# Tracial weights on topological graph $C^*$-algebras

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# Outline of talk

1. Topological graphs and their $C^*$-algebras

2. Traces on topological graph $C^*$-algebras

3. Loops in graphs

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# Topological graphs

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**Definition**

*A topological graph* $E = (E^0, E^1, r, s)$ *consists of*

- *two locally compact Hausdorff spaces* $E^0$ *and* $E^1$,
- *a local homeomorphism* $s : E^1 \to E^0$, *and*
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## Definition

*A map* $\varphi : X \to Y$ *is a local homeomorphism if it satisfies that for every* $x \in X$ *there exists an open neighbourhood* $U$ *of* $x$ *with* $\varphi(U)$ *open and* $\varphi|_U : U \to \varphi(U)$ *a homeomorphism.*

# Topological graphs

**Example (Directed graphs)**

Recall: A countable directed graph $E = (E^0, E^1, r, s)$ is

- a countable set of vertices $E^0$ and edges $E^1$,
- two maps $r, s : E^1 \to E^0$ identifying the range and source of an edge.

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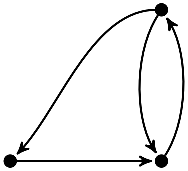

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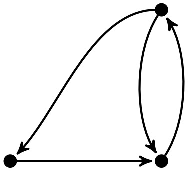

$\rightsquigarrow$ A directed graph is a second countable discrete topological graph.

**Example ($\mathbb{Z}$-actions)**

Let $X$ be a locally compact Hausdorff space and let $\varphi : X \to X$ be a homeomorphism. Then we can define a topological graph $E_X = (E_X^0, E_X^1, r, s)$ where

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- $E_X^0 = E_X^1 = X$,
- $s = \mathrm{id}_X : E_X^1 \to E_x^0$ and
- $r = \varphi : E_X^1 \to E_x^0$.

# $C^*$-algebras of topological graphs

**Construction (Katsura 2004)**

*We can associate a topological graph $C^*$-algebra $C^*(E)$ to any topological graph $E$.*

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**Remark**

*The $C^*$-algebra $C^*(E)$ is separable if and only if $E^0$ and $E^1$ are second countable.*

# C*-algebras of topological graphs

**Construction (Katsura 2004)**

*We can associate a topological graph C*-algebra $C^*(E)$ to any topological graph $E$.*

**Remark**

*The C*-algebra $C^*(E)$ is separable if and only if $E^0$ and $E^1$ are second countable.*

☞ We assume that $E^0$ and $E^1$ are second countable

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Let $E = (E^0, E^1, r, s)$ be a countable directed graph, and consider it as a topological graph with the discrete topology on $E^0$ and $E^1$.

# C*-algebras of topological graphs

**Example (Directed graphs)**

Let $E = (E^0, E^1, r, s)$ be a countable directed graph, and consider it as a topological graph with the discrete topology on $E^0$ and $E^1$. Then the topological graph C*-algebra $C^*(E)$ is the Cuntz-Krieger graph C*-algebra.

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Let $X$ be a locally compact Hausdorff space and let $\varphi : X \to X$ be a homeomorphism. Consider the associated topological graph

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Then the topological graph C*-algebra is isomorphic to the crossed product C*-algebra of the homeomorphism $\varphi$, i.e.

$$C^*(E) = C(X) \rtimes_\varphi \mathbb{Z} .$$

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- All Kirchberg algebras satisfying the UCT.

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The following $C^*$-algebras can be realised as topological graph $C^*$-algebras

- All commutative $C^*$-algebras.
- All approximately finite-dimensional (AF) $C^*$-algebras.
- All Kirchberg algebras satisfying the UCT.

Can all nuclear separable $C^*$-algebras satisfying the UCT be realised as topological graph algebras?

# Vertices and paths

**Notation**

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- $E^0$ *the space of vertices,*
- $E^1$ *the space of edges,*
- $E^*$ *the set of finite paths, i.e. concatenations* $\alpha_1\alpha_2\cdots\alpha_n$ *with* $n \in \mathbb{N}$ *and* $s(\alpha_i) = r(\alpha_{i+1})$ *for all* $i < n$.

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- $E^\infty$ *the set of infinite paths* $\alpha_1\alpha_2\cdots$ *with* $s(\alpha_i) = r(\alpha_{i+1})$ *for all* $i$.

# Groupoid C*-algebra

**Theorem (Yeend 2007)**

*Let* $E = (E^0, E^1, r, s)$ *be a second countable topological graph. There exists an amenable locally compact second countable Hausdorff étale groupoid* $\mathcal{G}_E$ *such that*

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- *there exists a canonical groupoid homomorphism $\Phi : \mathcal{G}_E \to \mathbb{Z}$.*

*$\Phi$ gives rise to an action of $\mathbb{T}$ on $C^*(E)$ which we denote the gauge-action.*

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**Definition**

*Let $\mathcal{A}$ be a $C^*$-algebra. We call a state $\psi$ on $\mathcal{A}$ a tracial state if*

$$\psi(ab) = \psi(ba) \quad \text{for all } a, b \in \mathcal{A}.$$

# Tracial states on topological graphs

**Definition**

*Let $\mu$ be a regular Borel measure on $E^0$. We call $\mu$ a vertex-invariant measure if it satisfies*

$$\int_{E^1} f \circ r \,\mathrm{d}s^*\mu \leq \int_{E^0} f \,\mathrm{d}\mu$$

*for all positive functions $f \in C_c(E^0)$, and with equality when $supp(f) \subseteq E^0_{reg}$.*

