Quantum Geodesics on Graphs

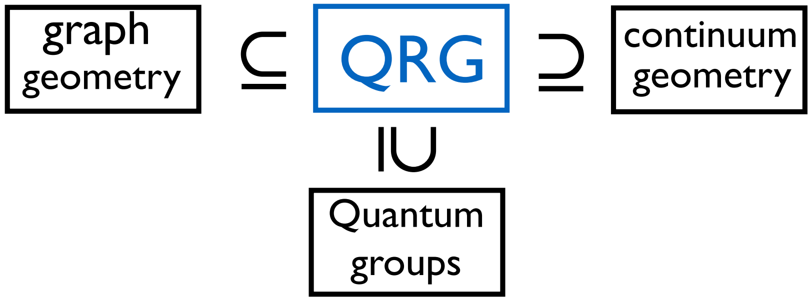
Shahn Majid (QMUL)

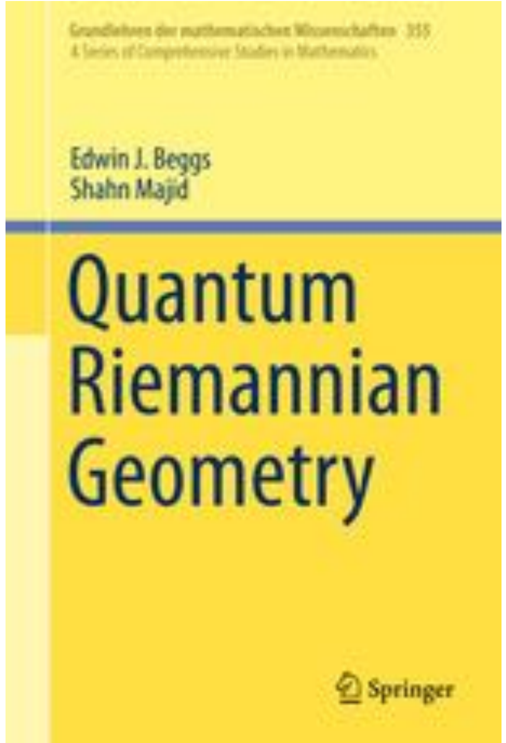
Ghent (online) 2024

I Formalism of QRG

$$g \in \Omega^1 \otimes_A \Omega^1, \quad \nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$$

Leibniz rule ➡ $\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$





II Quantum geodesics

Beggs JGP 2020 (key original work)

Beggs and SM JMP 2024 (Schrödinger eqn as a quantum geodesic flow on Heisenberg algebra)

Beggs & SM LMP 2023 (curved QRG examples eg fuzzy spheres)

Beggs and SM arXiv:2312 (application to graphs)

Liu and SM, JPhysA 2022 (quantum spacetime example)

Differential calculus

Classically, $C^\infty(M) = \Omega^0(M) \subset \Omega(M) = \oplus_i \Omega^i(M)$

Ω^1 space of 1-forms, e.g. `differentials' $\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i} \mathrm{d}x^i$

$$f\mathrm{d}g = (\mathrm{d}g)f \in \Omega^1$$

$$\wedge : \Omega \otimes_A \Omega \to \Omega, \quad \mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge \mathrm{d}\eta$$

$$\omega \wedge \eta = (-1)^{|\omega||\eta|}\eta \wedge \omega, \quad \mathrm{d}^2 = 0$$

graded Leibniz rule

- Algebra A over k; drop the (graded) commutativity but keep:

