Averaging multipliers on locally compact quantum groups

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Locally compact quantum groups

Definition (Kustermans-Vaes)

A locally compact quantum group is a von Neumann algebra M together with a normal unital $*$-homomorphism $\Delta : M \to M \bar{\otimes} M$ satisfying the coassociativity relation

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

and normal faithful semifinite weights φ, ψ on M such that

$$\varphi((\omega \otimes \mathrm{id})\Delta(f)) = \omega(1)\varphi(f), \qquad \psi((\mathrm{id} \otimes \omega)\Delta(f)) = \omega(1)\psi(f)$$

for all $f \in M_+$ and all positive linear functionals $\omega \in M_*$.

Example

a) Let G be a locally compact group. The algebra $M = L^\infty(G)$ of essentially bounded measurable functions on G becomes a locally compact quantum group with comultiplication

$$\Delta : L^\infty(G) \to L^\infty(G)\overline{\otimes}L^\infty(G) = L^\infty(G \times G), \quad \Delta(f)(s,t) = f(st).$$

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The Haar weights φ, ψ are given by integration with respect to left/right Haar measure on G.
b) Let G be a locally compact group and let

$$M = vN(G) = \lambda(G)'' \subset B(L^2(G))$$

be the *group von Neumann algebra* of G. Then M is a locally compact quantum group with comultiplication

$$\widehat{\Delta} : vN(G) \to vN(G)\overline{\otimes}vN(G), \quad \widehat{\Delta}(u_s) = u_s \otimes u_s.$$

The *Plancherel weight* on $vN(G)$ is a left and right Haar weight for M.

Locally compact quantum groups

Motivated by the first example on the previous slide, we will use the notation $M = \mathrm{L}^\infty(\mathbb{G})$ for a *general* locally compact quantum group.

Also, by slight abuse of terminology, we will refer to $\mathbb{G}$ as a locally compact quantum group, instead of M.

Motivated by the first example on the previous slide, we will use the notation $M = \mathrm{L}^\infty(\mathbb{G})$ for a *general* locally compact quantum group.

Also, by slight abuse of terminology, we will refer to $\mathbb{G}$ as a locally compact quantum group, instead of M.

Let $\mathrm{L}^2(\mathbb{G})$ be the GNS Hilbert space associated to the left Haar weight φ.

We consider

$$\mathcal{N}_\varphi = \{f \in \mathrm{L}^\infty(\mathbb{G}) \mid \varphi(f^*f) < \infty\}$$

and let $\Lambda : \mathcal{N}_\varphi \to \mathrm{L}^2(\mathbb{G})$ be the GNS-map.

It is a nontrivial fact that one obtains a unitary operator W on $\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{G})$ by defining

$$\mathrm{W}^*(\Lambda(f) \otimes \Lambda(g)) = (\Lambda \otimes \Lambda)(\Delta(g)(f \otimes 1))$$

for $f, g \in \mathcal{N}_\varphi$.

The operator W, called the multiplicative unitary of $\mathbb{G}$, satisfies the *pentagon equation* $\mathrm{W}_{12}\mathrm{W}_{13}\mathrm{W}_{23} = \mathrm{W}_{23}\mathrm{W}_{12}$ on $\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{G})$.

Locally compact quantum groups

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In particular,

$$\mathrm{C}_0(\mathbb{G}) = [(\mathsf{id} \otimes \omega)(\mathrm{W}) \mid \omega \in B(\mathrm{L}^2(\mathbb{G}))_*]$$

is a C^*-algebra with weak closure $\mathrm{L}^\infty(\mathbb{G})$, and

$$\Delta(f) = \mathrm{W}^*(1 \otimes f)\mathrm{W}$$

for all $f \in \mathrm{L}^\infty(\mathbb{G})$.

