}(\mathcal{H})$ is a commutator $\Leftrightarrow$ X has no form $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in \mathcal{B}(\mathcal{H})$ is a compact operator [Brown, Pearcy, 1965]. Hence, every compact operator $K \in \mathcal{B}(\mathcal{H})$ has the form $K = [A, B]$ with some $A, B \in \mathcal{B}(\mathcal{H})$. For a suitable K from $\mathfrak{S}_1$ of trace class operators we have $[A, B] \in \mathfrak{S}_1$ with $\mathrm{tr}([A, B]) \neq 0$.

Introduction

If $\mathcal{H}$ is separable and $\dim \mathcal{H} = +\infty$, then $X \in \mathcal{B}(\mathcal{H})$ is a commutator $\Leftrightarrow$ X has no form $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in \mathcal{B}(\mathcal{H})$ is a compact operator [Brown, Pearcy, 1965]. Hence, every compact operator $K \in \mathcal{B}(\mathcal{H})$ has the form $K = [A, B]$ with some $A, B \in \mathcal{B}(\mathcal{H})$. For a suitable K from $\mathfrak{S}_1$ of trace class operators we have $[A, B] \in \mathfrak{S}_1$ with $\mathrm{tr}([A, B]) \neq 0$.

[Fan P., 1984]: Equivalence (i)$\Leftrightarrow$(ii) holds for $X \in \mathfrak{S}_1$.

[Fan P., Fong C. K., 1980]: a Hermitian compact operator $X \in \mathcal{B}(\mathcal{H})$ is a selfcommutator $[A^*, A]$ of a compact operator $A \in \mathcal{B}(\mathcal{H})$ $\Leftrightarrow$ condition (i) holds.

[Fan P., Fong C. K., Che K., Herrero D. A., 1987]: for a Hermitian operator $X \in \mathcal{B}(\mathcal{H})$:
Condition (i)$\Leftrightarrow$ $\mathrm{tr}(X_+) = \mathrm{tr}(X_-)$, where $X_+ = (|X| + X)/2$, $X_- = |X| - X_+$.

Introduction

[Fan P., Fong C. K., Che K. 1994]: for $X \in \mathcal{B}(\mathcal{H})$
Condition (i)$\Leftrightarrow \mathrm{tr}(\mathrm{Re}(e^{\mathrm{i}\theta}X)_+) = \mathrm{tr}(\mathrm{Im}(e^{\mathrm{i}\theta}X)_-)$ for all $\theta,\ 0 \le \theta < 2\pi$.

If $U \in \mathcal{B}(\mathcal{H})$ is a non-unitary isometry and $\dim(I - UU^*)\mathcal{H} < +\infty$ then the trace of a selfcommutator $[U^*, U] = I - UU^*$ is non-zero. If $U = X + \mathrm{i}Y$ is the Cartesian decomposition of U with $X, Y \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ then $[U^*, U] = 2\mathrm{i}[X, Y]$, i.e., there exist Hermitian bounded operators such that their commutator lies in $\mathfrak{S}_1$ and has non-zero trace.

Introduction

[Fan P., Fong C. K., Che K. 1994]: for $X \in \mathcal{B}(\mathcal{H})$
Condition (i)$\Leftrightarrow \mathrm{tr}(\mathrm{Re}(e^{\mathrm{i}\theta}X)_+) = \mathrm{tr}(\mathrm{Im}(e^{\mathrm{i}\theta}X)_-)$ for all θ, $0 \le \theta < 2\pi$.

If $U \in \mathcal{B}(\mathcal{H})$ is a non-unitary isometry and $\dim(I - UU^*)\mathcal{H} < +\infty$ then the trace of a selfcommutator $[U^*, U] = I - UU^*$ is non-zero. If $U = X + \mathrm{i}Y$ is the Cartesian decomposition of U with $X, Y \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ then $[U^*, U] = 2\mathrm{i}[X, Y]$, i.e., there exist Hermitian bounded operators such that their commutator lies in $\mathfrak{S}_1$ and has non-zero trace.