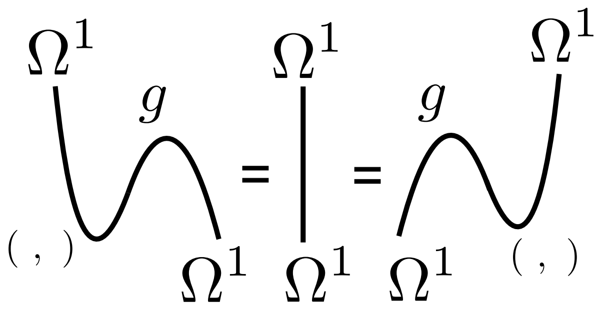
Ω^1	$a((\mathrm{d}b)c)=(a(\mathrm{d}b))c$	bimodule
$\mathrm{d} : A \to \Omega^1$	$\mathrm{d}(ab)=(\mathrm{d}a)b+a(\mathrm{d}b)$	Leibniz rule
$\{\sum a\mathrm{d}b\} = \Omega^1$		surjectivity

- Extend to DGA of differential forms (generated by A, $\mathrm{d}A$) $\quad \Omega = T_A\Omega^1/\mathcal{I} = \oplus_n \Omega^n, \quad \mathrm{d}^2 = 0$

<u>Propn</u>. Every Ω^1 has a maximal prolongation Ω_{max}

metrics and connections

- Metric $\quad g \in \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \mathrm{d}s^2 = \sum_{\mu,\nu} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$



(i) bimodule map inverse $\qquad (\ , \ ) : \Omega^1 \underset{A}{\otimes} \Omega^1 \to A$

(ii) some form of quantum symmetry e.g. $\wedge(g) = 0$

Lemma. a quantum metric is necessarily central

Proof: $\quad (\omega, ag^1)g^2 = (\omega a, g^1)g^2 = \omega a = (\omega, g^1)g^2 a$

- Connection/covariant derivative

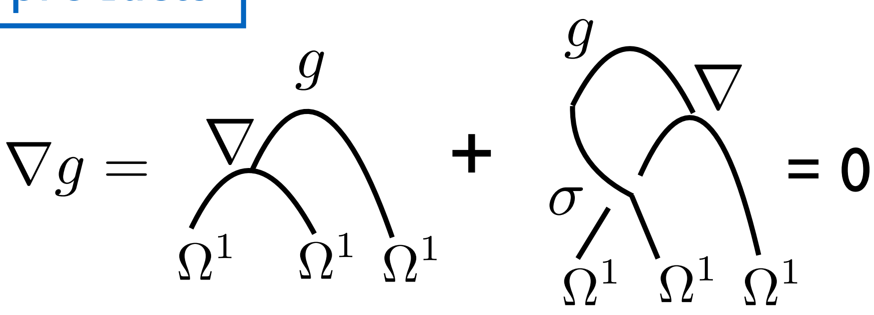
$$\nabla : \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \sigma : \Omega^1 \underset{A}{\otimes} \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1$$

$$\nabla(f\omega) = \mathrm{d}f \otimes \omega + f\nabla\omega \qquad \nabla(\omega f) = \sigma(\omega \otimes \mathrm{d}f) + (\nabla\omega)f$$

Lemma: bimodule connections extend to tensor products

$$\nabla(\omega \otimes \eta) = \nabla\omega \otimes \eta + (\sigma \otimes \mathrm{id})(\omega \otimes \nabla\eta)$$

→ metric compatibility



QLCs, curvature, *-structure and Dirac op

- Quantum Levi-Civita connection (QLC) $T_\nabla = \nabla g = 0$

$$T_\nabla : \Omega^1 \to \Omega^2 \qquad T_\nabla = \wedge\nabla - \mathrm{d}$$ torsion tensor

- Curvature tensor $R_\nabla : \Omega^1 \to \Omega^2 \otimes_A \Omega^1, \quad R_\nabla = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\nabla$

For *A* a *-algebra, require

- $*$ extends to (Ω, d) as an antilinear graded anti-involution commuting with d
- $\dagger(g) = g, \quad \dagger = \mathrm{flip}(* \otimes *) : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ `real'
- $\dagger\nabla = \sigma^{-1}\nabla *$ ➡ $\dagger\sigma = \sigma^{-1}\dagger$ *-preserving ➡ *-compatible
- $\int : A \to \mathbb{C}$ positive linear functional and e.g. trace, $\int \delta = 0, \quad \delta = (\ ,\ )\nabla : \Omega^1 \to A$
- Geometric spectral triples $D = \triangleright \circ \nabla_S$ for Cliff action and spinor bundle/conn

