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Ergodic states on type III_1 factors and ergodic actions

Noncommutative Geometry, Analysis on Groups, and Mathematical Physics

Universiteit Gent – 26-27 February 2024

Stefaan Vaes – KU Leuven

Classification of factors: Murray and von Neumann

Definition

A **factor** is a von Neumann algebra $M \subset B(H)$ with trivial center: $\mathcal{Z}(M) = \mathbb{C}1$.

Type classification of Murray and von Neumann

- **Type I.** There exist **minimal** projections, i.e. $M \cong B(K)$.
- **Type II.** There are no minimal projections, but there are nonzero **finite** projections p : if $v^*v = p$ and $vv^* \leq p$, then $vv^* = p$.
- **Type III.** Every nonzero projection is **infinite**.

Theorem (Murray and von Neumann)

A factor of type II_1, meaning that 1 is a finite projection, admits a unique faithful normal tracial state $\tau : M \to \mathbb{C} : \tau(ab) = \tau(ba)$.

Classification of factors: Tomita-Takesaki-Connes

Let M be a factor and $\varphi : M \to \mathbb{C}$ a faithful normal state.

- (Tomita-Takesaki) There is canonical 1-parameter group of automorphisms $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ of M such that $\varphi(ab) = \varphi(\sigma_i^\varphi(b)a)$.
- (Connes) If ω is another faithful normal state, there are unitaries $[D\omega : D\varphi]_t \in \mathcal{U}(M)$ such that $\sigma_t^\omega = \mathrm{Ad}[D\omega : D\varphi]_t \circ \sigma_t^\varphi$. This is the **Radon-Nikodym cocycle**.
- (Connes-Takesaki) We have a **canonical** algebra of type II_∞ $\mathrm{core}(M) = M \rtimes_{\sigma^\varphi} \mathbb{R}$.

Type classification of Connes

- Finer classification into type III_λ with $\lambda \in [0, 1]$. Flow of weights of type III_0.
- **Type III_1 if $\mathrm{core}(M)$ is a factor.**
- Complete classification of amenable factors.

Ergodic states and factors of type III_1

Notation: $\mathcal{S}(M)$ is the set of faithful normal states on a von Neumann algebra M.

- **Modular automorphism group** $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ of $\varphi \in \mathcal{S}(M)$.
- **Centralizer** $M_\varphi = \{x \in M \mid \forall t : \sigma_t^\varphi(x) = x\} = \{x \in M \mid \forall y : \varphi(xy) = \varphi(yx)\}$.
- We say that $\varphi \in \mathcal{S}(M)$ is **ergodic** if $M_\varphi = \mathbb{C}1$. **Notation:** $\mathcal{S}_{\mathrm{erg}}(M)$.

Early results

- (Herman-Takesaki, 1970) First examples of factors M with an ergodic state.
- (Longo, 1978) If M admits an ergodic state, then M is a factor of type III_1.

Question: what about the converse?

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Ergodic states and factors of type III_1

Theorem (Marrakchi-V, 2023)

Let $M \neq \mathbb{C}1$ be a von Neumann algebra with separable M_*. The following are equivalent.

- M admits an ergodic faithful normal state.
- $\mathcal{S}_{\mathrm{erg}}(M)$ is a dense G_δ subset of $\mathcal{S}(M)$.
- M is a factor of type III_1.

$\rightsquigarrow$ On type III_1 factors, a generic state is ergodic.

Key ideas for the proof

Fix a type III_1 factor M with separable predual M_*.

- For any $\varphi \in \mathcal{S}(M)$, we have the conditional expectation $E_\varphi : M \to M_\varphi$.
- Note that φ is ergodic iff $E_\varphi(x) = \varphi(x)1$ iff $\|E_\varphi(x)\|_\varphi = |\varphi(x)|$ for all $x \in M$.

Here $\|a\|_\varphi = \sqrt{\varphi(a^* a)}$.

Key lemma

For every $x \in M$ and $\varepsilon > 0$, the set $U(x, \varepsilon) = \{\varphi \in \mathcal{S}(M) \mid \|E_\varphi(x)\|_\varphi < |\varphi(x)| + \varepsilon\}$ is an open dense subset of $\mathcal{S}(M)$.

Then: $\mathcal{S}_{\text{erg}}(M) = \bigcap_{k,n} U(x_n, 1/k)$ is a dense G_δ subset of $\mathcal{S}(M)$.

Key ideas for the proof

Key lemma

For every $x \in M$ and $\varepsilon > 0$, the set $U(x, \varepsilon) = \{\varphi \in \mathcal{S}(M) \mid \|E_\varphi(x)\|_\varphi < |\varphi(x)| + \varepsilon\}$ is an open dense subset of $\mathcal{S}(M)$.