There exists a *dual locally compact quantum group* $\widehat{\mathbb{G}}$ with C^*-algebra

$$\mathrm{C}_0(\widehat{\mathbb{G}}) = [(\omega \otimes \mathsf{id})(\mathrm{W}) \mid \omega \in B(\mathrm{L}^2(\mathbb{G}))_*]$$

and weak closure $\mathrm{L}^\infty(\widehat{\mathbb{G}})$, and the comultiplication

$$\widehat{\Delta}(x) = \widehat{\mathrm{W}}^*(1 \otimes x)\widehat{\mathrm{W}},$$

where $\widehat{\mathrm{W}} = \Sigma \mathrm{W}^* \Sigma$, and Σ is the tensor flip.

Discrete quantum groups

Discrete quantum groups

Definition

A locally compact quantum group $\mathbb{\Gamma}$ is called discrete if $\mathrm{C}_0(\widehat{\mathbb{\Gamma}})$ is unital.

In this case we write $\mathrm{C}(\widehat{\mathbb{\Gamma}})$ and $\mathrm{c}_0(\mathbb{\Gamma})$ for the C^*-algebras associated to $\widehat{\mathbb{\Gamma}}$ and $\mathbb{\Gamma}$. The C^*-algebra $\mathrm{c}_0(\mathbb{\Gamma})$ is a (typically infinite c_0-) direct sum of matrix algebras.

We also write $\ell^\infty(\mathbb{\Gamma})$ instead of $\mathrm{L}^\infty(\mathbb{\Gamma})$. This is a (typically infinite) direct product of matrix algebras.

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Usually, discrete quantum groups are most easily described by defining their dual (compact) quantum groups...

The quantum group $SU_q(2)$

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Let $q \in (0,1]$. The C^*-algebra of functions on $SU_q(2)$ is the universal $*$-algebra $\mathrm{C}(SU_q(2))$ generated by elements α and γ satisfying

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$
$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

These relations are equivalent to saying that

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is a unitary matrix.

The comultiplication $\Delta : \mathrm{C}(SU_q(2)) \to \mathrm{C}(SU_q(2)) \otimes \mathrm{C}(SU_q(2))$ is defined by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

For $q = 1$ this reproduces the algebra of continuous functions on $SU(2)$ with the comultiplication coming from the group structure of $SU(2)$.

The Fourier algebra

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Write $\mathrm{L}^1(\widehat{\mathbb{G}})$ for the predual of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. This becomes a Banach algebra with the multiplication

$$(\omega \star \eta, x) = (\omega \otimes \eta, \widehat{\Delta}(x)),$$

induced by the comultiplication for $\mathrm{L}^\infty(\widehat{\mathbb{G}})$.

We get an algebra homomorphism $\widehat{\lambda} : \mathrm{L}^1(\widehat{\mathbb{G}}) \to B(\mathrm{L}^2(\mathbb{G}))$ by

$$\widehat{\lambda}(\omega) = (\omega \otimes \mathrm{id})(\widehat{\mathrm{W}}).$$

Definition

The Fourier algebra $\mathcal{A}(\mathbb{G})$ of $\mathbb{G}$ is the image $\widehat{\lambda}(\mathrm{L}^1(\widehat{\mathbb{G}})) \subset \mathrm{C}_0(\mathbb{G})$, which becomes a Banach algebra via the identification with $\mathrm{L}^1(\widehat{\mathbb{G}})$ given by $\widehat{\lambda}$.

A *left centraliser* of $\mathbb{G}$ is a right module map $L\colon \mathrm{L}^1(\widehat{\mathbb{G}}) \to \mathrm{L}^1(\widehat{\mathbb{G}})$,

$$L(\omega \star \eta) = L(\omega) \star \eta \quad (\omega, \eta \in \mathrm{L}^1(\widehat{\mathbb{G}})).$$

A *left multiplier* of $\mathbb{G}$ is an element $a \in \mathrm{L}^\infty(\mathbb{G})$ with

$$a\widehat{\lambda}(\omega) \in \widehat{\lambda}(\mathrm{L}^1(\widehat{\mathbb{G}})) = \mathcal{A}(\mathbb{G}), \quad (\omega \in \mathrm{L}^1(\widehat{\mathbb{G}})).$$

Every left multiplier a induces a (unique) left centraliser L via $a\widehat{\lambda}(\omega) = \widehat{\lambda}(L(\omega))$.