Defer functional analysis to the point where actually solve equations

Quantum differentials on finite sets

<u>Propn</u>: X a finite set $A = \mathbb{C}(X)$, Ω^1 ⟷ Directed graph with vertices X

$$\Omega^1 = \mathrm{span}_{\mathbb{C}}\{e_{x\to y}\} \quad f.e_{x\to y} = f(x)e_{x\to y}, \quad e_{x\to y}.f = e_{x\to y}f(y) \quad e^*_{x\to y} = -e_{y\to x}$$

if bidirected

$$\mathrm{d}f = \sum_{x\to y}(f(y)-f(x))e_{x\to y}$$

bilocal object ➜ bimodule

$T_A\Omega^1$ = Path algebra, in degree *i* $\Omega^{1\otimes i} = \{e_{x_1\to x_2\to\cdots\to x_i}\}$

- $\Omega_{max} = T_A\Omega^1$/ relations $\displaystyle\sum_{y:p\to y\to q} e_{p\to y}\wedge e_{y\to q} = 0$

  $p \neq q, \quad \text{not } p \to q$

  For Ω_{min} take all *p*,*q*

- <u>metric</u> $g = \sum_{x\to y} g_{x\to y}e_{x\to y}\otimes e_{y\to x}, \quad g_{x\to y}\in\mathbb{R}\setminus\{0\}$ if bidirected

edge symmetric if $g_{x\to y} = g_{y\to x}$ ➜ real \`square-length' on each edge

Classical geodesic flows

- dust particles moving on geodesics → tangents define vector field X_s obeying *geodesic velocity equation*

$$\dot{X}_s + \nabla_{X_s} X_s = 0$$

we reverse usual concept and first solve for this X_s for flow to time s

density ρ obeys continuity *equation*

$$\dot{\rho} = -X_s(\mathrm{d}\rho) - \rho\,\mathrm{div}(X_s)$$

- Let $\rho = |\psi|^2$ for a wave function ψ obeying the amplitude flow equation

$$\dot{\psi} = -X_s(\mathrm{d}\psi) - \frac{1}{2}\psi\,\mathrm{div}(X_s)$$

→ Convective derivative $\frac{D}{Ds} := \frac{\partial}{\partial s} + X_s$ of the divergence is the Ricci tensor

$$\frac{D\mathrm{div}(X_s)}{Ds} = -X^j{}_{;i}X^i{}_{;j} - X^i X^j \mathrm{Ricci}_{ij}$$

- Wave function $\psi(x,t)$ on spacetime, s is external geodesic proper time

Quantum geodesic equations

$A,\ \Omega^1,\ \mathrm{d},\ g,\ \nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ left conn, eg. QLC

➡ *right* conn $\nabla_\chi : \chi \to \chi \otimes_A \Omega^1$ on $\Omega^{-1} := {}_A\mathrm{Hom}(\Omega^1, A)$

- $\int$ non-deg ➡ $\mathrm{div}_\int$ defined by $\int (a\,\mathrm{div}_\int(X) + X(\mathrm{d}a)) = 0 \quad \forall a \in A,$

$$\kappa_s = \frac{1}{2}\mathrm{div}_\int(X_s)$$ flow divergence

$$\dot{X}_s + [X_s, \kappa_s] + (\mathrm{id} \otimes X_s)\nabla_\chi(X_s) = 0$$ velocity flow

$$\dot{\psi}_s = -\psi_s\kappa_s - X_s(\mathrm{d}\psi_s)$$ amplitude flow

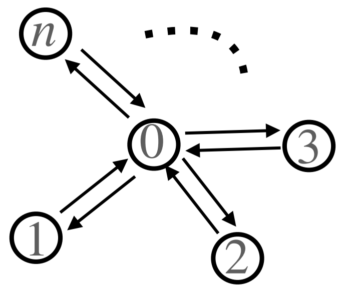
- Need $\int X_s(\omega^*) - X_s(\omega)^* = 0 \quad \forall \omega \in \Omega^1$ say X_s real with respect to $\int$

