KU LEUVEN

Harmonic decompositions of cocycles with coefficients in Banach spaces

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0 Outline

1. Cohomology with coefficients in Banach spaces

2. Weakly uniquely stationary representations

3. Application to cohomology

1 Outline

1 Motivation

Group cohomology is a set of mathematical tools used to study groups. It...

- ... encodes interesting connections between algebraic and geometric or topological properties of groups.
- ... touches many other important notions, e.g. amenability, Kazhdan's property (T), a-T-menability, and the Haagerup property.

Personal motivation: Study of 1-cocycles that naturally occur in compact quantum metric space theory.

1 Cohomology with coefficients in Banach spaces

Setting: Let G be a topological group, V a Banach space, and let $\rho : G \to \mathsf{GL}(V) \subseteq \mathcal{B}(V)$ be a uniformly bounded, strongly operator continuous linear representation.

Equip

$$C^n(G,V)^G := \{f : G^n \to V \mid f \text{ continuous and equivariant}\}$$

with the topology of uniform convergence on compact subsets. Consider the cochain complex

$$C^0(G,V)^G \xrightarrow{\partial^0} C^1(G,V)^G \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-1}} C^n(G,V)^G \xrightarrow{\partial^n} \dots,$$

with the *standard differentials*

$$(\partial^n f)(g_1, \dots, g_{n+1}) := \sum_{i=1}^{n+1} (-1)^{i+1} f(g_1, \dots, \widehat{g}_i, \dots, g_{n+1}).$$

The ∂^n are continuous with $\partial^{n+1} \circ \partial^n = 0$ for all $n \in \mathbb{N}$.

Definition

The subspace of *cocycles* is given by $Z^n(G,V) := \ker(\partial^{n+1})$, whereas $B^n(G,V) := \text{im}(\partial^n)$ denotes the subspace of *coboundaries*. Define the *n-th continuous reduced cohomology group* by

$$\overline{H}_c^n(G,V) := Z^n(G,V)/\overline{B^n(G,V)}.$$

Cohomology has mostly been studied for Hilbert spaces. In recent years the study of general Banach spaces has gained increased attention.

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1. Cohomology with coefficients in Banach spaces
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2 Weakly almost periodic representations

In 2014 the study of (the cohomology of) weakly almost periodic representations was initiated.

Definition (Bader-Rosendal-Sauer, 2014)

The representation ρ is called *weakly almost periodic* (*wap*) if the orbit $\{\rho_g(v) \mid g \in G\} \subseteq V$ is relatively weakly compact for every $v \in V$.

- Representations on reflexive Banach spaces are wap;
- For an action $G \curvearrowright X$ by measure preserving transformations on a probability space (X, μ) the induced representation $\rho : G \to \mathsf{GL}(L^1(X, \mu))$ is wap.

However: Many examples (especially many representations on C^*-algebras) are not covered.

2 Weakly uniquely stationary representations

For a probability measure $\mu \in \mathrm{Prob}(G)$ and $v \in V$ define an element in the bidual V^{**} via $V^* \to \mathbb{C}, \phi \mapsto \int_G \phi\left(\rho_g(v)\right) d\mu(g)$.

Definition

Call ρ *BP-integrable* if the element above is contained in the isometric image of V in V^{**} for all $\mu \in \mathrm{Prob}(G)$, $v \in V$. In this case write $\rho_\mu(v)$ for its preimage in V. Further define

$$\begin{aligned} V^\mu &:= \{v \in V \mid \rho_\mu(v) = v\}, \\ (V^*)^\mu &:= \{\phi \in V^* \mid \phi \circ \rho_{\check{\mu}} = \phi\}, \end{aligned}$$

where $\check{\mu}$ is the *symmetric opposite* of μ.

Definition

Assume that ρ is BP-integrable. For $\mu \in \mathrm{Prob}(G)$ we say that ρ is *weakly uniquely μ-stationary*, if every non-trivial functional $\phi \in (V^*)^\mu$ admits an element $v \in V^\mu$ with $\phi(v) \neq 0$.

