A duality theorem for non-unital operator systems

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Note: for exposition, everything will be separable, although this is not essential.

The goal of the following talk is to understand the following theorem:

Theorem (Kennedy-K-Manor)

There is a categorical duality between non-unital operator systems and pointed nc convex sets.

First let us start off by understanding the commutative C* case.

Example (Gelfand duality)

Unital commutative C*-algebras are dual to compact (hausdorff) spaces.

This duality is given by

$$
\begin{aligned}
A &\mapsto \mathrm{Irreps}(A) \\
X &\mapsto C(X) \,.
\end{aligned}
$$

In the non-unital case, Irreps(A) will always have the zero map $0 : A \to \mathbb{C}$, which one usually removes from Irreps(A).

If we allow for $0 \in \text{Irreps}(A)$,

Example (Gelfand duality for non-unital C*-algebras)

Commutative C*-algebras are dual to *pointed* compact hausdorff spaces (X, z).

We are identifying $C_0(X \setminus \{z\})$ with the C*-algebra $C(X, z)$ of continuous functions on X that evaluate to 0 on z.

One nice thing about this approach is that the minimal unitization can be identified with the forgetful functor $(X, z) \mapsto X$.

Unital Operator Systems

Definition

A subspace $E \subseteq B(H)$ is a unital operator system if $1 \in E$ and $E^* = E$.

Unital operator systems admit an axiomatization due to Choi and Effros.

This axiomatization states that the essential structure for operator systems are which elements of $M_n(E)$ are positive. For example:

Example

Let $T \in B(H)$. Then $\|T\| \leq 1$ if and only if

$$\begin{bmatrix} 1 & T \\ T^* & 1 \end{bmatrix} \geq 0 \,.$$

Definition

We say that an operator system E is a function system if $xy = yx$ for all $x, y \in E$.

Example

Given a compact convex set K, the space $A(K)$ of continuous affine $f : K \to \mathbb{C}$ is a unital function system in $C(K)$.

Example

The unital operator system $A([-1, 1])$ of continuous affine $f : [-1, 1] \to \mathbb{C}$ are always of the form $ax + b$ for $a, b \in \mathbb{C}$.

Theorem (Kadison's duality)

Unital function systems are dual to compact convex sets.

The duality is implemented by

$$E \mapsto S(E)$$
$$K \mapsto A(K)$$

Example

The C*-algebra $\mathbb{C}^2$ has state space $S(\mathbb{C}^2) \cong [-1, 1]$. Therefore, $A([-1, 1]) \cong \mathbb{C}^2$.

Definition

Let K be a compact convex set. For $x \in K$, we say a probability measure $\mu \in P(K)$ is a representing measure for x if $\mu(f) = f(x)$ for all $f \in A(K)$.

We say that μ is maximal if it is furthermore supported on ∂K.

The set K is called a (Choquet) simplex if every point in K has a unique maximal representing measure.

Example

The closed disk $\mathbb{D} \subseteq \mathbb{C}$ is not a simplex. For example, for any $z \in \mathbb{T}$,

$$\mu_z(f) := \frac{1}{2}(f(z) + f(-z))$$

is a maximal representing measure for 0. On the other hand, $[-1, 1]$ is a simplex.

Definition

We say that a simplex K is a Bauer simplex if ∂K is closed.

Theorem (Bauer)

A compact convex set K is a Bauer simplex iff $A(K)$ is a C-algebra.*

Example

Since $[-1, 1]$ is a Bauer simplex, $A([-1, 1])$ is a C*-algebra. On the other hand, $A(\mathbb{D})$ is not a C*-algebra, since $\mathbb{D}$ is not a simplex. The smallest C*-algebra containing $A(\mathbb{D})$ is $C(\mathbb{T})$.

The non-commutative setting

We wish to define the appropriate analogue of convex sets for general operator systems.

Definition

A compact nc convex set over an operator space E is a graded subset $K = \bigsqcup_{1 \leq n \leq \aleph_0} K_n$ with $K_n \subseteq M_n(E^*) = CB(E, M_n)$ such that

1. all K_n are compact in the point-weak* topology on $M_n(E^*)$ and
2. K is closed under nc convex combinations:

   if $\alpha_i \in M_{n,n_i}$ and $x_i \in K_{n_i}$ for all i such that $\sum_i \alpha_i \alpha_i^* = 1$, then

$$\sum_i \alpha_i x_i \alpha_i^* \in K .$$

Notice that if we take $\alpha_i \in \mathbb{C}$ then $\sum_i \alpha_i x_i \alpha_i^*$ is just a scalar convex combination.

The set K is closed under compressions: if $\beta \in M_{m,n}$ is an isometry and $x \in K_m$ then $\beta^* x \beta \in K$.

Let $x \in K_n$ and $y \in K_m$. Notice

$$x \oplus y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y \begin{bmatrix} 0 & 1 \end{bmatrix} .$$

Example

The set $\bigsqcup_n \{\alpha \in (M_n)_{sa} : -1 \le \alpha \le 1\}$ is a compact nc convex set over $\mathbb{C}$. This is the nc convex set associated to $\mathbb{C}^2$.

Example

Let E be a unital operator system. Let $S_n(E)$ denote the set of ucp maps from E into M_n. The nc state space is $S(E) = \bigsqcup_n S_n(E)$.