But if $X \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ and an operator $Y \in \mathcal{B}(\mathcal{H})$ is compact with $[X, Y] \in \mathfrak{S}_1$ then $\mathrm{tr}([X, Y]) = 0$ [Helton J., Howe R., 1975].

Introduction

[Weiss G., 1978] showed that $\mathrm{tr}([T, X]) = 0$ for normal operator $T \in \mathcal{B}(\mathcal{H})$ and $X \in \mathfrak{S}_2$ with $[T, X] \in \mathfrak{S}_1$.

[Kittaneh F., 1986] generalized this result to some non normal operators.

[Kittaneh F., 1991] for $T \in \mathcal{B}(\mathcal{H})$ and $X \in \mathfrak{S}_2$ with $[T, X] \in \mathfrak{S}_1$ proved that $\mathrm{tr}([T, X]) = 0$ under one of conditions: a) T^2 is normal, or b) T^n is normal for some integer $n > 2$ and $[T^*, T] \in \mathfrak{S}_1$.

Notation and Definitions

Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, let $\mathcal{M}^+$ be the positive part of $\mathcal{M}$, let I be the unit of $\mathcal{M}$. Let $\mathcal{M}^{\rm pr}$ be the projection lattice of $\mathcal{M}$ and let $P^\perp = I - P$ for $P \in \mathcal{M}^{\rm pr}$.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if

- $\varphi(X + Y) = \varphi(X) + \varphi(Y)$ for all $X, Y \in \mathcal{M}^+$;
- $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$);
- $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$.

Notation and Definitions

Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, let $\mathcal{M}^+$ be the positive part of $\mathcal{M}$, let I be the unit of $\mathcal{M}$. Let $\mathcal{M}^{\mathrm{pr}}$ be the projection lattice of $\mathcal{M}$ and let $P^\perp = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if

- $\varphi(X + Y) = \varphi(X) + \varphi(Y)$ for all $X, Y \in \mathcal{M}^+$;
- $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$);
- $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$.

A trace φ is called

- *faithful*, if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$;
- *normal*, if $X_i \nearrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$;
- *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : \ Y \in \mathcal{M}^+, \ Y \leq X, \ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

Definitions

Let τ be a faithful normal semifinite trace on $\mathcal{M}$. An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possesing a domain $\mathfrak{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be *τ-measurable* if, for any $\varepsilon > 0$, there exists a projection $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$.

Definitions

Let τ be a faithful normal semifinite trace on $\mathcal{M}$. An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possesing a domain $\mathfrak{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be τ*-measurable* if, for any $\varepsilon > 0$, there exists a projection $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$.

The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations.

Definitions

The generalized singular value function $\mu(\cdot; X) : t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(t; X) = \inf\{\|XP\| : \quad P \in \mathcal{M}^{\mathrm{pr}} \ \text{ and } \ \tau(P^{\perp}) \leq t\}, \quad t > 0.$$

Definitions

The generalized singular value function $\mu(\cdot; X) : t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(t; X) = \inf\{\|XP\| : \ P \in \mathcal{M}^{\rm pr} \ \text{and} \ \tau(P^{\perp}) \le t\}, \quad t > 0.$$

Let m be the linear Lebesgue measure on $\mathbb{R}$. Noncommutative Lebesgue L_p-space $(0 < p < \infty)$, assosiated with $(\mathcal{M}, \tau)$, may be defined as

$$L_p(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \ \mu(\cdot; X) \in L_p(\mathbb{R}^+, m)\}$$

with the F-norm (norm for $1 \le p < \infty$) $\|X\|_p = \|\mu(\cdot; X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. The extension of τ to the unique linear functional on the whole space $L_1(\mathcal{M}, \tau)$ we denote by the same letter τ.

