Gelfand–Phillips type properties in noncommutative analysis

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Abstract

(i) We characterize noncommutative symmetric spaces $E(\mathcal{M}, \tau)$ having the Gelfand–Phillips property;

(ii) We characterize noncommutative symmetric spaces $E(\mathcal{M}, \tau)$ having the WCG-property.

τ-measurable operators

A closed and densely defined operator $x : D(x) \to H$ is said to be

1. **affiliated** with the von Neumann algebra $\mathcal{M}$ if it commutes with every unitary operator from the commutant $\mathcal{M}'$ of $\mathcal{M}$;
2. τ-**measurable** if it is affiliated with $\mathcal{M}$ and if there exists $\lambda > 0$ such that
$$\tau(e^{|x|}(\lambda, \infty)) < \infty.$$

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The collection of all τ-measurable operators with respect to $\mathcal{M}$ is denoted by $S(\mathcal{M}, \tau)$.

NC-version of decreasing rearrangement

Definition

Let $x \in S(\mathcal{M}, \tau)$. The generalised singular value function (decreasing, right-continuous) $\mu(x) : t \to \mu(t; x)$, $t \geq 0$, of the operator x is defined by setting

$$\mu(t; x) = \inf \left\{ \|xp\|_\infty : \ p \in \mathcal{P}(\mathcal{M}), \ \tau(\mathbf{1} - p) \leq t \right\}.$$

Noncommutative Symmetric Spaces

Definition (V.I. Ovčinnikov, Symmetric spaces of measurable operators, Dokl. Akad. Nauk SSSR, 1970)

[a] We say that $(E(\mathcal{M},\tau), \|\cdot\|_E)$ is a **noncommutative symmetric space affiliated with** $\mathcal{M}$ if the following hold:

1. $E(\mathcal{M},\tau)$ is a subset of $S(\mathcal{M},\tau)$;
2. $(E(\mathcal{M},\tau), \|\cdot\|_E)$ is a Banach space;
3. If $x \in E(\mathcal{M},\tau)$ and if $y \in S(\mathcal{M},\tau)$ are such that $\mu(y) \leq \mu(x)$, then $y \in E(\mathcal{M},\tau)$ and $\|y\|_E \leq \|x\|_E$.

[a] P. G. Dodds , B. de Pagter , F. A. Sukochev, Noncommutative Integration and Operator Theory, Springer Nature, PM volume 349, 2023.

Order continuity

Let $E(\mathcal{M}, \tau)$ be a noncommutative symmetric space.
The norm $\|\cdot\|_E$ is called order continuous if $\|x_\alpha\|_E \downarrow 0$ whenever $0 \leq x_\alpha \downarrow 0 \subset E(\mathcal{M}, \tau)$.

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Proposition

$E(0, \infty)$ is separable if and only if it has order continuous norm. [a]

[a] Aliprantis–Burkinshaw, Positive Operators, 1985.

Gelfand-Phillips property

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Recall that a (norm) bounded subset A of a Banach space E is called *limited* if given a weak*-null sequence $(f_n)_n$ in E^* we have

$$\lim_{n \to \infty} \sup_{x \in A} |f_n(x)| = 0.$$

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Each relatively (norm) compact subset of E is limited. If, conversely, every limited subset of E is relatively (norm) compact, then E is said to have the *Gelfand–Phillips property*.

Buhvalov's theorem[1]

Theorem

For a σ-Dedekind complete Banach lattice E the following statements are equivalent:
(i) The norm on E is order continuous;
(ii) E has the Gelfand–Phillips property.

[1] A.V. Buhvalov, *Locally convex spaces that are generated by weakly compact sets,* Vestnik Leningrad. Univ. No. 7 Mat. Meh. Astronom. Vyp. 2 (1973), 11–17 (in Russian).

Noncommutative version of Buhvalov's theorem

Theorem (Huang–N–Pliev–Sukochev, TAMS, 2024)

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. A strongly symmetric space $E(\mathcal{M}, \tau)$ has the Gelfand–Phillips property if and only if its norm is order continuous.

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Proof.

($\Rightarrow$) Since every closed subspace of a space having the Gelfand–Phillips property inherits the Gelfand–Phillips property [a], it follows from Buhvalov's Theorem and the fact that $\|\cdot\|_{\ell_\infty}$ is not order continuous that $E(\mathcal{M}, \tau)$ does not have a subspace isomorphic to ℓ_∞. Hence, $E(\mathcal{M}, \tau)$ has order continuous norm. [b]

□

[a] G. Emmanuele, *Gelfand–Phillips property in a Banach space of vector valued measures,* Math. Nachr. 127 (1986), 21–23.

[b] P. Dodds, B. de Pagter, *Properties* (u) *and* (V^*) *of Pelczynski in symmetric spaces of* τ*-measurable operators*, Positivity 15 (2011), 571–594. Theorem 3.7

WCG property

WCG property

Let us recall that a Banach space E is said to be a *WCG-space* (or E has WCG property) if E contains a linearly dense weakly compact subset K, i.e., $E = \overline{\mathrm{span} K}$ [2].

[2] W. Wnuk, *Banach Lattices with Order Continuous Norms*, Advanced Topics in Math, Adam Mickiewicz Univ., Poznan, Polish Scientific Publishers PWN 1999.

A symmetric function space $E(0,\infty)$ is separable if and only if it has order continuous norm. However, a noncommutative symmetric space $E(\mathcal{M},\tau)$ is not necessarily separable when $\|\cdot\|_{E(\mathcal{M},\tau)}$ is order continuous [3].

