Pseudodifferential operators in the noncommutative setting

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Classical Ψdo

A Ψdo differential operator is formally defined as

$$Au(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\langle x-y,\theta\rangle} a(x,y,\theta) u(y)\, dy\, d\theta,$$

where $a(x,y,\theta) \in S^m_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$, meaning that

$$|\partial_\theta^\alpha \partial_x^\beta a(x,y,\theta)| \leq C_{\alpha,\beta,K}(1+|\theta|^2)^{\frac{m-\rho|\alpha|+\delta|\beta|}{2}}$$

for $(x,y) \in K$ compact and $\theta \in \mathbb{R}^d$.

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We usually fix $(\rho,\delta) = (1,0)$, more generally $0 \leq \delta \leq \rho \leq 1$ but $\delta \neq 1$, singular case $(\rho,\delta) = (1,1)$.

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- **Example :** $A = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$, where $a_\alpha(x) \in C^\infty(\mathbb{R}^d)$. Formally

$$Au(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle}\Big(\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha\Big)\,dy\,d\xi.$$

Symbol

Formally the symbol of a Ψdo is determined as

$$\sigma_A(x,\xi) = e^{-ix\cdot\xi} A e^{ix\cdot\xi}.$$

Then

$$Au(x) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} \sigma_A(x,\xi)\widehat{u}(\xi)\, d\xi = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle}\sigma_A(x,\xi)u(y)\, dy\, d\xi.$$

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- **Torus case :** For $\sigma \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$

$$P_\sigma f(x) = \sum_{m\in\mathbb{Z}^d} \sigma(x,m)\widehat{f}(m)e^{im\cdot x}.$$

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$$P_\sigma f(x) = \sum_{m\in\mathbb{Z}^d} \sigma(x,m)\widehat{f}(m)e^{im\cdot x}.$$

If $\sigma = \sigma(x)$ then P_σ is a pointwise multiplier, if $\sigma = \sigma(m)$ then P_σ is a Fourier multiplier.

Key theorems

Symbol calculus

Theorem

ρ_1, ρ_2 are symbols in $S^{n_1}(\mathbb{R}^d \times \mathbb{R}^d)$ and $S^{n_2}(\mathbb{R}^d \times \mathbb{R}^d)$ resp. Then there exists a symbol ρ_3 in $S^{n_1+n_2}(\mathbb{R}^d \times \mathbb{R}^d)$ such that $P_{\rho_3} = P_{\rho_1} P_{\rho_2}$. Moreover, for any $N_0 \geq 0$,

$$\rho_3 - \sum_{|\alpha|_1 < N_0} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha \rho_1(x,\xi) D_x^\alpha \rho_2(x,\xi) \in S^{n_1+n_2-N_0}(\mathbb{R}^d \times \mathbb{R}^d).$$

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Regularity on function spaces (L_p, Sobolev, Besov, local Hardy), we focus on Hilbert-Sobolev spaces $\|f\|_{H_2^s} := \|(1-\Delta)^{s/2} f\|_2$

Theorem

For $\rho \in S^n$, P_σ is bounded from H_2^s to H_2^{s-n}. For $n \leq 0$, P_σ is bounded on H_2^s, in particular on L_2.

Ψdo meets C^*-algebra

- **Operator valued setting :** Let X be a Banach space. $u : \mathbb{R}^d \to X$, $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to B(X)$, we may define

$$Au(x) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\sigma_A(x,\xi)\widehat{u}(\xi)\,d\xi.$$

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• **Group action setting :** Let $(s,u) \to \alpha_s(u)$ be a C^*-action on the C^*-algebra $\mathcal{A}$. For $u \in \mathcal{A}$ and smooth $\sigma : \mathbb{R}^d \to \mathcal{A}$, Ψdo is defined as

$$P_\sigma u = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\langle s,\xi\rangle} \sigma(\xi)\alpha_{-s}(u)\, ds\, d\xi.$$

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• **Example :** Quantum torus $\mathcal{A}_\theta$, with a periodic group action.

Quantum tori

• **Noncommutative tori :** $d \geq 2$ and $\theta = (\theta_{kj})$ real skew-symmetric $d \times d$-matrix. The quantum torus $\mathcal{A}_\theta$ is the universal C^*-algebra generated by d unitaries $U_1, \ldots, U_d$ satisfying the following commutation relation

$$U_k U_j = e^{2\pi \mathrm{i} \theta_{kj}} U_j U_k, \ j, k = 1, \ldots, d.$$

• **Trace :** Let $\mathcal{P}_\theta$ denote the involutive subalgebra of polynomials, dense in $\mathcal{A}_\theta$. For any polynomial $x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m$ define $\tau(x) = \alpha_{\mathbf{0}}$. Then τ extends to a faithful tracial state on $\mathcal{A}_\theta$.