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**Definition**

*Let $\nu$ be a regular Borel measure on $\partial E$. We let $\sigma$ on $\partial E (\subseteq E^* \sqcup E^\infty)$ denote the shift map, i.e. $\sigma(\alpha_1\alpha_2\cdots) = \alpha_2\cdots$.*

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**Definition**

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$$\nu(\sigma(A)) = \nu(A) \quad \text{for all Borel } A \subseteq \partial E \setminus E^0 \text{ with } \sigma \text{ injective on } A.$$

# Tracial states on topological graphs

**Theorem (Schafhauser 2016)**

*Let $E = (E^0, E^1, r, s)$ be a topological graph. There is a bijection between*

(1) *The vertex-invariant probability measures $\mu$ on $E^0$.*

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(3) *The gauge-invariant tracial states on $C^*(E)$.*

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# Traces on topological graphs

**Conjecture (Schafhauser 2016)**

*Let $E$ be a topological graph. All tracial states on $C^*(E)$ are gauge-invariant when $E$ is free.*

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*Free is a simultaneous generalisation of the notion of condition (K) for directed graphs and the notion of freeness for actions of $\mathbb{Z}$.*

**Theorem (Katsura 2006)**

*Simplicity of the $C^*$-algebra $C^*(E)$ implies that the topological graph $E$ is free.*

# Traces on topological graphs

**Theorem (C. 2018)**

*We can, loosely speaking, associate a non-gauge invariant tracial weight to a $\sigma$-invariant measure $\nu$ on $\partial E$ if and only if $Per(x) := \{k - l \in \mathbb{Z} \mid \sigma^k(x) = \sigma^l(x)\} \neq \{0\}$ on a set of positive $\nu$-measure.*

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**Definition**

*Let $E$ be a topological graph. We call $\alpha = \alpha_1 \cdots \alpha_n \in E^*$ a loop if $|\alpha| = n \geq 1$ and $s(\alpha) = r(\alpha)$ where $s(\alpha) = s(\alpha_n)$ and $r(\alpha) = r(\alpha_1)$.*

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**Definition**

*We call an element $x \in \partial E$ an eventually cyclic path if there exists a loop $\alpha \in E^*$ and a $\beta \in E^*$ with $s(\beta) = r(\alpha)$ such that*

$$x = \beta\alpha^\infty .$$

# Loops

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*Let E be a topological graph. We call* $\alpha = \alpha_1 \cdots \alpha_n \in E^*$ *a loop if* $|\alpha| = n \geq 1$ *and* $s(\alpha) = r(\alpha)$ *where* $s(\alpha) = s(\alpha_n)$ *and* $r(\alpha) = r(\alpha_1)$*.*

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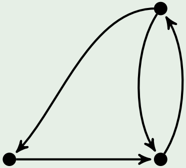
$$x = \beta\alpha^\infty .$$

## Observation

*An element* $x \in \partial E$ *satisfies* $Per(x) \neq \{0\}$ *if and only if x is an eventually cyclic path.*

**Example**



# Eventually cyclic paths

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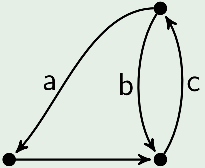

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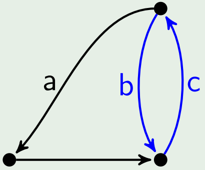


The path $cb$ is an example of a loop.

# Eventually cyclic paths

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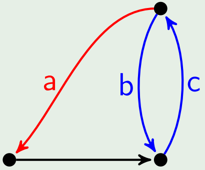


The infinite path $acbcbcb\cdots$ is an eventually cyclic path.

# Eventually cyclic paths

**Definition (C. 2022)**

*Let $E$ be a topological graph and let $\alpha$ be a loop in $E$. Set*

$$E^*_\alpha = \{\beta \in E^* \mid s(\beta) = r(\alpha)\ ,\ \beta \neq \beta'\alpha \text{ for some } \beta'\}\ .$$

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$$E_\alpha^* = \{\beta \in E^* \mid s(\beta) = r(\alpha)\ ,\ \beta \neq \beta'\alpha \text{ for some } \beta'\}\ .$$

*We call a loop $\alpha \in E^*$ summable if*

- *$\alpha$ is not on the form $\gamma^n$ for a $\gamma \in E^*$ and a $n > 1$, and*
- *for all $w \in E^0$ there exists an open neighbourhood $V_w$ of $w$ with*

$$|\{x \in E_\alpha^* \mid r(x) \in V_w\}| < \infty\ .$$

# Main results

**Theorem (C. 2022)**

*Let $E$ be a topological graph. All tracial weights on $C^*(E)$ are gauge-invariant if and only if there does not exists a summable loop $\alpha$ in $E$.*

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**Theorem (C. 2022)**

*Let $E$ be a topological graph. All tracial states on $C^*(E)$ are gauge-invariant if and only if there does not exists a summable loop $\alpha$ in $E$ with $|E_\alpha^*| < \infty$.*

# Main results

**Theorem**

*Let $E$ be a topological graph. If $E$ is free, then all tracial weights on $C^*(E)$ are gauge-invariant.*

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**Theorem (C. 2022)**

*Let $E = (E^0, E^1, r, s)$ be a topological graph. If $E$ is free then all tracial weights are gauge-invariant. This implies that there is a bijection between*

(1) *The vertex-invariant regular measures $\mu$ on $E^0$.*

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Thank you for your attention