We say that a left multipler a is *completely bounded (CB)/completely positive (CP)* if the associated map $L^* : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}})$ is completely bounded/completely positive.

We denote the space of CB left multipliers by $\mathcal{M}^l_{cb}(\mathcal{A}(\mathbb{G}))$. Note that $\mathcal{A}(\mathbb{G}) \subset \mathcal{M}^l_{cb}(\mathcal{A}(\mathbb{G}))$ naturally.

Approximation properties

Definition

Let $\mathbb{G}$ be a locally compact quantum group. Then $\mathbb{G}$

- is *strongly amenable* if there exists a bounded approximate identity of $\mathcal{A}(\mathbb{G})$ consisting of CP multipliers of $\mathbb{G}$.
- has the *Haagerup property* if there exists a bounded approximate identity of $\mathrm{C}_0(\mathbb{G})$ which consists of CP multipliers of $\mathbb{G}$.
- is *weakly amenable* if there exists a left approximate identity $(e_i)_{i\in I}$ of $\mathcal{A}(\mathbb{G})$ satisfying $\limsup_{i\in I} \|e_i\|_{cb} < \infty$. In this case, the smallest M such that we can choose $\|e_i\|_{cb} \leq M$ for all $i \in I$ is the *Cowling–Haagerup constant* of $\mathbb{G}$, denoted $\Lambda_{cb}(\mathbb{G})$.
- has the *approximation property* if there exists a net $(e_i)_{i\in I}$ in $\mathcal{A}(\mathbb{G})$ which converges to $\mathbb{1}$ in the weak* topology of $\mathcal{M}^l_{cb}(\mathcal{A}(\mathbb{G}))$.

Central approximation properties

Central approximation properties

Definition

Let $\mathbb{\Gamma}$ be a discrete quantum group. Then $\mathbb{\Gamma}$

- is *centrally strongly amenable* if there is a net $(e_i)_{i\in I}$ consisting of finitely supported central CP multipliers of $\mathbb{\Gamma}$ converging to $\mathbb{1}$ pointwise.
- has the *central Haagerup property* if there exists a bounded approximate identity of $\mathrm{c}_0(\mathbb{\Gamma})$ which consists of central CP multipliers of $\mathbb{\Gamma}$.
- is *centrally weakly amenable* if there exists a net $(e_i)_{i\in I}$ of finitely supported central multipliers converging to $\mathbb{1}$ pointwise and satisfying $\limsup_{i\in I}\|e_i\|_{cb}<\infty$. In this case, the smallest M such that we can choose $\|e_i\|_{cb}\leq M$ for all $i\in I$ is the *central Cowling–Haagerup constant* of $\mathbb{\Gamma}$, denoted $\mathcal{Z}\Lambda_{cb}(\mathbb{\Gamma})$.
- has the *central approximation property* if there exists a net $(e_i)_{i\in I}$ of finitely supported central multipliers which converges to $\mathbb{1}$ in the weak* topology of $\mathcal{M}^l_{cb}(\mathcal{A}(\mathbb{G}))$.

The Drinfeld double

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Definition

The *Drinfeld double* $D(\mathbb{\Gamma})$ of $\mathbb{\Gamma}$ is given by

$$\mathrm{L}^\infty(D(\mathbb{\Gamma})) = \ell^\infty(\mathbb{\Gamma})\overline{\otimes}\mathrm{L}^\infty(\widehat{\mathbb{\Gamma}})$$

with the coproduct

$$\Delta_{D(\mathbb{\Gamma})} = (\mathrm{id}\otimes\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{ad}(\mathrm{W})\otimes\mathrm{id})(\Delta\otimes\widehat{\Delta}),$$

where σ is the tensor flip map.
A left and right Haar weight on $D(\mathbb{\Gamma})$ is given by $\varphi\otimes\widehat{\varphi}$.

