Schatten properties of commutators on noncommutative Euclidean space

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Preliminaries: noncommutative Euclidean space (quantum Euclidean space)

- θ will be a fixed antisymmetric $d \times d$ matrix with $d \geq 2$; together with θ, we define

$$\sigma(s,t) = \exp\left(\frac{\mathrm{i}}{2}\langle s, \theta t\rangle\right), \quad s,t \in \mathbb{R}^d. \tag{1}$$

Then $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{T}$ is a 2-cocycle.

- The *noncommutative euclidean space (quantum Euclidean space)* associated to θ, denoted by $\mathbb{R}^d_\theta$, is the von Neumann subalgebra $\mathcal{L}_\sigma(G)$ of $B(L_2(\mathbb{R}^d))$ generated by $\{\lambda_\theta(s)\}_{s\in\mathbb{R}^d}$ in the following form:

$$\lambda_\theta(s)\xi(t) = \sigma(-t,s)\xi(t-s), \; \xi \in L_2(\mathbb{R}^d), \; s,t \in \mathbb{R}^d.$$

Preliminaries: Trace on noncommutative Euclidean space

• Let $f \in L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. Define

$$\tau_\theta(\lambda_\theta(f)) = f(0).$$

The functional $\tau_\theta : L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d) \to \mathbb{C}$ admits an extension to a semifinite normal faithful trace on $\mathbb{R}^d_\theta$.

• By the Plancherel formula, the map $f \mapsto \lambda_\theta(f)$ establishes an isometry from $L_2(\mathbb{R}^d)$ onto $L_2(\mathbb{R}^d_\theta)$. Later, we will identify $L_2(\mathbb{R}^d)$ and $L_2(\mathbb{R}^d_\theta)$.

Preliminaries: Distribution space

• The class of Schwartz functions on $\mathbb{R}^d_\theta$ is defined as the image of the usual Schwartz class $\mathcal{S}(\mathbb{R}^d)$ under λ_θ. That is,

$$\mathcal{S}(\mathbb{R}^d_\theta) = \{\lambda_\theta(f) : f \in \mathcal{S}(\mathbb{R}^d)\}. \tag{2}$$

• The space of *tempered distributions* on $\mathbb{R}^d_\theta$ is the topological dual space $\mathcal{S}'(\mathbb{R}^d_\theta)$ of $\mathcal{S}(\mathbb{R}^d_\theta)$, i.e., the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d_\theta)$.

Preliminaries: Derivatives on noncommutative Euclidean space

- For $x = \lambda_\theta(f) \in \mathcal{S}(\mathbb{R}^d_\theta)$, $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$, we set

$$\partial^\alpha x = \int_{\mathbb{R}^d} s^\alpha f(s) \lambda_\theta(s) ds,$$

where $s^\alpha = s_1^{\alpha_1} \cdots s_d^{\alpha_d}$.

- $\partial^\alpha x$ belongs to $\mathcal{S}(\mathbb{R}^d_\theta)$ too. By duality, these partial derivations extend to distributions.

• Let $\Delta = \partial_1^2 + \cdots + \partial_d^2$ be the Laplacian. We will frequently use the Bessel and Riesz operators $(1+\Delta)^{\frac{1}{2}}$ and $\Delta^{\frac{1}{2}}$ which will be abbreviated as J and I respectively. More generally, for $a \in \mathbb{R}$, define $J^a = (1+\Delta)^{\frac{a}{2}}$ and $I^a = \Delta^{\frac{a}{2}}$.

• The Bessel potential J^a operates on $\mathcal{S}'(\mathbb{R}^d_\theta)$. While for the Riesz potential I^a. Let

$$\mathcal{S}_0(\mathbb{R}^d) = \{x : \widehat{\partial^\alpha x}(0) = 0 \quad \forall\, \alpha \in \mathbb{N}_0^d\}.$$

Then I^a operates on $\mathcal{S}_0(\mathbb{R}^d_\theta) = \lambda_\theta\big(\mathcal{S}_0(\mathbb{R}^d)\big)$, and by duality, on the dual space $\mathcal{S}_0'(\mathbb{R}^d_\theta)$.

