What is Quantum Harmonic Analysis?

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February 26 2024
Noncommutative Geometry, Analysis on Groups, and Mathematical Physics
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Convolutions and Fourier transforms in Quantum Harmonic Analysis

Essentially everything that will be said can be done on abelian phase spaces Ξ with a symplectic self-duality. Further, the Hilbert space setting can be extended to so-called coorbit spaces (or, in the standard case $\Xi = G \times \widehat{G}$, modulation spaces $M^q(G)$).

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By $W_{(x,\xi)}$ we will denote the *Weyl operators* on $\mathcal{H} := L^2(\mathbb{R})$:

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They depend continuously in SOT on (x, ξ) and satisfy:

$$W_{(x,\xi)} W_{(y,\eta)} = e^{i\sigma((x,\xi),(y,\eta))/2} W_{(x+y,\xi+\eta)} = e^{i\sigma((x,\xi),(y,\eta))} W_{(y,\eta)} W_{(x,\xi)}.$$

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We will usually write $W_z = W_{(x,\xi)}$ for $z = (x,\xi) \in \Xi$.

Convolutions and Fourier transforms in Quantum Harmonic Analysis

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$$\alpha_z(f)(w) = f(w - z), \quad z, w \in \Xi, \quad \beta_-(f)(w) = f(-w).$$

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The shifts are strongly continuous on $L^p(\Xi)$ $(1 \leq p < \infty)$ and $C_0(\Xi)$ resp. $\mathcal{T}^p(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$, but only weak* continuous on $L^\infty(\Xi)$ resp. $\mathcal{L}(\mathcal{H})$.

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The idea is now to investigate elements of $L^\infty(\Xi) \oplus \mathcal{L}(\mathcal{H})$, and more specifically elements from $\mathcal{L}(\mathcal{H})$, from the perspective of this harmonic analysis structure.

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Theorem (Wiener's approximation theorem for functions)

Let $f \in L^1(\Xi)$. Then, the following are equivalent:

1. $\mathrm{span}\{\alpha_z(f) : z \in \Xi\}$ *is dense in* $L^1(\Xi)$.
2. $\mathcal{F}_\sigma(f)$ *vanishes nowhere.*
3. $L^1(\Xi) \ni g \mapsto f * g \in L^1(\Xi)$ *has dense range.*
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A function $f \in L^1(\Xi)$ satisfying the above equivalent properties is called *regular*. The analogous result for operators is a key point in quantum harmonic analysis.

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4. $L^1(\Xi) \ni g \mapsto A * g \in \mathcal{T}^1(\mathcal{H})$ *has dense range.*
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An operator $A \in \mathcal{T}^1(\mathcal{H})$ satisfying the above properties will be called *regular*.

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Example

The possibly single most important regular operator is

$$A_0 = \frac{1}{\pi}\varphi_0 \otimes \varphi_0,$$

where φ_0 *is the ground state of the quantum harmonic oscillator:* $\varphi_0(x) = e^{-x^2}$. $\mathcal{F}_W(A_0)$ *is a Gaussian, so* A_0 *is a regular operator.*

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When $f \in L^\infty(\Xi)$, *then* $A_0 * f$ *is (modulo the Bargmann-transform) the Toeplitz operator with symbol* f. *For* $B \in \mathcal{L}(\mathcal{H})$, $\widetilde{B} := A_0 * B$ *is the Berezin transform of* B.

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Example

More generally, $(\varphi \otimes \psi) * f$ is a localization operator from time-frequency analysis.

The algebra $\mathcal{C}_1$

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$$\begin{aligned}\mathcal{C}_0 :=& \{f \in L^\infty(\Xi) : \ z \mapsto \alpha_z(f) \text{ is } \|\cdot\|_\infty\text{-cont.}\}\\ =& \operatorname{BUC}(\Xi)\\ \mathcal{C}_1 :=& \{A \in \mathcal{L}(\mathcal{H}) : \ z \mapsto \alpha_z(A) \text{ is } \|\cdot\|_{op}\text{-cont.}\}\end{aligned}$$

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By Calderón-Vaillancourt: $\mathrm{Op}^W(f) \in \mathcal{C}_1$ whenever $f \in C_b^\infty(\Xi)$.

The Correspondence Theorem

In the language of Banach modules, $\mathrm{BUC}(\Xi) \oplus \mathcal{C}_1$ is an *essential* $L^1(\Xi) \oplus \mathcal{T}^1(\mathcal{H})$-module, while $L^\infty(\Xi) \oplus \mathcal{L}(\mathcal{H})$ is not.