Let $\mathbb{T}^d_\theta$ be the w*-closure of $\mathcal{A}_\theta$ in the GNS representation of τ. Then τ becomes a normal faithful tracial state on $\mathbb{T}^d_\theta$. Thus $(\mathbb{T}^d_\theta, \tau)$ is a noncommutative(=quantum) probability space.

• **Noncommutative L_p-spaces :** For $1 \leq p < \infty$ and $x \in \mathbb{T}^d_\theta$ let $\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}$ with $|x| = (x^* x)^{\frac{1}{2}}$. This defines a norm on $\mathbb{T}^d_\theta$. The corresponding completion is denoted by $L_p(\mathbb{T}^d_\theta)$. We also set $L_\infty(\mathbb{T}^d_\theta) = \mathbb{T}^d_\theta$.

Correspondence : torus and quantum torus.

$$\begin{aligned}
\text{probability space } (\mathbb{T}^d, \mu) &\leftrightarrow \text{noncom probability space}(\mathbb{T}^d_\theta, \tau) \\
\text{commutative algebra } L_\infty(\mathbb{T}^d) &\leftrightarrow \text{noncommutative algebra } \mathbb{T}^d_\theta \\
\text{integration against } \mu \int_{\mathbb{T}^d} &\leftrightarrow \text{trace } \tau \\
\int_{\mathbb{T}^d} f d\mu &\leftrightarrow \tau(x) \\
\|f\|_p = \Big(\int_{\mathbb{T}^d} |f|^p d\mu \Big)^{\frac{1}{p}} &\leftrightarrow \|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}} \\
L_p(\mathbb{T}^d) &\leftrightarrow L_p(\mathbb{T}^d_\theta)
\end{aligned}$$

Symbol calculus

$S^n(\mathbb{R}^d; \mathcal{S}(\mathbb{T}_\theta^d))$ consists of maps $\rho \in C^\infty(\mathbb{R}^d; \mathcal{S}(\mathbb{T}_\theta^d))$ s.t.

$$\|D_\theta^\alpha D_\xi^\beta \rho(\xi)\| \leq C_{\alpha,\beta}(1+|\xi|^2)^{\frac{n-|\beta|}{2}}.$$

Theorem (Baaj, Connes, 1980s)

ρ_1, ρ_2 are symbols in $S^{n_1}(\mathbb{R}^d; \mathcal{S}(\mathbb{T}_\theta^d))$ and $S^{n_2}(\mathbb{R}^d; \mathcal{S}(\mathbb{T}_\theta^d))$ resp. Then there exists a symbol ρ_3 in $S^{n_1+n_2}(\mathbb{R}^d; \mathcal{S}(\mathbb{T}_\theta^d))$ such that $P_{\rho_3} = P_{\rho_1} P_{\rho_2}$. Moreover, for any $N_0 \geq 0$,

$$\rho_3 - \sum_{|\alpha|_1 < N_0} \frac{(2\pi i)^{-|\alpha|_1}}{\alpha!} D_\xi^\alpha \rho_1 D_\theta^\alpha \rho_2 \in S^{n_1+n_2-N_0}(\mathbb{R}^d; \mathcal{S}(\mathbb{T}_\theta^d)).$$

Mapping property

Theorem (Xia-X, Ha-Lee-Ponge)

For $\rho \in S^n$, P_σ is bounded from $H_2^s(\mathbb{T}_\theta^d)$ to $H_2^{s-n}(\mathbb{T}_\theta^d)$. For $n \leq 0$, P_σ is bounded on $H_2^s(\mathbb{T}_\theta^d)$, in particular on $L_2(\mathbb{T}_\theta^d)$.

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Easy observation :

1. $\theta = 0$, for $\rho_1, \rho_2 \in S^0$, $[P_{\rho_1}, P_{\rho_2}] \in S^{-1}$, is compact, since

$$\mathrm{sym}(P_{\rho_1} \circ P_{\rho_2}) - \rho_1\rho_2 \in S^{-1}, \quad \mathrm{sym}(P_{\rho_2} \circ P_{\rho_1}) - \rho_2\rho_1 \in S^{-1};$$

2. general θ, if $\rho_1, \rho_2 \in S^0$ are commutative, $[P_{\rho_1}, P_{\rho_2}]$, is compact.
3. special case : $\rho_1 : \mathbb{R}^d \to \mathbb{C}$, $\rho_2 \in \mathcal{S}(\mathbb{T}_\theta^d)$, then $[P_{\rho_1}, P_{\rho_2}]$ is compact .