- For every $x \in M$, the map $\psi \mapsto \|E_\psi(x)\|_\psi$ is upper semicontinuous.
  Thus, $U(x, \varepsilon)$ is open.
- Fix $\psi \in \mathcal{S}(M)$, $x \in M$ and $\varepsilon > 0$. We need to find a $\varphi \in U(x, \varepsilon)$ that is close to ψ.
- Write $x = y + z + \psi(x)1$ where $y = E_\psi(x) - \psi(x)1$ and $z = x - E_\psi(x)$.
- Whenever φ is close to ψ, we have $\varphi(x) \approx \psi(x)$ and we have $\|E_\varphi(z)\|_\varphi$ small.
- We need to prove: if $y \in M_\psi$ and $\psi(y) = 0$, there exists a φ close to ψ with $\|E_\varphi(y)\|_\varphi$ small. **Here, type III_1 and separability will come in !**

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Key ideas for the proof

We fix a type III_1 factor M with separable predual, $\psi \in \mathcal{S}(M)$ and $y \in M_\psi$ with $\psi(y) = 0$.

$\rightsquigarrow$ We need to find φ close to ψ such that $E_\varphi(y)$ is small.

Connes-Størmer approximate transitivity

On a type III_1 factor M with separable predual, all faithful normal states are **approximately unitarily conjugate :** for all $\omega, \psi \in \mathcal{S}(M)$, there exist $u_n \in \mathcal{U}(M)$ such that $\|\omega - u_n \psi u_n^*\| \to 0$. $\rightsquigarrow$ The asymptotic centralizer of ψ is a II_1 factor.

Popa's local quantization

If N is a II_1 factor with trace ψ and $y \in N$ with $\psi(y) = 0$, there exist projections $p_1, \dots, p_k \in N$ with $\sum_i p_i = 1$ and $\|\sum_i p_i y p_i\|_\psi$ small.

We can find our φ of the form $\varphi = \sum_i \lambda_i \, p_i \psi p_i$.

Remarks

Separability is essential

Let M be a type III_1 factor and $\varphi \in \mathcal{S}(M)$. Let $M^{\mathcal{U}}$ be the Ocneanu ultrapower.

- Ando-Haagerup, 2012: all faithful normal states on $M^{\mathcal{U}}$ are unitarily conjugate.
- The centralizer of $\varphi^{\mathcal{U}}$ is a II_1 factor.
- So, $M^{\mathcal{U}}$ is a countably decomposable III_1 factor without ergodic state.

The Connes bicentralizer problem

- Does there exist $\varphi \in \mathcal{S}(M)$ with $(M_\varphi)' \cap M = \mathbb{C}1$?
- Ergodic states are generic. States with large centralizer are rare (but dense if they exist).

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Cocycle perturbations of group actions

Recall Connes' Radon-Nikodym cocycle: $\sigma_t^\varphi = \mathrm{Ad}[D\varphi : D\psi]_t \circ \sigma_t^\psi$.

Let $\Gamma \curvearrowright^\alpha M$ be any action of a group Γ by automorphisms of M.

- **Ergodic** if $M^\Gamma = \{x \in M \mid \forall g : \alpha_g(x) = x\}$ equals $\mathbb{C}1$.
- **Outer** if for all $g \in \Gamma \setminus \{e\} : \alpha_g \neq \mathrm{Ad}\, u$.
- A **1-cocycle** is a map $v : \Gamma \to \mathcal{U}(M) : v_{gh} = v_g \alpha_g(v_h)$.
- Then $\beta = \mathrm{Ad}\, v \circ \alpha$ given by $\beta_g(x) = v_g \alpha_g(x) v_g^*$ is again a group action.

$\rightsquigarrow$ Which group actions admit an ergodic cocycle perturbation?

Ergodic perturbations of single automorphisms

Theorem (Marrakchi-V, 2023)

Let M be a II_1 factor with separable predual. Let $\alpha \in \operatorname{Aut} M$.

Then the following are equivalent.

- α has infinite order in $\operatorname{Out} M = \operatorname{Aut} M / \operatorname{Inn} M$.
- There exists a $u \in \mathcal{U}(M)$ such that $\operatorname{Ad} u \circ \alpha$ is ergodic.
- The set $\{u \in \mathcal{U}(M) \mid \operatorname{Ad} u \circ \alpha \text{ is ergodic}\}$ is a dense G_δ subset of $\mathcal{U}(M)$.

Ergodic cocycle perturbations of group actions

Theorem (Marrakchi-V, 2023)

For the following discrete groups, **any** outer action on **any** II_1 factor with separable predual admits an ergodic cocycle perturbation.

- Infinite amenable groups.
- Stable under free products (with amalgamation over finite subgroups).

But: "rigid groups" like infinite property (T) groups or nonamenable nontrivial product groups $\Gamma_1 \times \Gamma_2$ do not have this property.

Open questions. What about actions on factors of other types?

What about actions of locally compact groups?