Preliminaries: Fourier multipliers and convolution

- We denote $\check{\phi}$ as the inverse Fourier transform of ϕ. Now assume that $\check{\phi} \in L_1(\mathbb{R}^d)$. Define

$$\check{\phi} * x = \int_{\mathbb{R}^d} \check{\phi}(t)\widehat{\alpha}_{-t}(x)dt. \tag{3}$$

- For $x = \lambda_\theta(f)$ with $f \in \mathcal{S}(\mathbb{R}^d)$, we have for the Fourier multiplier T_ϕ defined by mapping $\lambda_\theta(f)$ to $\lambda_\theta(\phi f)$,

$$T_\phi(x) = \check{\phi} * x.$$

Preliminaries: Commutators

• Given $x \in \mathbb{R}_\theta^d$, denote by $M_x : y \mapsto xy$ the left multiplication on $L_2(\mathbb{R}_\theta^d)$. Then M_x is a bounded linear operator on $L_2(\mathbb{R}_\theta^d)$. We now define the commutator

$$\mathbf{C}_{\phi,x} = [T_\phi, M_x].$$

This is the so-called Calderón-Zygmund transform on $\mathbb{R}_\theta^d$, it is bounded on $L_2(\mathbb{R}_\theta^d)$.

Function spaces on noncommutative Euclidean space

•The *homogeneous Sobolev space* $\dot{W}_p^m(\mathbb{R}_\theta^d)$ consists of those $x \in \mathcal{S}'(\mathbb{R}_\theta^d)$ such that every partial derivative of order m is in $L_p(\mathbb{R}_\theta^d)$, equipped with the seminorm:

$$\|x\|_{\dot{W}_p^m} = \Big(\sum_{|\alpha|=m} \|\partial^\alpha x\|_p \Big)^{\frac{1}{p}}.$$

- Besov spaces are defined by using a fixed test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\begin{cases} \operatorname{supp}\varphi \subset \{\xi : 2^{-1} \leq |\xi| \leq 2\}, \\ \varphi > 0 \text{ on } \{\xi : 2^{-1} < |\xi| < 2\}, \\ \sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1,\ \xi \neq 0. \end{cases} \tag{4}$$

The sequence $\{\varphi(2^{-k}\cdot)\}_{k\in\mathbb{Z}}$ is a Littlewood-Paley decomposition of $\mathbb{R}^d$, modulo constant functions. Denote by φ_k the inverse Fourier transform of $\varphi(2^{-k}\cdot)$.

Definition 1

Let $1 \leq p, q \leq \infty$ and $a \in \mathbb{R}$. The *homogeneous Besov space* on $\mathbb{R}^d_\theta$ is defined by

$$B^a_{p,q}(\mathbb{R}^d_\theta) = \{x \in \mathcal{S}'_0(\mathbb{R}^d_\theta) : \|x\|_{B^a_{p,q}} < \infty\},$$

where

$$\|x\|_{B^a_{p,q}} = \Big(\sum_{k\in\mathbb{Z}} 2^{qka}\|\varphi_k * x\|_p^q\Big)^{\frac{1}{q}}.$$

Let $B^a_{p,c_0}(\mathbb{R}^d_\theta)$ be the subspace of $B^a_{p,\infty}(\mathbb{R}^d_\theta)$ consisting of all x such that $2^{kr}\|\varphi_k * x\|_p \to 0$ as $|k| \to \infty$.

Function spaces on noncommutative Euclidean space

- $\mathcal{S}_0(\mathbb{R}^d_\theta)$ is dense in $B^a_{p,q}(\mathbb{R}^d_\theta)$ for $1 \leq p < \infty$ and $1 \leq q < \infty$.