<u>Lemma</u> Then $\int \psi_s^*\psi_s$ constant in s ➡ probabilistic picture

- Can add driving force F to velocity equation to ensure X_0 real ➡ X_s real
- Can extend to $E = C^\infty(\mathbb{R}, \mathcal{H}) \ni \psi$, A acts on $\mathcal{H}$ e.g. Schroedinger repn

QRG of the n-star graph

w/ Beggs arXiv 2023



$$\sum_{i=1}^{n} e_{0\to i\to 0} = 0, \qquad \Omega^2_{min} \text{ is } n-1 \text{ dimensional} \quad \Omega^{i>3}_{min}=0$$

Thm. There exists QLC iff $n \le 4$ and $\frac{g_{i\to 0}}{g_{0\to i}} = \sqrt{n}$

$$\sigma(e_{0\to i}\otimes e_{i\to 0}) = \frac{q^{-1}}{\sqrt{n}} e_{0\to i}\otimes e_{i\to 0} + (1+\frac{q^{-1}}{\sqrt{n}})\sum_{j\neq i} e_{0\to j}\otimes e_{j\to 0},$$

$$\sigma(e_{i\to 0}\otimes e_{0\to i}) = q e_{i\to 0}\otimes e_{0\to i},$$

$$\sigma(e_{i\to 0}\otimes e_{0\to j}) = -\frac{g_{j\to 0}}{g_{i\to 0}} q^{-1} e_{i\to 0}\otimes e_{0\to j}$$

$$q = \begin{cases} e^{\frac{3i\pi}{4}} & n=2 \\ e^{\frac{5i\pi}{6}} & n=3 \\ -1 & n=4 \end{cases}$$

$$\nabla e_{0\to i} = \sum_j e_{j\to 0}\otimes e_{0\to i} - \sigma(e_{0\to i}\otimes e_{i\to 0})$$

$$\nabla e_{i\to 0} = e_{0\to i}\otimes e_{i\to 0} - \sum_j \sigma(e_{i\to 0}\otimes e_{0\to j})$$

(extends to $U(1)$ moduli of QLCs for $n=2$)

Similarly for A_n graph, QRG needs greater metric pointing into the bulk

Quantum geodesic flow on n-star graph

$$\int f = \sum_X \mu(x) f(x) \quad \rightarrow \quad (X^{y\leftarrow x})^* = -\frac{\mu_y}{\mu_x} X^{x\leftarrow y} \qquad \text{real w.r.t.} \int$$

$$\mathrm{div}_\int(X)(x) = \sum_{y:x\to y} X^{y\leftarrow x} - \sum_{y:y\to x} \frac{\mu_y}{\mu_x} X^{x\leftarrow y} \quad \rightarrow \quad \kappa_s = \frac{1}{2}\mathrm{div}_\int(X_s)$$