If E is not necessarily unital, then we shall instead take $Q_n(E)$ to be the space of cpcc maps from E into M_n. The nc quasistate space of E is $Q(E) = \bigsqcup_n Q_n(E)$.

Definition

Let K be a compact nc convex set. Let $\mathcal{M} := \bigsqcup_n M_n$. We say a graded function $f := \bigsqcup_n f_n : K \to \mathcal{M}$ is an affine nc function if f is closed under nc convex combinations:

$$f(\sum_i \alpha_i x_i \alpha_i^*) = \sum_i \alpha_i f(x_i) \alpha_i^* \ .$$

Let $A(K)$ denote the unital operator system of continuous affine nc functions.

Theorem (Webster–Winkler, Davidson–Kennedy)

Compact nc convex sets are dual to unital operator systems. The duality is given by

$$K \mapsto A(K)$$
$$E \mapsto S(E) \ .$$

Webster and Winkler's duality only take $\bigsqcup_{n<\aleph_0} K_n$. With the appropriate notion of extreme points, this will not give us enough extreme points to generate our convex set.

Theorem (NC Krein–Milman theorem)

Every compact nc convex set is the closed nc-convex hull of its extremal points. In particular, there are enough extremal points to generate the nc convex set.

We also have:

Theorem (Milman's Converse)

Let $X \subseteq K$ be a closed and compression closed subset of K. Suppose that K is the smallest compact nc convex set containing X. Then $\partial K \subseteq X$.

The non-unital case

Definition

A (non-unital) operator system is a norm closed subspace $E \subseteq B(H)$ such that $E^* = E$.

Morphisms in this category are cpcc (completely positive and completely contractive) maps.
Operator systems have an abstract characterization due to Werner.

Example

Let K be a compact nc convex set and fix a point $z \in K_1$. The space $A(K, z) := \{f \in A(K) : f(z) = 0\} \subseteq A(K)$ is an operator system.

Example

The operator system $A([-1,1],0)$ is an operator system consisting of functions of the form $f_a : [-1,1] \to \mathbb{C} : x \mapsto ax$ for $a \in \mathbb{C}$.

We have $f_a(x) \geq 0$ for all x iff $a = 0$.

Thus, $A([-1,1],0) \cong \mathbb{C}$ as a vector space and $A([-1,1],0)_+ = 0$.

Example

Similarly, $A(\mathbb{D},0) \cong \mathbb{C}^2$ with $A(\mathbb{D},0)_+ = 0$.

The restriction map $A(\mathbb{D},0) \to A([-1,1],0)$ is then a complete order isomorphism but the operator systems are not isomorphic.

One might think that in the non-unital case, we take pairs (K, z) where K is a compact nc convex subset and $z \in K_1$ is a distinguished point.

It is *not* the case that $E \mapsto (Q(E), 0)$ and $(K, z) \mapsto A(K, z)$ form a duality.

Example

We have $A([-1/2, 1], 0) \cong A([-1, 1], 0)$.
In fact, $Q(A([-1/2, 1], 0)) = Q(A([-1, 1], 0)) = ([-1, 1], 0)$.

Definition

We call a pair (K, z) a pointed compact nc convex set if $(K, z) = (Q(A(K, z)), 0)$.

Theorem (Kennedy–K–Manor)

Operator systems are dual to pointed compact nc convex sets. The duality is given by

$$E \mapsto (Q(E), 0)$$
$$(K, z) \mapsto A(K, z) .$$

The forgetful functor $(K, z) \mapsto K$ corresponds to the canonical unitization.

Theorem (Werner)

Let (K, z) be a pointed compact nc convex set. There is a canonical unitization $A(K, z)^\sharp$ with state space K.

For an nc convex set K, we denote by ∂K of extreme points of K. If $K = S(A)$ for a unital C*-algebra A, then $\partial K = \text{Irreps}(A)$. We can then say

Definition

For a compact nc convex set K, we say K is a Choquet simplex if every point in K has a unique maximal representing measure. It is a Bauer simplex if ∂K is closed.

Theorem (Kennedy–Shamovich, Kennedy–K–Manor)

The space (K, z) corresponds to a C-algebra iff $z \in \partial K$ and K is a Bauer simplex.*

Finally we come to an application in dynamics. Fix a second countable group G.

Definition

A (continuous) representation $\rho : G \to U(H)$ is said to have almost invariant vectors if for every $\epsilon > 0$ and for every compact $C \subseteq G$, there is a unit vector $h \in H$ such that

$$\sup_{g \in C} \|\rho(g)h - h\| < \epsilon \, .$$

We say G has Kazhdan's property (T) if every representation with *almost* invariant vectors admits an invariant unit vector.

With our non-unital duality, we get the following characterization of Property (T):

Theorem (Glasner–Weiss, Kennedy–Shamovich, Kennedy–K–Manor)

Let G be a second countable group. The following are equivalent:

1. *G has property (T).*
2. *For every unital G-C^*-algebra A, $S_1(A)^G$ is the state space of some C^*-algebra.*
3. *For every G-C^*-algebra A, $(Q_1(A)^G, 0)$ is the pointed quasistate space of some C^*-algebra.*