Spectral analysis of terrestrial planets and gas giants

Characterization of the spectra of rotating truncated gas planets, part II

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Riesz projectors and acoustic mode decomposition

- expanding u in acoustic modes: eigenfunctions of L_{11} corresponding to discrete eigenvalues (by Proposition 1)
- spectral theory on Krein spaces (Langer et al. [2008], Azizov and Iokhidov [1981]) and Proposition 1 lead to resolution of the identity for L_{11} using its eigenfunctions
- however, these eigenfunctions are not modes for the operator L; suppose $\lambda \in \rho_2$, which is outside of the essential spectrum of L (Proposition 2); using the Schur decomposition (30), we have that

$$\mathscr{L}(\lambda)\begin{pmatrix} u \\ v \end{pmatrix} = 0$$

if and only if

$$S_1(\lambda)u = 0, \quad L_{22}^{-1}(\lambda)L_{21}(\lambda)u = -v$$

- thus, eigenvalues of L outside the essential spectrum and their corresponding modes actually correspond to eigenfunctions of S_1, and contain a component in $\mathrm{Ker}(T)$
- to develop a true expansion for L away from the essential spectrum we should use the eigenfunctions of S_1

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Riesz projectors and acoustic mode decomposition

- assume secular stability: $\gamma(a_2) = 0$ and $A_2^{1/2}$ is well defined on $D(A_2)$, with a nontrivial $\mathrm{Ker}(A_2^{1/2})$ coinciding with $\mathrm{Ker}(A_2)$
- we let

$$B_2 = \begin{pmatrix} 0 & \mathrm{i}A_2^{1/2} \\ \mathrm{i}A_2^{1/2} & -2R_\Omega \end{pmatrix}, \qquad D(B_2) = D(A_2) \times D(A_2) \tag{34}$$

- it is immediate that $\mathrm{i}B_2$ is *self adjoint* on $H \times H$, equipped with the original inner product:

$$\left(B_2 \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right) = (\mathrm{i}A_2^{1/2} v, u')_H + (\mathrm{i}A_2^{1/2} u - 2R_\Omega v, v')_H = -\left(\begin{pmatrix} u \\ v \end{pmatrix}, B_2 \begin{pmatrix} u' \\ v' \end{pmatrix} \right) \tag{35}$$

- we introduce (noting the minus sign)

$$\widetilde{L}(\lambda) = B_2 - \lambda\,\mathrm{Id} = \begin{pmatrix} -\lambda & \mathrm{i}A_2^{1/2} \\ \mathrm{i}A_2^{1/2} & -\lambda - 2R_\Omega \end{pmatrix} \quad \text{and} \quad \widetilde{R}(\lambda) = \widetilde{L}(\lambda)^{-1} \tag{36}$$

Riesz projectors and acoustic mode decomposition

- from an analysis of geostrophic modes Remark 4, $0 \notin \rho(L)$, so for $\lambda \in \rho(L)$ we can invert the previous equation to obtain

$$\widetilde{R}(\lambda) = \widetilde{L}(\lambda)^{-1} = \begin{pmatrix} -\lambda^{-1}(\mathrm{Id} - A_2^{1/2} R(\lambda) A_2^{1/2}) & -\mathrm{i} A_2^{1/2} R(\lambda) \\ -\mathrm{i} R(\lambda) A_2^{1/2} & -\lambda R(\lambda) \end{pmatrix} \tag{37}$$

- on the other hand, if $\lambda \in \rho(\widetilde{L})$ we have an inverse

$$\widetilde{R}(\lambda) = \begin{pmatrix} \widetilde{R}_{11}(\lambda) & \widetilde{R}_{12}(\lambda) \\ \widetilde{R}_{12}(\lambda) & \widetilde{R}_{22}(\lambda) \end{pmatrix} \tag{38}$$

- the resolvents are related; if $\lambda \neq 0$, $R(\lambda) = -\lambda^{-1} \widetilde{R}_{22}(\lambda)$
- the presence of geostrophic modes also imply that $0 \notin \rho(\widetilde{L})$ and so we see that $\rho(L) = \rho(\widetilde{L})$; hence, the spectra are the same

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Riesz projectors and acoustic mode decomposition