Examples

Example 1. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, and $\tau = \mathrm{tr}$ is the canonical trace, then $S(\mathcal{M}, \tau)$ coincides with $\mathcal{B}(\mathcal{H})$, the space $L_p(\mathcal{M}, \tau)$ coincides with the Shatten – von Neumann *-ideal $\mathfrak{S}_p$ of compact operators in $\mathcal{B}(\mathcal{H})$ and

$$\mu(t; X) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of X; χ_A is the indicator function of the set $A \subset \mathbb{R}$.

Examples

Example 1. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, and $\tau = \mathrm{tr}$ is the canonical trace, then $S(\mathcal{M}, \tau)$ coincides with $\mathcal{B}(\mathcal{H})$, the space $L_p(\mathcal{M}, \tau)$ coincides with the Shatten – von Neumann *-ideal $\mathfrak{S}_p$ of compact operators in $\mathcal{B}(\mathcal{H})$ and

$$\mu(t; X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of X; χ_A is the indicator function of the set $A \subset \mathbb{R}$.

Example 2. If $\mathcal{M}$ is Abelian (i. e., commutative), then $\mathcal{M} \simeq L^{\infty}(\Omega, \Sigma, \nu)$ and $\tau(f) = \int\limits_{\Omega} f \, \mathrm{d}\, \nu$, where (Ω, Σ, ν) is a localized measure space, the *-algebra $S(\mathcal{M}, \tau)$ coincides with the algebra of all complex measurable functions f on (Ω, Σ, ν), bounded everywhere but for a set of finite measure. The function $\mu(t; f)$ coincides with the nonincreasing rearrangement of the function $|f|$.

Results

Theorem 1. If the trace τ is infinite, then the positive selfcommutator $[A^*, A]$ $(A \in S(\mathcal{M}, \tau))$ cannot have the form $\lambda I + K$, where λ is a non-zero complex number and an operator K is τ-compact.

Results

Theorem 1. If the trace τ is infinite, then the positive selfcommutator $[A^*, A]$ $(A \in S(\mathcal{M}, \tau))$ cannot have the form $\lambda I + K$, where λ is a non-zero complex number and an operator K is τ-compact.

Theorem 2. (a generalization of C. Putnam Theorem, 1951):
a positive selfcommutator $[A^*, A]$ $(A \in S(\mathcal{M}, \tau))$ cannot have the inverse in $\mathcal{M}$.

We have two different proofs of Theorem 2.
The first proof is published in [Bikchentaev, LJM, 2023]; the second proof will be published in [Bikchentaev, Siberian Math. J., 2024].

The main Question

Let $L_1(\mathcal{M},\tau)$ be the Banach space of all τ-integrable operators.
Let $A, B \in S(\mathcal{M},\tau)$ and $[A,B] \in L_1(\mathcal{M},\tau)$.

Question: under which conditions $\tau([A,B]) = 0$?

[Bikchentaev, Proc. Steklov Inst. Math. of RAS, 2016, **Theorem 4.8**]:

If $\tau(I) = 1$ and $X \in L_1(\mathcal{M},\tau)$ then

$\tau(X) = 0 \Longleftrightarrow \|I + zX\|_1 \geq 1$ for all $z \in \mathbb{C}$.

Theorems 3 and 4

Theorem 3. If $X \in S(\mathcal{M}, \tau)$, $Y = Y^3 \in \mathcal{M}$ and $[X, Y] \in L_1(\mathcal{M}, \tau)$, then $\tau([X, Y]) = 0$.

Theorems 3 and 4

Theorem 3. If $X \in S(\mathcal{M}, \tau)$, $Y = Y^3 \in \mathcal{M}$ and $[X, Y] \in L_1(\mathcal{M}, \tau)$, then $\tau([X, Y]) = 0$.

Theorem 4. If $A^2 = A \in S(\mathcal{M}, \tau)$ and $[A^*, A] \in L_1(\mathcal{M}, \tau)$, then $\tau([A^*, A]) = 0$.