<u>Propn</u>. For *n*-star graph the geodesic velocity eqn with driving force is

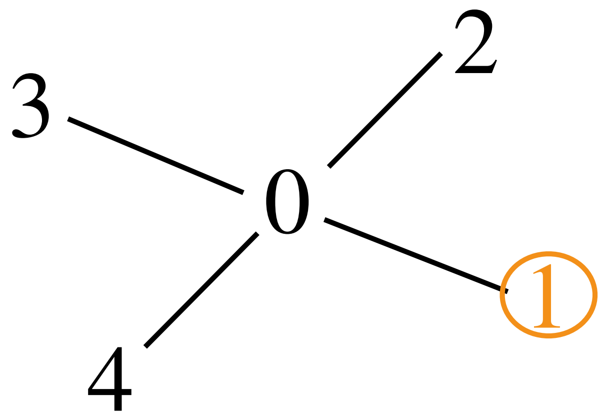
$$-\dot{X}^{0\leftarrow y} = \frac{1}{2} X^{0\leftarrow y}\left(- X^{0\leftarrow y} + \sum_i \frac{\mu_i}{\mu_0} X^{0\leftarrow i} - \sum_i (X^{0\leftarrow i})^* \frac{\mu_i\, g_{i\to 0}}{\mu_0 g_{y\to 0}} + \left(2\frac{\mu_y}{\mu_0} - 1\right)(X^{0\leftarrow y})^* \right) + \frac{1}{4}\sum_i \frac{\mu_i}{\mu_y} |X^{0\leftarrow i}|^2$$