- suppose $\lambda \in \sigma_{disc}(L) = \sigma_{disc}(\widetilde{L})$; a corresponding eigenfunction $(u, v) \in H \times H$ satisfies

$$\mathrm{i}A_2^{1/2} v = \lambda u, \quad \mathrm{i}A_2^{1/2} u - 2R_\Omega v = \lambda v$$

- restricting to acoustic modes, $v = 0$ is not possible since $\lambda \neq 0$ ($\lambda = 0$ is an eigenvalue but does not correspond with an acoustic mode)
- we combine these formulae:

$$L(\lambda)v = 0, \quad u = \lambda^{-1}\mathrm{i}A_2^{1/2} v \tag{39}$$

- we introduce Riesz projectors onto the space of acoustic modes, which are the spectrum of S_1: let $\lambda \in \sigma(S_1)$ and Γ_λ be a contour surrounding λ and no other part of $\sigma(B_2)$; then consider the standard formula for the projection onto the eigenspace of λ

$$\widetilde{P}_\lambda = \frac{1}{2\pi i} \oint_{\Gamma_\lambda} \widetilde{R}(\omega)\, \mathrm{d}\omega$$

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Riesz projectors and acoustic mode decomposition

- we further let π_v be projection onto the v component and define $P_\lambda = \pi_v \widetilde{P}_\lambda \pi_v^*$; using (37),

$$P_\lambda = -\frac{\lambda}{2\pi i} \oint_{\Gamma_\lambda} R(\omega) \, \mathrm{d}\omega$$

- we can now use these projectors to define the projection onto (part of) the acoustic part of the spectrum, which is

$$E = \sum_{\lambda \in \sigma(S_1)} P_\lambda$$

- we conclude that the projection onto the eigenspace of λ for $\widetilde{L}$ gives a corresponding projection, by taking the v component as in (39), onto the space $\mathrm{Ker}(L(\lambda))$ of an acoustic mode
- this projection E shows it is possible to express the acoustic part of the wavefield as a sum of normal modes (apart from acoustic eigenvalues embedded in σ_2)

Riesz projectors and acoustic mode decomposition

- using the above mentioned Riesz projectors, we obtain a partial spectral decomposition of $\widetilde{R}_{22}(\lambda)$:

$$\widetilde{R}_{22}(\lambda)|_{acoustic} = \sum_{\omega \in \sigma(S_1)} \frac{P_\omega}{(\omega - \lambda)}$$

- this induces a corresponding partial spectral decomposition of $R(\lambda)$ from (37):

$$R(\lambda)|_{acoustic} = \frac{1}{\lambda} \sum_{\omega \in \sigma(S_1)} \frac{P_\omega}{(\lambda - \omega)}$$

(commonly used in computations)

Inertia-gravity modes and essential spectrum

essential spectrum of L

- because $L_{11}^{-1}(\lambda)$ is compact and the $L_{ij}(\lambda)$ are bounded from Proposition 1, using Proposition 2 we have that
$$\sigma_{ess}(L) = \sigma_{ess}(L_{22})$$
- using the formula for L_{22} given in Remark 2 and Lemma 4, this further reduces to
$$\sigma_{ess}(L) = \sigma_{ess}\left(\pi_2\left(F(\lambda) + N^2 \hat{g}_0' \hat{g}_0'^T\right)\pi_2^*\right) \tag{40}$$
- thus, referring to (14), we are led to consider the spectrum of
$$M(\lambda) = \pi_2(\lambda^2 \operatorname{Id} + 2\lambda R_\Omega + N^2 \hat{g}_0' \hat{g}_0'^T)\pi_2^* : \ \operatorname{Ker}(T) \to \operatorname{Ker}(T)$$

Inertia-gravity modes and essential spectrum

- solutions $u \in \mathrm{Ker}(T)$ of

$$\partial_t^2 u + 2\Omega \times \partial_t u + N^2 \hat{g}_0' \hat{g}_0'^T u = 0 \tag{41}$$

  are modes of M, referred to as **inertia-gravity modes**
- restoring force of inertial modes is the Coriolis force: $2\Omega \times \partial_t(\rho_0 u)$
- restoring force of gravity modes is the buoyancy: $(\nabla \cdot \rho_0 u) g_0' = N^2 \hat{g}_0' \hat{g}_0'^T \rho_0 u$