- If ρ is wap, then ρ is BP-integrable and weakly uniquely μ-stationary for every $\mu \in \mathrm{Prob}(G)$.
- Uniquely μ-stationary C*-dynamical systems give BP-integrable and weakly uniquely μ-stationary representations.

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3 Application to cohomology

Idea: Combine the notion of weak unique μ-stationarity with "commutativity".

3 Decomposition of cocycles

Theorem

Let $N, C \leq G$ be subgroups with $C \subseteq C_G(N)$ and $\mu \in \mathrm{Prob}(C)$. Assume that $\rho|_C$ is weakly uniquely μ-stationary. Then for every $b \in Z^n(G, V)$ the restriction $b|_{N^n} \in Z^n(N, V)$ is almost cohomologous to a cocycle with values in V^μ.

Corollary

Let $\mu \in \mathrm{Prob}(\mathcal{Z}(G))$ and $\rho|_{\mathcal{Z}(G)}$ weakly uniquely μ-stationary. Then, $\overline{H}_c^n(G, V) \cong \overline{H}_c^n(G, V^\mu)$ for all $n \in \mathbb{N}$.

3 Cohomology of products of groups

The statement of the previous slide generalizes a theorem by Bader-Rosendal-Sauer (2014). It is an important ingredient in the proof of the following theorem.

Theorem (Shalom, 2004; Bader-Rosendal-Sauer, 2014)

If $\rho : G_1 \times G_2 \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then

$$\overline{H}_c^1(G_1 \times G_2, \mathcal{H}) \cong \overline{H}_c^1(G_1, \mathcal{H}^{G_2}) \oplus \overline{H}_c^1(G_2, \mathcal{H}^{G_1}).$$

It is open whether this holds for arbitrary wap (or even uniformly convex) representations. The statement is used to prove many rigidity theorems.

A partial generalization for arbitrary Banach spaces:

Theorem

Let $G = G_1 \times G_2$, $\mu_1 \in \mathrm{Prob}(G_1)$, $\mu_2 \in \mathrm{Prob}(G_2)$. Assume that $\rho|_{G_1}$ is weakly uniquely μ_1-stationary and that $\rho|_{G_2}$ is weakly uniquely μ_2-stationary without almost invariant vectors. Then,

$$\overline{H}_c^1(G_1 \times G_2, V) \cong \overline{H}_c^1(G_1, V^{\mu_2}) \oplus \overline{H}_c^1(G_2, V^{\mu_1}).$$

Corollary

Let $\rho : G_1 \times G_2 \to \mathrm{GL}(V)$ be wap and assume that $\rho|_{G_2}$ has no almost invariant vectors. Then, $\overline{H}_c^1(G_1 \times G_2, V) \cong \overline{H}_c^1(G_2, V^{G_1})$.

3 Other results

Theorem

Let $\mu \in \mathrm{Prob}(\mathcal{Z}(G))$ and assume that $\rho|_{\mathcal{Z}(G)}$ is weakly uniquely μ-stationary. If $\mathrm{Hom}(\mathcal{Z}(G)_\mu, \mathbb{R}) = \{0\}$, where $\mathcal{Z}(G)_\mu$ is the smallest closed subgroup of $\mathcal{Z}(G)$ that contains the support of μ, then

$$\overline{H}_c^1(G, V) \cong \overline{H}_c^1(G/\mathcal{Z}(G)_\mu, V^\mu).$$

Corollary

Let G be nilpotent, V separable, and ρ wap. Then,

$$\overline{H}_c^1(G, V) \cong \overline{H}_c^1(G^{\mathsf{ab}}, V^{[G,G]}).$$

Theorem

Assume that G is nilpotent, Hausdorff and second-countable, and let $H \leq \overline{[G,G]}$ be a subgroup. Then the image of the restriction map

$$\overline{H}_c^1(G,V) \to \overline{H}_c^1(H,V)$$

is zero.

For wap representations the topological assumptions can be dropped.

This generalizes a theorem by Fernós-Valette-Martin (2012) to arbitrary Banach spaces.

Thank you!