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Let $\mathcal{D}_0 \subset L^\infty(\Xi)$, $\mathcal{D}_1 \subset \mathcal{L}(\mathcal{H})$ be α-invariant subspaces. $(\mathcal{D}_0, \mathcal{D}_1)$ is a *corresponding pair* (we also say: $\mathcal{D}_0$ and $\mathcal{D}_1$ correspond to each other) if $\mathcal{D}_0 \oplus \mathcal{D}_1$ is an $L^1(\Xi) \oplus \mathcal{T}^1(\mathcal{H})$-module.

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Theorem (The Correspondence Theorem)

Let $\mathcal{D}_0 \subset \mathrm{BUC}(\Xi)$ *be an* α*-invariant, closed subspace.*

The Correspondence Theorem

In the language of Banach modules, $\mathrm{BUC}(\Xi) \oplus \mathcal{C}_1$ is an *essential* $L^1(\Xi) \oplus \mathcal{T}^1(\mathcal{H})$-module, while $L^\infty(\Xi) \oplus \mathcal{L}(\mathcal{H})$ is not.

Definition

Let $\mathcal{D}_0 \subset L^\infty(\Xi)$, $\mathcal{D}_1 \subset \mathcal{L}(\mathcal{H})$ be α-invariant subspaces. $(\mathcal{D}_0, \mathcal{D}_1)$ is a *corresponding pair* (we also say: $\mathcal{D}_0$ and $\mathcal{D}_1$ correspond to each other) if $\mathcal{D}_0 \oplus \mathcal{D}_1$ is an $L^1(\Xi) \oplus \mathcal{T}^1(\mathcal{H})$-module.

Theorem (The Correspondence Theorem)

Let $\mathcal{D}_0 \subset \mathrm{BUC}(\Xi)$ be an α-invariant, closed subspace.

1. *There is a unique closed, α-invariant subspace $\mathcal{D}_1$ of $\mathcal{C}_1$ corresponding to $\mathcal{D}_0$. $\mathcal{D}_1$ is given by $\mathcal{D}_1 = \overline{\mathcal{T}^1(\mathcal{H}) * \mathcal{D}_0}$.*

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2. *Let $A \in \mathcal{T}^1(\mathcal{H})$ be regular. For $f \in \mathrm{BUC}(\Xi)$ and $B \in \mathcal{C}_1$ it is:*

$$f \in \mathcal{D}_0 \Leftrightarrow A * f \in \mathcal{D}_1,$$
$$B \in \mathcal{D}_1 \Leftrightarrow A * B \in \mathcal{D}_0.$$

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1. For $\mathcal{D}_0 = \mathrm{BUC}(\Xi)$ we get $\mathcal{D}_1 = \mathcal{C}_1$. In particular, for $A \in \mathcal{T}^1(\mathcal{H})$ regular:

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2. For $\mathcal{D}_0 = C_0(\Xi)$ we get $\mathcal{D}_1 = \mathcal{K}(\mathcal{H})$. This yields the following result: For $B \in \mathcal{L}(\mathcal{H})$ it is $B \in \mathcal{K}(\mathcal{H})$ if and only if $B \in \mathcal{C}_1$ and $\widetilde{B} \in C_0(\Xi)$.

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4. Let $\mathcal{D}_0 = C^*(e^{i\sigma(z,\cdot)}, e^{i\sigma(w,\cdot)})$. Then, $\mathcal{D}_1$ is the corresponding non-commutative torus A_θ with $\theta = e^{-i\sigma(z,w)/2}$. An operator $A \in \mathcal{C}_1$ is contained in A_θ if and only if $\widetilde{A} \in C^*(e^{i\sigma(z,\cdot)}, e^{i\sigma(w,\cdot)})$.

Some research in that area so far:

1. Fredholm theory of $\mathcal{C}_1$,
2. Characterizations of the algebra $\mathcal{C}_1$,
3. Applications in operator theory, for example Toeplitz operators,
4. Applications in time-frequency analysis,
5. Investigation of commutative operator algebras,
6. Wiener's Tauberian theorem for operators,
7. Harmonic analysis of $L^1(\Xi) \oplus \mathcal{T}^1(\mathcal{H})$,
8. Extensions to other phase spaces Ξ, as well as extension to Banach spaces instead of $\mathcal{H}$,
9. Explaining old theorems through the lens of QHA.

Thank you for your attention!