Inertia-gravity modes and essential spectrum

- $P_\xi^\perp = \sigma_p(\pi_2)$ defined by (22), which is the projection onto the space orthogonal to ξ
- for $\Omega \in \mathbb{R}^3$ let Ω_ξ be the component of Ω in the direction ξ given by

$$\Omega_\xi = \frac{\xi \cdot \Omega}{|\xi|}$$

Definition 2

For $x \in M$ and $\xi \in \mathbb{R}^3 \setminus \{0\}$, let $\sigma_{pt}(x, \xi)$ be the set of $\lambda \in \mathbb{C}$ such that

$$\mathbb{C}^3 \ni \eta \mapsto \lambda^2 P_\xi^\perp \eta + 2\lambda P_\xi^\perp (\Omega \times P_\xi^\perp \eta) + N^2 (\hat{g}_0' \cdot P_\xi^\perp \eta) P_\xi^\perp \hat{g}_0' \tag{42}$$

has rank less than two (note that two is the largest possible rank due to $P_\xi^\perp$).

Characterization of $\sigma_{pt}(x, \xi)$

Lemma 5

If $\lambda \in \sigma_{pt}(x, \xi)$, *then* $\lambda = 0$ *or*

$$\lambda = \pm i\sqrt{4\Omega_\xi^2 + N^2 |P_\xi^\perp \hat{g}_0'|^2}. \tag{43}$$

Link to proof

if λ satisfies (43), then

$$\lambda^2 = -\frac{1}{|\xi|^2} \underbrace{\left(4(\Omega \cdot \xi)^2 + N^2|\xi|^2 - N^2(\hat{g}_0' \cdot \xi)^2\right)}_{\text{quadratic form in } \xi} \tag{46}$$

eigenvalues of the matrix corresponding to this quadratic form: N^2 and

$$\beta_\pm = \frac{1}{2}\left(4|\Omega|^2 + N^2 \pm \sqrt{(N^2 + 4|\Omega|^2)^2 - 16(\Omega \cdot \hat{g}_0')^2 N^2}\right) \tag{47}$$

Range of possible λ^2, varying ξ

-1 times the interval between the min/max of these eigenvalues:

- if $N^2 \geq 0$, then this range will be $\lambda^2 \in -[\beta_-, \beta_+]$ which leads to $\lambda \in \pm i[\sqrt{\beta_-}, \sqrt{\beta_+}]$
- if $N^2 < 0$, then the range of possible values is $\lambda^2 \in -[N^2, \beta_+]$, which gives $\lambda \in [-\sqrt{(-N^2)}, \sqrt{(-N^2)}] \cup i[-\sqrt{\beta_+}, \sqrt{\beta_+}]$