[3]F. Sukochev, *Linear–topological classification of separable L_p-spaces associated with von Neumann algebras of type I*, Israel. J. Math. 115 (2000), 137–156.

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Proposition

Let $E(\mathcal{M},\tau)$ be a strongly symmetric space which is a WCG-space, then $E(\mathcal{M},\tau)$ has order continuous norm.

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Proposition

Let $E(\mathcal{M},\tau)$ be a strongly symmetric space which is a WCG-space, then $E(\mathcal{M},\tau)$ has order continuous norm.

Proof.

Any strongly symmetric WCG-space is automatically Gelfand–Phillips[a]. Hence, $E(\mathcal{M},\tau)$ has order continuous norm. □

[a] W. Wnuk, *Banach Lattices with Order Continuous Norms*, Advanced Topics in Math, Adam Mickiewicz Univ., Poznan, Polish Scientific Publishers PWN 1999.

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Theorem

For a strongly symmetric space $E(\mathcal{M},\tau)$ *affiliated with a von Neumann algebra* $\mathcal{M}$ *equipped with a semifinite faithful normal trace* τ *such that* $E(\mathcal{M},\tau)^{\times} \subset S_0(\mathcal{M},\tau)$*, we have*

$E(\mathcal{M},\tau)$ *has order continuous norm if and only if it is a WCG-space.*

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For a strongly symmetric space $E(\mathcal{M},\tau)$ affiliated with a von Neumann algebra $\mathcal{M}$ equipped with a semifinite faithful normal trace τ such that $E(\mathcal{M},\tau)^{\times} \subset S_0(\mathcal{M},\tau)$, we have

$E(\mathcal{M},\tau)$ has order continuous norm if and only if it is a WCG-space.

Proof.

($\Rightarrow$) $E(\mathcal{M},\tau)$ is fully symmetric. Note for $0 \neq p \in \mathcal{P}(\mathcal{M})$ we have $\Omega_{\mathcal{M}}(p) = \{y \in S(\mathcal{M},\tau) : y \prec\prec p\}$ is $\sigma(E(\mathcal{M},\tau), E(\mathcal{M},\tau)^{\times})$-compact in $E(\mathcal{M},\tau)$. For any $y \in S(\mathcal{M},\tau)$ with $\tau(l(y)) < \infty$ ($y \in \mathcal{F}(\tau)$), we have

$$\mu(y) \leq \|y\|_{\mathcal{M}}\, \chi_{(0,\tau(l(y)))} \prec\prec \|y\|_{\mathcal{M}} \frac{\tau(l(y))}{\min\{\tau(l(y)),\tau(p)\}}\mu(p).$$

That is, $\Omega_{\mathcal{M}}(p)$ generates $\mathcal{F}(\tau)$. Hence $\overline{\mathrm{span}(\Omega_{\mathcal{M}}(p))} = E(\mathcal{M},\tau)$. □

Theorem

Let $\mathcal{M}$ be a σ-finite von Neumann algebra equipped with a semifinite faithful normal trace τ, then a strongly symmetric space $E(\mathcal{M}, \tau)$ has order continuous norm if and only if it is a WCG-space.

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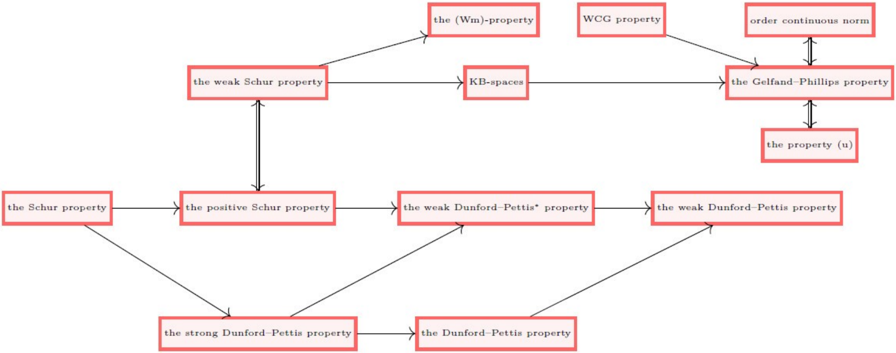
Proof.

($\Rightarrow$) There exists an increasing sequence of τ-finite projections p_n's such that $p_n \uparrow_n \mathbf{1}$. It follows from Theorem above that $E(\mathcal{M}_{p_n},\tau)$ is a WCG-space. Hence, there is a weakly compact $K_n \subset E(\mathcal{M}_{p_n},\tau)$ such that $E(\mathcal{M}_{p_n},\tau) = \overline{\mathrm{span} K_n}$, $n \in \mathbb{N}$. We claim that $\cup_{n\geq 1} \frac{1}{n} K_n$ is a weakly compact subset generating $E(\mathcal{M},\tau)$. For each $x \in E(\mathcal{M},\tau)$, we have $\|p_n x p_n - x\|_E \to 0$ as $n \to \infty$. It suffices to prove that $\cup_{n\geq 1} \frac{1}{n} K_n$ is weakly compact. Indeed, for any sequence $(x_k)_k$ in $\cup_{n\geq 1} \frac{1}{n} K_n$ either

1. all x_k's belong to $\cup_{1\leq k\leq n} \frac{1}{k} K_k$ for some $n < \infty$, then done.
2. there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} \in \frac{1}{m_k} K_{m_k}$ with $n_k, m_k \to_k \infty$. In this case, we have $x_{n_k} \to_k 0$ in norm.

□

Interrelations between properties in NC setting



$E(\mathcal{M}, \tau)$ is a strongly symmetric space affiliated with a semifinite von Neumann algebra $\mathcal{M}$