- $\mathcal{S}(\mathbb{R}^d_\theta)$ is norm-dense in $W^m_p(\mathbb{R}^d_\theta)$ when $m \geq 0$ and $1 \leq p < \infty$; the density of $\mathcal{S}(\mathbb{R}^d_\theta)$ in $\dot{W}^m_p(\mathbb{R}^d_\theta)$ holds only when $m \geq 0$ and $1 < p < \infty$

- The dual space of $B^a_{p,q}(\mathbb{R}^d_\theta)$ coincides isomorphically with $B^{-a}_{p',q'}(\mathbb{R}^d_\theta)$ for $1 \leq p < \infty$ and $1 \leq q < \infty$

- J^b and I^b are isomorphisms between $B^a_{p,q}(\mathbb{R}^d_\theta)$ and $B^{a-b}_{p,q}(\mathbb{R}^d_\theta)$.

• The first results [Mcdonald, Sukochev and Xiong, Commun. Math. Phys. 2019] concerning quantum differentiability in the noncommutative euclidean space are the characterizations of the Schatten $S_{d,\infty}$ properties of

$$đx := \sum_{j=1}^{d} \gamma_j \otimes đx_j \tag{5}$$

on noncommutative euclidean space $\mathbb{R}^d_\theta$.

• γ_j's denote the d-dimensional euclidean gamma matrices, and $đx_j := \mathrm{i}[R_j, M_x]$, where for $1 \le j \le d$, $R_j = T_\phi$ for $\phi(s) = \frac{s_j}{|s|}$ denote the quantum counterpart of Riesz transforms on $\mathbb{R}^d_\theta$.

Backgrounds and motivations

- Our research is motivated by the following:

Theorem 2 (Mcdonald, Sukochev and Xiong, 2019)

$\,\mathrm{đ}x_i$ has bounded extension in $S_{d,\infty}$ for every $1 \leq i \leq d$ iff x belongs to the homogeneous Sobolev space $\dot{W}^1_d(\mathbb{R}^d_\theta)$.

- One related result is the formula on Dixmier Trace. For any continuous normalised trace tr on $S_{1,\infty}$ we have

$$\mathrm{Tr}_\omega(|\mathrm{đ}x|^d) = c_d \Big\| \sum_{j=1}^{d} \gamma_j \otimes \Big(\partial_j x - s_j \sum_{k=1}^{d} s_k \partial_k x\Big) \Big\|_d^d. \tag{6}$$

Main results

- We aim to extend the aforementioned results to a more general setting. Here are our results.

Theorem 3

Let $d < p < \infty$. *If* $x \in B^{\frac{d}{p}}_{p,p}(\mathbb{R}^d_\theta)$, *then* $\mathbf{C}_{\phi,x}$ *has a bounded extension in* S_p *and*

$$\big\|\mathbf{C}_{\phi,x}\big\|_{S_p} \lesssim_{d,p} \Big[\sup_{s\in\mathbb{S}^{d-1}} |\phi(s)| + \sup_{s\in\mathbb{S}=^{d-1}} |\nabla\phi(s)|\Big] \big\|x\big\|_{B^{\frac{d}{p}}_{p,p}}.$$

Conversely, assume additionally that ϕ *is not constant. If* $x \in \mathbb{R}^d_\theta$ *and* $\mathbf{C}_{\phi,x} \in S_p$, *then* $x \in B^{\frac{d}{p}}_{p,p}(\mathbb{R}^d_\theta)$ *and*

$$\big\|x\big\|_{B^{\frac{d}{p}}_{p,p}} \lesssim_{d,p} \Big[\sup_{s\in\mathbb{S}^{d-1}} |\phi(s)| + \sup_{s\in\mathbb{S}^{d-1}} |\nabla\phi(s)|\Big] \big\|\mathbf{C}_{\phi,x}\big\|_{S_p}.$$

Main results

- For the critical case, i.e., the $S_{d,\infty}$ properties of $\mathbf{C}_{\phi,x}$ for $p \leq d$.

Theorem 4

If $x \in \dot{W}^1_d(\mathbb{R}^d_\theta)$, then $\mathbf{C}_{\phi,x}$ has bounded extension in $S_{d,\infty}$.

Main results

- The following trace formula is new even for classical setting.