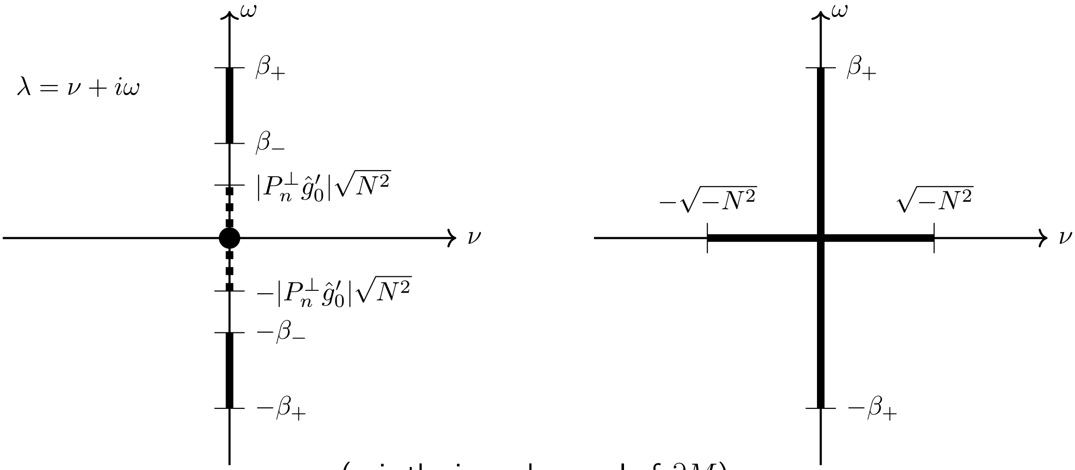
Lemma 6

Let $\beta_\pm$ be given by (47). Then

$$\bigcup_{\xi \in \mathbb{R}^3 \setminus \{0\}} \sigma_{pt}(x, \xi) = \bigcup_{\pm \in \{-1,1\}} \left(\left[-\sqrt{\max(0, -N^2)}, \sqrt{\max(0, -N^2)} \right] \cup \pm i \left[\sqrt{\max(0, \beta_-)}, \sqrt{\beta_+} \right] \right) \tag{48}$$

Furthermore, this set contains $\sqrt{-N^2}$.

Set (48) for $x \in M$ fixed



(n is the inward normal of ∂M)

(a) $N^2 \geq 0$ (b) $N^2 < 0$

The essential spectrum

Theorem 1

For $x \in \partial M$, let $n(x)$ denote the inward pointing unit normal vector. The essential spectrum $\sigma_{ess}(L)$ is given by

$$\sigma_{ess}(L) = \left(\bigcup_{x \in M,\ \pm \in \{-1,1\}} \left[-\sqrt{\max(0,-N^2)}, \sqrt{\max(0,-N^2)} \right] \cup \pm i \left[\sqrt{\max(0,\beta_-)}, \sqrt{\beta_+} \right] \right)$$
$$\bigcup \left(\bigcup_{x \in \partial M} i |P_n^{\perp} \hat{g}_0'| \left[-\sqrt{\max(0,N^2)}, \sqrt{\max(0,N^2)} \right] \right). \tag{49}$$

Proof:

- Part 1 ($\supset$): Link to Part 1
- Part 2 ($\subset$): Link to Part 2
- Part 3 (Lemma 7): Link to Part 3

Elements of the proof: interior

- suppose that $\lambda \in \mathbb{C}$ is contained in $\sigma_{pt}(x_0, \xi_0)$ such that $x_0 \in M^{int}$; thus, there exists nonzero η orthogonal to ξ_0 such that

$$\lambda^2 P_{\xi_0}\eta + 2\lambda P_{\xi}(\Omega \times P_{\xi_0}\eta) + N^2(P_{\xi_0}\hat{g}_0 \cdot P_{\xi_0}\eta)P_{\xi_0}\hat{g}_0 = 0$$

- then, for any $\epsilon > 0$, choose a neighborhood $U \subset M^{int}$ of x_0 such that at all $x \in U$

$$|\lambda^2 P_{\xi_0}\eta + 2\lambda P_{\xi_0}(\Omega \times P_{\xi_0}\eta) + N^2(P_{\xi_0}\hat{g}_0 \cdot P_{\xi_0}\eta)P_{\xi_0}\hat{g}_0| < \epsilon$$

- let $\phi \in C_c^\infty(U)$ be such that $\|\phi\|_{L^2(\rho^0\,\mathrm{d}\,x)} = 1$ and consider

$$u(x) = \eta\phi(x)e^{itx\cdot\xi_0}$$

  as $t \to \infty$, u converges to zero weakly
- using the fact that ξ_0 is orthogonal to η

$$\pi_2(u)(x) = \phi(x)e^{itx\cdot\xi_0}\eta + O\left(\frac{1}{t}\right)$$

  Weyl sequence (even though such a sequence is normalized, its mass can move around the Hilbert space so that it doesn't overlap with any fixed finite part)

Elements of the proof: boundary

- introduce a certain system of PDEs, then show that this system satisfies the Lopatinskii conditions Agmon et al. [1964] if and only if λ is in the complement of the right side of (49)
- when the Lopatinskii conditions are satisfied, the system is a Fredholm operator which implies $M(\lambda)$ is also Fredholm; therefore, in this case $\lambda \in \sigma_{ess}(L)^c$
- the Lopatinskii conditions fail if either the system is not elliptic in the interior, or at the boundary