$D(\mathbb{\Gamma})$ contains $\widehat{\mathbb{\Gamma}}$ naturally as a compact (open) quantum subgroup.

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Example

For $\mathbb{\Gamma} = \widehat{SU_q(2)}$ for $q \in (0,1)$ the Drinfeld double $D(\mathbb{\Gamma})$ can be interpreted as a quantum deformation of $SL(2,\mathbb{C})$.

The Main Theorem

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Theorem (Daws-Krajczok-V. 2023)

Let $\mathbb{\Gamma}$ be a discrete quantum group and let $D(\mathbb{\Gamma})$ be its Drinfeld double. Then the following conditions are equivalent.

1) *$\mathbb{\Gamma}$ is centrally strongly amenable (respectively, is centrally weakly amenable, has the central Haagerup property, has central AP).*
2) *$D(\mathbb{\Gamma})$ is strongly amenable (respectively, is weakly amenable, has the Haagerup property, has AP).*

Furthermore, we have $\Lambda_{cb}(D(\mathbb{\Gamma})) = \mathcal{Z}\Lambda_{cb}(\mathbb{\Gamma})$ in the weakly amenable case.

The key technique in the proof is an *averaging argument* for multipliers on $D(\mathbb{\Gamma})$ with respect to the compact quantum subgroup $\widehat{\mathbb{\Gamma}} \subset D(\mathbb{\Gamma})$.

Averaging

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Let $\mathbb{G}$ be a locally compact quantum group with a compact quantum subgroup $\mathbb{K} \subseteq \mathbb{G}$.

There is a normal conditional expectation $\Xi\colon \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G})$ given by averaging with respect to left and right translations of $\mathbb{K}$.

Theorem (Daws-Krajczok-V. 2023)

The averaging map $\Xi\colon \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G})$ restricts to a contractive weak*-weak*-continuous map $\mathcal{M}^l_{cb}(\mathcal{A}(\mathbb{G})) \to \mathcal{M}^l_{cb}(\mathcal{A}(\mathbb{G}))$, mapping CP multipliers to CP multipliers.

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In the case of $\mathbb{G} = D(\mathbb{\Gamma})$ for a discrete quantum group $\mathbb{\Gamma}$, this yields an isometric identification

$$\Xi(\mathcal{M}^l_{cb}(\mathcal{A}(D(\mathbb{\Gamma})))) \cong \mathcal{Z}\mathcal{M}^l_{cb}(\mathcal{A}(\mathbb{\Gamma})) \otimes \mathbb{1}$$

of the $\widehat{\mathbb{\Gamma}}$-biinvariant CB-multipliers of $D(\mathbb{\Gamma})$ and the central CB-multipliers of $\mathbb{\Gamma}$.

Counter-examples

Counter-examples

The averaging map also maps $\mathcal{A}(D(\mathbb{\Gamma}))$ to $\mathcal{A}(D(\mathbb{\Gamma}))$ and there is a natural inclusion $\Xi(\mathcal{A}(D(\mathbb{\Gamma})) \subset \mathcal{Z}\mathcal{A}(\mathbb{\Gamma}) \otimes \mathbb{1}$.

Theorem (Daws-Krajczok-V. 2024)

Let $\mathbb{\Gamma}$ be a discrete quantum group which is strongly amenable and non-unimodular. If $\mathbb{\Gamma}$ is centrally weakly amenable, then the inclusion

$$\Xi(\mathcal{A}(D(\mathbb{\Gamma}))) \subsetneq \mathcal{Z}\mathcal{A}(\mathbb{\Gamma}) \otimes \mathbb{1}$$

is strict.

This applies in particular to the dual of $SU_q(2)$ for $q \in (0, 1)$.