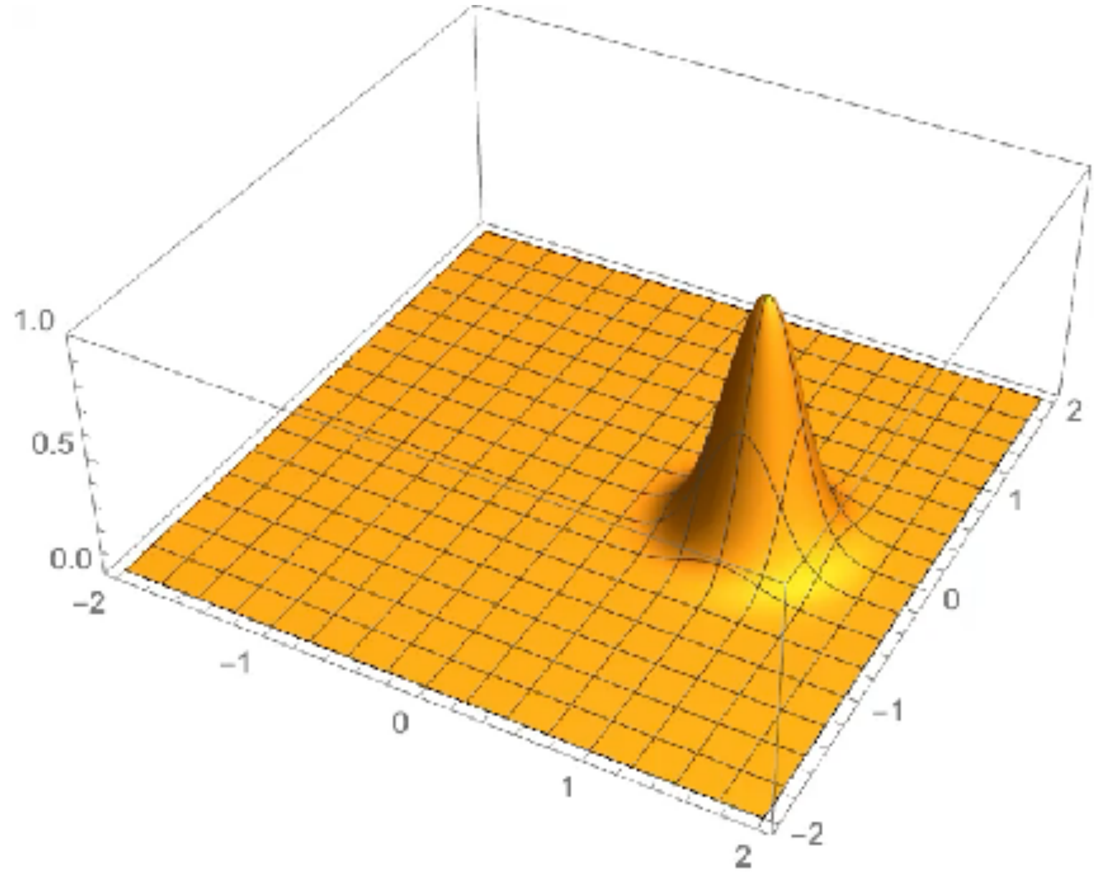
Then solve amplitude flow

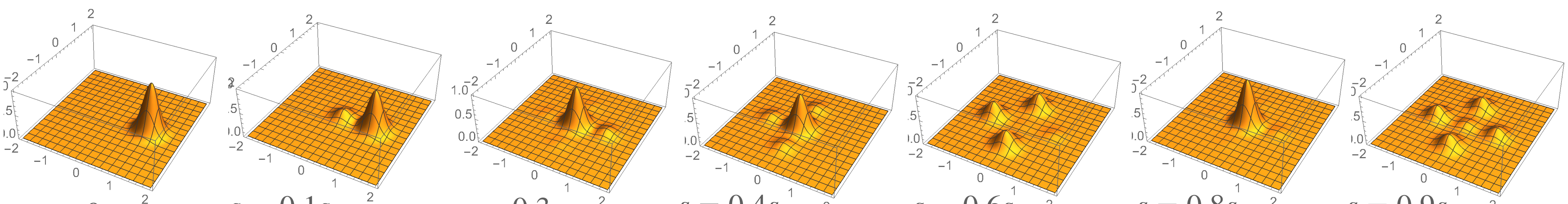
$$\dot{\psi}_x = -\frac{1}{2}\psi_x \mathrm{div}_\int(X)_x - \sum_{p\leftarrow x} (\psi_p - \psi_x) X^{p\leftarrow x}$$

Gaussian-interpolated movie of flow for n=4

$\mu_i, g_{i\to 0}$ constant   initial $X^{0\leftarrow i} = \psi_i = \delta_{i,1}$







$s = 0$ $s = 0.1s_{max}$ $s = 0.3s_{max}$ $s = 0.4s_{max}$ $s = 0.6s_{max}$ $s = 0.8s_{max}$ $s = 0.9s_{max}$

Strict quantum geodesic flows (no driving force)

$\hat{\nabla} = \sigma_\chi^{-1} \nabla_\chi$ left connection → $\mathrm{div}_{\hat{\nabla}} = \mathrm{ev}\, \hat{\nabla}$

<u>Lemma.</u> $\int ab = \int ba$ and $\int \mathrm{div}_{\hat{\nabla}} = 0$ then Ω^{-1} gets a $*$ structure such that X real w.r.t $\int$ ↔ $X^* = X$

Fuzzy sphere $[x_i, x_j] = 2\imath \lambda_p \epsilon_{ijk} x_k$ $\sum_i x_i^2 = 1 - \lambda_p^2$

Ω^1 central basis s^i, $i = 1, 2, 3$. $\int$ = spin 0 component

→ $\{f_i\}$ dual basis to $\{s^i\}$ has $f_i^* = f_i$ → $X = f_i X^i$ real iff $X^{i*} = X^i$

$$\dot{X}^i = \frac{1}{2}[\partial_j X^j, X^i] - \Gamma^i{}_{jk} X^k X^j - (\partial_j X^i) X^j.$$ velocity flow eqn

$$\partial_j [X^i, X^j] = 2\epsilon_{ijk} X^j X^k$$ aux eqn (from conjugate velocity flow eqn)

$$\dot{\psi} = -X^i \partial_i \psi - \frac{\psi}{2} \partial_i X^i$$ amplitude flow eqn

We focus on $X^i \propto 1$

$$\dot{X}^i = -\Gamma^i{}_{jk} X^j X^k = g^{il} g_{mj} \epsilon_{lmk} X^j X^k$$

$g = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ ➡

$$\boxed{\dot{X}^1 = \mu_1 X^2 X^3, \quad \dot{X}^2 = \mu_2 X^1 X^2, \quad \dot{X}^3 = \mu_3 X^1 X^2}$$