Theorem 5

Let $x \in \dot{W}^1_d(\mathbb{R}^d_\theta)$*. Then for every continuous normalised trace* Tr_ω *on* $S_{1,\infty}$*, we have*

$$\mathrm{Tr}_\omega(|\mathbf{C}_{\phi,x}|^d) = C_d \int_{\mathbb{S}^{d-1}} \tau_\theta(\Big|\sum_{1\leq k\leq d} \partial_{s_k}\phi\, \partial_k x\Big|^d)ds.$$

Here the integral over $\mathbb{S}^{d-1}$ *is taken with respect to the rotation-invariant measure* ds *on* $\mathbb{S}^{d-1}$*.*

Proof of Theorem 3: Upper bounds estimate

We use the complex interpolation to obtain the desired estimate. Indeed, we have the following three endpoint cases.

- Let $a > 0, b > 0$ and $a + b < 1$. If $x \in B^{a+b}_{\infty,\infty}(\mathbb{R}^d_\theta)$, then $I^a \mathbf{C}_{\phi,x} I^b \in S_\infty(L_2(\mathbb{R}^d_\theta))$ and

$$\|I^a \mathbf{C}_{\phi,x} I^b\|_{S_\infty} \lesssim_{d,a,b} \|x\|_{B^{a+b}_{\infty,\infty}}.$$

- Let $a > -\frac{d}{2}$, $b > -\frac{d}{2}$ and $a + b + d < 1$. If $x \in B^{a+b+d}_{1,1}(\mathbb{R}^d_\theta)$, then $I^a \mathbf{C}_{\phi,x} I^b \in S_1$ and

$$\left\|I^a \mathbf{C}_{\phi,x} I^b\right\|_{S_1} \lesssim_{d,a,b} \left\|x\right\|_{B^{a+b+d}_{1,1}}. \tag{7}$$

Proof of Theorem 3: Upper bounds estimate

- Let $a, b > -\frac{d}{2}$ and $a + b + \frac{d}{2} < 1$. If $x \in B_{2,2}^{a+b+\frac{d}{2}}(\mathbb{R}_\theta^d)$, then $I^a \mathbf{C}_{\phi,x} I^b \in S_2$ and

$$\left\| I^a \mathbf{C}_{\phi,x} I^b \right\|_{S_2} \lesssim_{d,a,b} \|x\|_{B_{2,2}^{a+b+\frac{d}{2}}}.$$

Theorem 6

Let $1 \leq p \leq \infty$, $a + b + \frac{d}{p} < 1$ *and* $a, b > \max(-\frac{d}{p}, -\frac{d}{2})$. *If* $x \in B_{p,p}^{a+b+\frac{d}{p}}(\mathbb{R}_\theta^d)$, *then* $I^a \mathbf{C}_{\phi,x} I^b$ *belongs to* $B_{p,p}^{a+b+\frac{d}{p}}(\mathbb{R}_\theta^d)$ *and*

$$\left\| I^a \mathbf{C}_{\phi,x} I^b \right\|_{S_p} \lesssim_{d,p,a,b} \|x\|_{B_{p,p}^{a+b+\frac{d}{p}}}.$$

Proof of Theorem 3: Upper bounds estimate

• We end this part with a generalization to higher commutators. Namely, let $\phi_1, \cdots, \phi_N \in C^\infty(\mathbb{S}^{d-1})$ be N non-constant functions. Define

$$\mathbf{C}_{\phi_1,\cdots,\phi_N,x} = [T_{\phi_N}, ..., [T_{\phi_1}, M_x]...] \tag{8}$$

• Theorem 6 extends to higher commutators.

Proof of Theorem 3: Lower bounds estimate

- This part is devoted to the converse results of those in the previous part.

- We need the following nondegeneracy condition:

$$\forall\, s \in \mathbb{R}^d \setminus \{0\}\ \exists\, t \in \mathbb{R}^d \setminus \{0\} \text{ such that } \prod_{i=1}^{N} (\phi_i(s) - \phi_i(t)) \neq 0. \tag{9}$$

For $N = 1$, this condition means that ϕ_1 is not a constant function.