Abstract construction/definition of 0-order Ψdo

Theorem (McDonald-Sukochev-Zanin)

Let $\pi_1 : \mathcal{A}_1 \to \mathcal{B}(H)$ *and* $\pi_2 : \mathcal{A}_2 \to \mathcal{B}(H)$ *be representations of* C^**-algebras,* Π *be the* C^**-algebra generated by* $\pi_1(\mathcal{A}_1)$ *and* $\pi_2(\mathcal{A}_2)$*. If*

1. $\mathcal{A}_1, \mathcal{A}_2$ *are unital and* $\mathcal{A}_2$ *is abelian ;*
2. $[\pi_1(a_1), \pi_2(a_2)]$ *is compact ;*
3. $\sum_{k=1}^n \pi_1(a_k)\pi_2(b_k)$ *compact implies* $\sum a_k \otimes b_k = 0$*,*

then $\exists \, \mathrm{sym} : \Pi \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ *such that*

$$\mathrm{sym}(\pi_1(a_1)) = a_1 \otimes 1, \quad \mathrm{sym}(\pi_2(a_2)) = 1 \otimes a_2.$$

Abstract Ψdo

Example :

1. $\mathcal{A}_1 = C(\mathbb{T}^d), \mathcal{A}_2 = C(\mathbb{S}^{d-1}), \pi_1(f) = M_f, \pi_2(g) = g(\frac{\nabla}{(-\Delta)^{1/2}})$;
2. $\mathcal{A}_1 = \mathbb{C} + C_0(\mathbb{R}^d), \mathcal{A}_2 = C(\mathbb{S}^{d-1}), \pi_1(f) = M_f, \pi_2(g) = g(\frac{\nabla}{(-\Delta)^{1/2}})$;
3. $\mathcal{A}_1 = C(\mathbb{T}^d_\theta), \mathcal{A}_2 = C(\mathbb{S}^{d-1}), \pi_1(f) = M_f, \pi_2(g) = g(\frac{\nabla}{(-\Delta)^{1/2}})$.

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Proof : Denote $q : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$. We have a natural $*$-isomorphism $\pi : \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \to \mathcal{B}(H)$ determined by

$$\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2).$$

The map sym is defined as

$$\mathrm{sym} = \pi^{-1} \circ q : \Pi \to \mathcal{B}(H)/\mathcal{K}(H) \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2.$$

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Remark : Positive order Ψdos are not considered ; Negative order Ψdos are killed by the quotient map.

Abstract Ψdo of negative order

Ideals of compact op. $\mu_k \in \ell_p \leftrightarrow T \in \mathcal{S}_p$, and $\mu_k = O(k^{-\frac{1}{p}}) \leftrightarrow T \in \mathcal{S}_{p,\infty}$.
Observation : If $\sigma \in S^{-\alpha}$ then $P_\sigma \in \mathcal{S}_{d/\alpha,\infty}$, in particular

$$J^\alpha = (1-\Delta)^{-\frac{\alpha}{2}}, I^\alpha = (-\Delta)^{-\frac{\alpha}{2}} \in \mathcal{S}_{d/\alpha,\infty}.$$

Theorem (McDonald-Sukochev-X. 2020CMP)

Let $\alpha, \beta \in \mathbb{R}$. For smooth x, if $\alpha < \beta + 1$, then $[J^\alpha, x]J^{-\beta} \in \mathcal{L}_{\frac{d}{\beta-\alpha+1},\infty}$. If $\alpha = \beta + 1$, then the operator $[J^\alpha, x]J^{-\beta}$ has bounded extension.

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Theorem (Sukochev-X.-Zanin 2023JFA)

Let $\alpha, \beta \in \mathbb{R}$, and m be a homogeneous symbol of order 0. For smooth x, if $\alpha < \beta + 1$, then $[T_m J^\alpha, x]J^{-\beta} \in \mathcal{L}_{\frac{d}{\beta-\alpha+1},\infty}$. If $\alpha = \beta + 1$, then the operator $[T_m J^\alpha, x]J^{-\beta}$ has bounded extension.

Asymptotic limits of Ψdo

$T \in \mathcal{S}_{p,\infty}$ means $\sup_t t^{1/p}\mu(t, T) < \infty$.

Now we are interested in $\lim_{t\to\infty} t^{1/p}\mu(t, T)$.