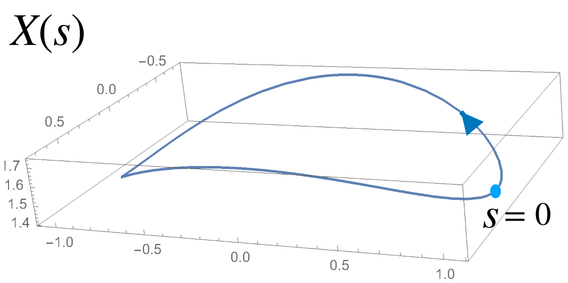
$$\mu_1 = \frac{\lambda_2 - \lambda_3}{\lambda_1}, \quad \mu_2 = \frac{\lambda_3 - \lambda_1}{\lambda_2}, \quad \mu_3 = \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

➡ solve with elliptic jacobi

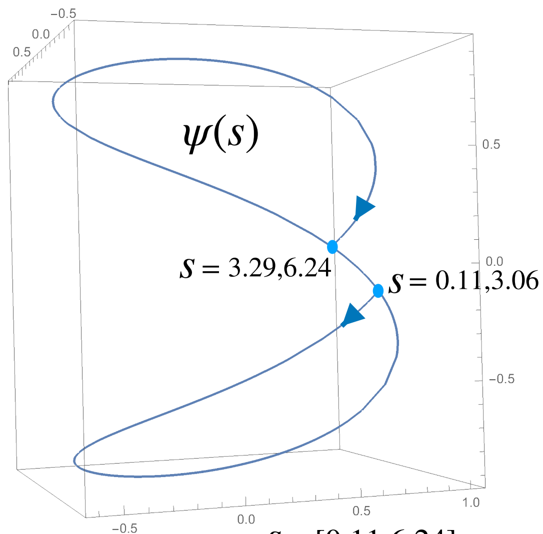
$$X^1(s) = i c_1 \sqrt{\mu_1} \mathrm{sn}\left(c_2 s | \mu\right), \quad X^2(s) = c_1 \sqrt{\mu_2} \mathrm{cn}\left(c_2 s | \mu\right), \quad X^3(s) = c_1 \sqrt{\frac{\mu_3}{\mu}} \sqrt{1 - \mu\, \mathrm{sn}^2\left(c_2 s | \mu\right)}$$

E.g. linear fields $\psi = \psi^i x_i$ $\quad \dot{\psi}^k = -\epsilon_{kij} X^i \psi^j$ $\quad \mu = -\mu_1 \mu_2 \mu_3 \frac{c_1^2}{c_2^2}$

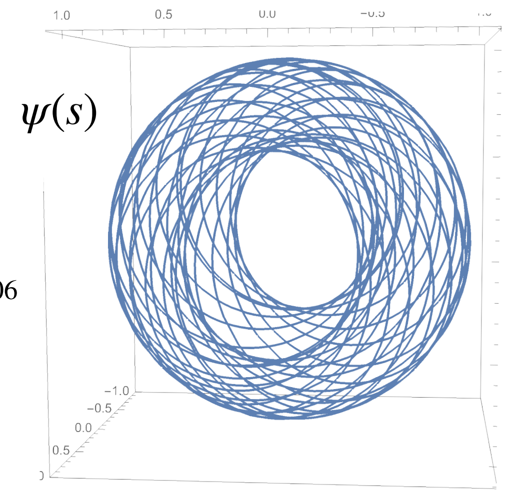
$g = \mathrm{diag}(4, 3, 1)$ initial $X = (0, 1, \sqrt{2}),\ \psi = (1, 0, 0)$



$s \in [0, 5.66]$



$s \in [0.11, 6.24]$



$s \in [0, 100]$

Thank you