Proof of Theorem 3: Lower bounds estimate

• Denote $\gamma = -(a + a_1 + b + b_1 + d)$ and set

$$\omega(s) = |s|^\gamma \int_{\mathbb{R}^d} \prod_{i=1}^{N} |\phi_i(s+t) - \phi_i(t)|^{2k} \, |s+t|^{a+a_1} |t|^{b+b_1} dt. \tag{10}$$

• Suppose that $\phi_1, ..., \phi_N$ satisfy condition 9, we can show that ω is a homogeneous function of order 0 and never vanishes for $s \neq 0$.

• ω is a Fourier multiplier on $B^r_{1,1}(\mathbb{R}^d_\theta))$ for some r. By a Tauberian result, we see that ω^{-1} is a Fourier multiplier on $B^a_{p,p}(\mathbb{R}^d_\theta)$ for any $a \in \mathbb{R}$.

Proof of Theorem 3: Lower bounds estimate

- For $k \geq 1$ set

$$\mathbf{C}_{N,k,y} = \mathbf{C}_{\underbrace{\phi_1, ..., \phi_N}_{k \text{ tuple}}, \underbrace{\bar{\phi}_1, ...\bar{\phi}_N}_{k-1 \text{ tuple}}, y},$$

- By the duality, we have

$$\langle I^a C_{\phi_1,...,\phi_N,x} I^b, \, I^{a_1} \mathbf{C}_{N,k,y} I^{b_1} \rangle = \langle I^{-\gamma} T_\omega(x), y \rangle.$$

Thus,

$$\|T_\omega(x)\|_{B_{p,p}^{a+b+\frac{d}{p}}} \leq C \|I^a \mathbf{C}_{\phi_1,...,\phi_N,x} I^b\|_{S_p}.$$

The trace formula: Pseudodifferential operator

- Given $f \in \mathcal{S}(\mathbb{R}^d)$ and $\rho \in S^m(\mathbb{R}^d; \mathcal{S}(\mathbb{R}^d_\theta))$, we set

$$P_\rho(\lambda_\theta(f)) = \int_{\mathbb{R}^d} f(\xi)\rho(\xi)\lambda_\theta(\xi)d\xi.$$

The operator P_ρ is called the pseudo-differential operator of symbol ρ.

The trace formula

- We replace T_ϕ by another Fourier multiplier $T_{\widetilde{\phi}}$ whose symbol is smooth on the whole $\mathbb{R}^d$.

- We put

$$A = \frac{1}{2\pi \mathrm{i}} \sum_{1 \le k \le d} T_{|\xi| \partial_{\xi_k} \widetilde{\phi}} M_{\partial_k x}. \tag{11}$$

We are going to reduce the computation of $\mathrm{Tr}_\omega(|\mathbf{C}_{\phi,x}|^d)$ to that of $\mathrm{Tr}_\omega(|A|^d(1+\Delta)^{-\frac{d}{2}})$.

The trace formula

- Compute the symbol of $\mathbf{C}_{\widetilde{\phi},x} - AJ^{-1}$ is of order -2. We see that

$$M_y \mathbf{C}_{\widetilde{\phi},x} - M_y A J^{-1} \in S_{\frac{d}{2},\infty}.$$

Then we have

$$|M_y \mathbf{C}_{\phi,x}|^d - |M_y A|^d J^{-d} \in S_1.$$

- We have we have

$$\mathrm{Tr}_\omega(|M_y \mathbf{C}_{\phi,x}|^d) = \mathrm{Tr}_\omega(|M_y A|^d J^{-d}).$$

So we can apply the trace formula in [McDonald, Sukochev and Zanin, Math. Ann. 2018] to deduce our trace formula.

Remark

Remark 7

We can replace the quantum Euclidean space $\mathbb{R}^d_\theta$ by general twisted crossed products of Euclidean space, namely, given a von Neumann algebra quipped with a normal semifinite faithful weight τ, we set

$$\mathcal{R} = \mathcal{M} \rtimes_{\alpha,\sigma} \mathbb{R}^d.$$

Then we can extend the previous results of Schatten p class memberships to the commutators on $\mathcal{R}$.