On $\mathbb{T}^d_\theta$, one can calculate

$$\lim_{t\to\infty} t^{1/d}\mu(t, J^{-1}) = d^{-\frac{1}{d}}.$$

- **Question** For 0-order Ψdo T, what is $\lim_{t\to\infty} t^{1/d}\mu(t, TJ^{-1})$?

Asymptotic limits of Ψdo

Another proof of $\lim_{t\to\infty} t^{1/d}\mu(t, J^{-1}) = d^{-\frac{1}{d}}$ is done using noncommutative Taubrian Theorem (Wiener-Ikehara Theorem) :

Theorem (McDonald-Sukochev-Zanin)

Let $p > 2$ and let $0 \leq A, B \in \mathcal{S}_\infty$ satisfy $B \in \mathcal{S}_{p,\infty}$ and $[B, A^{\frac{1}{2}}] \in \mathcal{S}_{\frac{p}{2},\infty}$. If there exists $0 \leq c \in \mathbb{R}$ such that the function

$$F_{A,B}(z) := \mathrm{Tr}(A^z B^z) - \frac{c}{z-p}, \quad z \in \mathbb{C}, \quad \Re(z) > p,$$

admits a continuous extension to the closed half plane $\{z \in \mathbb{C} : \Re(z) \geq p\}$, then there exists the limit

$$\lim_{t\to\infty} t^{\frac{1}{p}}\mu(t, AB) = \left(\frac{c}{p}\right)^{\frac{1}{p}}.$$

Asymptotic limits of Ψdo

The calculation $\lim_{t\to\infty} t^{1/d}\mu(t, J^{-1})$ is reduced to the meromorphic continuation of $\mathrm{Tr}(J^z)$ on $\{z \in \mathbb{C} : \Re(z) > d\}$ to $\{z \in \mathbb{C} : \Re(z) \geq d\}$. Here

$$\mathrm{Tr}(J^z) = \sum_{m\in\mathbb{Z}^d} (1 + |m|^2)^{\frac{z}{2}}$$

has a simple pole at $z = d$. (see Shubin's book for more general op.)

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Similarly we can establish the following

Theorem (Sukochev-X.-Zanin)

Let $d \geq 2$. If $T \in \Pi(C(\mathbb{T}^d_\theta), C(\mathbb{S}^{d-1}))$, then

$$\lim_{t\to\infty} t^{\frac{1}{d}}\mu(t, T(-\Delta)^{-\frac{1}{2}}) = d^{-\frac{1}{d}} \|\mathrm{sym}(T)\|_{L_d(L_\infty(\mathbb{T}^d_\theta)\bar{\otimes}L_\infty(\mathbb{S}^{d-1}))}.$$

Similar results for classical Ψdo are obtained by Birman, Solomyak and their coauthors.

Summary

$T(-\Delta)^{-\frac{1}{2}}$ is a pseudodifferential operator of order -1 :

$T(-\Delta)^{-\frac{1}{2}} \sim$	$\sigma_{-1}(x,\xi)$	$+\sigma_{-2}(x,\xi)$	$+\sigma_{-3}(x,\xi)$	$+\sigma_{-4}(x,\xi)$	$+\cdots$
Ψdo	S^{-1}	S^{-2}	S^{-3}	S^{-4}	$\cdots$
Cpt. op.	$\mathcal{S}_{d,\infty}$	$\mathcal{S}_{d/2,\infty}$	$\mathcal{S}_{d/3,\infty}$	$\mathcal{S}_{d/4,\infty}$	$\cdots$

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Ψdo		S^{-1}	S^{-2}	S^{-3}	S^{-4}	$\cdots$
Cpt. op.		$\mathcal{S}_{d,\infty}$	$\mathcal{S}_{d/2,\infty}$	$\mathcal{S}_{d/3,\infty}$	$\mathcal{S}_{d/4,\infty}$	$\cdots$

When taking the asymptotic limit, $\sigma_{-k}, k > 1$ vanish.

$$\lim_{t\to\infty} t^{\frac{1}{d}} \mu(t, T(-\Delta)^{-\frac{1}{2}}) = d^{-\frac{1}{d}} \|\sigma_{-1}\|_d.$$

Quantum differential of Connes

• **Def :** Let $\mathcal{A}$ be an involutive algebra over $\mathbb{C}$. Then a **Fredholm module** over $\mathcal{A}$ is given by

1. an involutive representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$,
2. an operator $F = F^*$, $F^2 = 1$, on $\mathcal{H}$ such that $[F, \pi(a)]$ is a compact operator for any $a \in \mathcal{A}$.

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• **Quantized calculus** of differential forms :

1. the differential df of $f \in \mathcal{A}$:

$$\text{đ}f = i[F, f] = i(Ff - fF);$$

2. the integration $T \mapsto \mathrm{Tr}_\omega(T)$.