Special cases of (48); upper bound on essential spectrum

if for some value of x we have $\Omega \cdot \hat{g}_0 = 0$, then from (47) we have

$$\beta_\pm = \min(0, 4|\Omega|^2 + N^2),\ \max(0, 4|\Omega|^2 + N^2)$$

also, for general points $\beta_+ \leq 4|\Omega|^2 + N^2$; therefore, considering (49), we see that the part of $\sigma_{ess}(L)$ along the imaginary axis must be contained in

$$i\left[-\sqrt{4|\Omega|^2 + \max(0, N^2_{\text{sup}})}, \sqrt{4|\Omega|^2 + \max(0, N^2_{\text{sup}})}\right]$$

on the other hand, directly from (49) we see that the part of $\sigma_{ess}(L)$ along the real axis must be contained in

$$\left[-\sqrt{\max(0, -N^2_{\text{inf}})}, \sqrt{\max(0, -N^2_{\text{inf}})}\right]$$

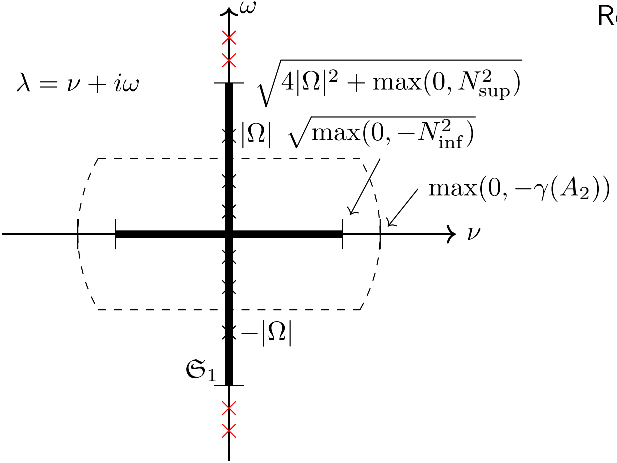
Full spectrum bound

Proposition 3 (Dyson and Schutz)

The spectrum $\sigma(L)$ satisfies

1. $$\sigma(L) \subseteq \mathrm{i}\mathbb{R} \cup \{\lambda \in \mathbb{C} \;:\; |\,\mathrm{Im}(\lambda)| \leq |\Omega|\};$$
2. *while A_2 is bounded below by $\gamma(A_2)$, $\lambda \in \sigma(L)$ and $\lambda \notin i\mathbb{R}$,*
$$|\lambda|^2 \leq \max(0, -\gamma(A_2)).$$

Overview



an illustration of the spectrum $\sigma(L)$ after Rogister and Valette [2009]

- dark cross: must contain the essential spectrum $\sigma_{ess}(L)$ (but may be larger)
- full spectrum $\sigma(L)$ is contained in the union of the imaginary axis and region surrounded by the dashed curve
- crosses on the imaginary axis: eigenvalues, which could also occur within the dashed curve
- red crosses: outside of the essential spectrum, part of $\sigma(S_1)$ which is (part of) the acoustic component of the spectrum

Encore: Hamiltonian

recall

$$\tilde{s} = \nabla \rho_0 - \frac{\rho_0}{c^2} g_0' \tag{78}$$

and the dynamic pressure

$$P = -c^2 [\nabla \cdot (\rho_0 u) - \tilde{s} \cdot u] \tag{79}$$

or

$$P = -\rho_0 \, [c^2 \nabla \cdot u + g_0' \cdot u] \tag{80}$$

using that

$$\tilde{s} \cdot u = \frac{\tilde{s} \cdot g_0'}{|g_0'|^2} (g_0' \cdot u) = \frac{N^2}{|g_0'|^2} (g_0' \cdot (\rho_0 u)) \tag{81}$$

as $\nabla \rho_0$ and g_0' must be parallel, we obtain

$$P = -c^2 \left[\nabla \cdot (\rho_0 u) - \frac{N^2}{|g_0'|^2} (g_0' \cdot (\rho_0 u)) \right] \tag{82}$$