Corollaries from Theorem 4

Corollary 1. If $X = X^3 \in S(\mathcal{M}, \tau)$, and if an operator $X^2 - X$ is Hermitian and $[X^*, X] \in L_1(\mathcal{M}, \tau)$ then $\tau([X^*, X]) = 0$.

Corollary 2. If $X \in S(\mathcal{M}, \tau)$ with $X^2 = I$ and $[X^*, X] \in L_1(\mathcal{M}, \tau)$ then $\tau([X^*, X]) = 0$.

Proof. The formula $X = 2P - I$ $(P \in S(\mathcal{M}, \tau)^{\mathrm{id}})$ establishes a bijection between $S(\mathcal{M}, \tau)^{\mathrm{id}}$ and the set of all symmetries from $S(\mathcal{M}, \tau)$.

Corollaries from Theorem 4

Corollary 1. If $X = X^3 \in S(\mathcal{M}, \tau)$, and if an operator $X^2 - X$ is Hermitian and $[X^*, X] \in L_1(\mathcal{M}, \tau)$ then $\tau([X^*, X]) = 0$.

Corollary 2. If $X \in S(\mathcal{M}, \tau)$ with $X^2 = I$ and $[X^*, X] \in L_1(\mathcal{M}, \tau)$ then $\tau([X^*, X]) = 0$.

Proof. The formula $X = 2P - I$ $(P \in S(\mathcal{M}, \tau)^{\mathrm{id}})$ establishes a bijection between $S(\mathcal{M}, \tau)^{\mathrm{id}}$ and the set of all symmetries from $S(\mathcal{M}, \tau)$.

For examples of unbounded idempotents $P \in S(\mathcal{M}, \tau)$ see [Bikchentaev, Math. Notes, 2015, 2016].

Theorems 5 and 6

Theorem 5. If a partial isometry U lies in $\mathcal{M}$ and $U^n = 0$ for some integer $n \geq 2$, then the operator U^{n-1} is a commutator and if $U^{n-1} \in L_1(\mathcal{M}, \tau)$, then $\tau(U^{n-1}) = 0$.

Corollary 3. If a partial isometry $U \in L_1(\mathcal{M}, \tau)$ and the projections $P = U^*U$, $Q = UU^*$ are mutually orthogonal then $U^2 = 0$. Hence, U is a commutator and $\tau(U) = 0$.

Theorems 5 and 6

Theorem 5. If a partial isometry U lies in $\mathcal{M}$ and $U^n = 0$ for some integer $n \geq 2$, then the operator U^{n-1} is a commutator and if $U^{n-1} \in L_1(\mathcal{M}, \tau)$, then $\tau(U^{n-1}) = 0$.

Corollary 3. If a partial isometry $U \in L_1(\mathcal{M}, \tau)$ and the projections $P = U^*U$, $Q = UU^*$ are mutually orthogonal then $U^2 = 0$. Hence, U is a commutator and $\tau(U) = 0$.

Theorem 6. If $P, Q \in S(\mathcal{M}, \tau)^{\mathrm{id}}$ and $P - Q \in L_1(\mathcal{M}, \tau)$, then $\tau(P - Q) \in \mathbb{R}$.

Corollary 4. If $A = A^3 \in L_1(\mathcal{M}, \tau)$, then $\tau(A) \in \mathbb{R}$.

Corollary 5. Assume that $A, B \in S(\mathcal{M}, \tau)$ are tripotents. If $A - B \in L_1(\mathcal{M}, \tau)$ and $A + B \in \mathcal{M}$, then $\tau(A - B) \in \mathbb{R}$.

Corollary 6

Corollary 6. Let $U, V \in S(\mathcal{M}, \tau)$ be symmetries. If $U - V \in L_1(\mathcal{M}, \tau)$, then $\tau(U - V) \in \mathbb{R}$.

Acknowledgments. The work is performed under the development program of Volga Region Mathematical Center (agreement no. 075-02-2023-944).

Thank you!