Traces on $\mathcal{S}_{1,\infty}$

Definition

A linear functional $\varphi : \mathcal{S}_{1,\infty} \to \mathbb{C}$ is called a trace if $\varphi(AB) = \varphi(BA)$ for every $A \in \mathcal{S}_{1,\infty}$ and for every $B \in B(H)$.

The trace φ is called normalised if $\varphi(\mathrm{diag}(\{\frac{1}{k+1}\}_{k\geq 0})) = 1$.
There exists a plethora of (normalised) traces on $\mathcal{S}_{1,\infty}$. The most famous ones are Dixmier traces.

Dixmier traces

An extended limit is a bounded functional ω in ℓ_∞ which extends the "limit" functional on the subspace c of convergent sequences. $(\ell_\infty)^* = (\ell_\infty)^*_n \oplus (\ell_\infty)^*_s$, then $\omega \in (\ell_\infty)^*_s$.

Definition (Dixmier)

If ω is an extended limit then the functional

$$T \to \omega\left(\frac{1}{\log(n+2)}\sum_{k=0}^{n}\mu(k,T)\right), \quad 0 \leq T \in \mathcal{L}_{1,\infty}$$

is finite and additive on the positive cone of $\mathcal{S}_{1,\infty}$. Thus, it uniquely extends to a unitarily invariant linear functional on $\mathcal{S}_{1,\infty}$. The latter is called a Dixmier trace and is denoted by Tr_ω.

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Thus for a compact operator $T \in \mathcal{S}_{1,\infty}$,

$$\lim_{t\to\infty} t\, \mu(t, T) = \mathrm{Tr}_\omega(T).$$

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C^*-algebra : $\mathcal{A}_\theta$;   Hilbert space : $L_2(\mathbb{T}^d_\theta)$.

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$$F = \mathrm{sgn}(\mathcal{D}) = \sum_j \gamma_j \otimes \frac{D_j}{\sqrt{D_1^2 + \cdots + D_d^2}}.$$

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• **Quantum derivative :** Let $M_x : y \mapsto xy$ left multiplication representing $\mathcal{A}_\theta$ on $L_2(\mathbb{T}_\theta^d)$.

$$đx = i[F, M_x].$$

Question in Connes' framework

Question 1 : characterise differentiable elements on $\mathbb{T}_\theta^d$.

Theorem (McDonald-Sukochev-X. 2019CMP)

For $x \in \mathbb{T}_\theta^d$, $đx \in \mathcal{S}_{d,\infty}$ iff $x \in \dot{W}_d^1(\mathbb{T}_\theta^d)$.

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For $x \in \mathbb{T}_\theta^d$, $\bar{d}x \in \mathcal{S}_{d,\infty}$ iff $x \in \dot{W}_d^1(\mathbb{T}_\theta^d)$.

Question 2 : calculate the quantum integration (Dixmier trace) of $\bar{d}x$.

Theorem (Sukochev-X.-Zanin 2023JFA)

Let $d \geq 2$. If $x \in \dot{W}_d^1(\mathbb{T}_\theta^d)$, then

$$\lim_{t\to\infty} t^{\frac{1}{d}}\mu(t, \bar{d}x) = d^{-\frac{1}{d}}\Big\|\sum_{j=1}^{d}\gamma_j \otimes D_j x \otimes 1 - \sum_{j,k=1}^{d}\gamma_j \otimes D_k x \otimes s_k s_j\Big\|_d.$$

Conclusion

• Let $x \in \dot{W}^1_d(\mathbb{T}^d_\theta)$. For any continuous normalized trace φ on $\mathcal{L}_{1,\infty}$ we have

$$\varphi(|\bar{d}x|^d) = c_d \int_{\mathbb{S}^{d-1}} \tau\Big(\big(\sum_{j=1}^{d} |\partial_j x - s_j \sum_{k=1}^{d} s_k \partial_k x|^2\big)^{\frac{d}{2}}\Big) ds \approx_d \|x\|_{\dot{W}^1_d}.$$

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• Alain Connes' idea (1988) : "*The next result shows how to pass from quantized 1-forms to ordinary forms, not by a classical limit, but by a direct application of the Dixmier trace.*"

$$\mathrm{Tr}_\omega(|\bar{d}f|^d) = c_d \int_{\mathbb{T}^d} (\sum_{j=1}^d |\partial_j f(\xi)|^2)^{\frac{d}{2}} d\xi = c_d \|f\|^d_{\dot{W}^1_d}.$$

Thank you !