Hamiltonian

while introducing the particle velocity, $v = \partial_t u$, equations (10) and

$$\nabla^2 \Phi' = -4\pi G \nabla \cdot (\rho_0 u) \qquad (\star)$$

are equivalent to the system

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho_0 v) &= 0, && (83) \\ \partial_t (\rho_0 v) + 2\Omega \times (\rho_0 v) &= -\nabla P + \rho g_0' - \rho_0 \nabla \Phi', && (84) \\ \partial_t P &= c^2 \left[\partial_t \rho + \frac{N^2}{|g_0'|^2} (g_0' \cdot (\rho_0 v)) \right] && (85) \end{aligned}$$

supplemented with $(\star)$

Hamiltonian

equivalent to the system in linearized hydrodynamics

$$\partial_t \rho + \nabla \cdot (\rho_0 v) = 0, \tag{86}$$

$$\partial_t(\rho_0 v) + 2\Omega \times (\rho_0 v) = -\nabla P + \rho g_0' - \rho_0 \nabla \Phi', \tag{87}$$

$$\partial_t P + v \cdot \nabla P_0 = c^2[\partial_t \rho + v \cdot \nabla \rho_0] \tag{88}$$

as $\nabla P_0 = -\rho_0 g_0'$ (in the Cowling approximation, one drops the term $-\rho_0 \nabla \Phi'$) if $u \in \ker(T)$ then $P = 0$ and

$$\rho g_0' = -(\nabla \cdot (\rho_0 u)) g_0' = -(\tilde{s} \cdot u) g_0' = -N^2 \hat{g}_0' (\hat{g}_0' \cdot \rho_0 u). \tag{89}$$

then, (84) is seen to be equivalent to

$$\partial_t v + 2\Omega \times v + N^2 \hat{g}_0' (\hat{g}_0' \cdot u) = 0, \quad T u = 0$$

which is closely related to (41)

upon first introducing

$$\rho' = N(\underbrace{\hat{g}_0' \cdot u}_{u_\parallel}) \tag{90}$$

this equation can be written as the system

$$(\partial_t + A)\begin{pmatrix} v \\ \rho' \end{pmatrix} = 0 \quad \text{with } A = \begin{pmatrix} 2\Omega\times & N\hat{g}_0' \\ -N\hat{g}_0'^T & 0 \end{pmatrix}, \quad Tv = 0 \tag{91}$$

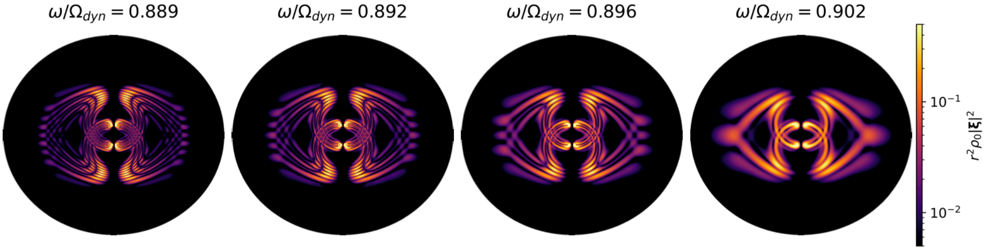
in Colin de Verdière and Vidal [2024], this system is formed by expressing v in an orthogonal basis where one of the basis vectors is $\hat{g}_0'$; including the projectors

$$\pi_2' \begin{pmatrix} v \\ \rho' \end{pmatrix} = \begin{pmatrix} \pi_2 v \\ \rho' \end{pmatrix}, \tag{92}$$

the system takes the form

$$(\partial_t + H)\begin{pmatrix} v \\ \rho' \end{pmatrix} = 0 \quad \text{with } H = \pi_2' A \pi_2' \tag{93}$$

Inertia gravity modes: Practice



[Dewberry et al 2021]

[Prat et al 2022]

Model of terrestrial planets

- bounded $M \subset \mathbb{R}^3$, smooth boundary ∂M
- M divided into two regions:
  - Ω_F (fluid outer core), annulus
  - Ω_S (solid), two components — inner core ("ball") and mantle (annulus)
- $\Sigma_{FS} = \partial\Omega_F$ interface between the fluid and solid regions; two smooth "spheres"
- $u = u(t, x) \in \mathbb{R}^3$ is displacement (as before)

Model of terrestrial planets

equation of motion for the oscillations of a rotating elastic and self-gravitating planet:

$$\rho^0[\partial_t^2 u + 2\ \Omega \times \partial_t u] + \rho^0 u \cdot \nabla\nabla(\Phi^0 + \Psi^s) + \rho^0 \nabla S(u) - \nabla \cdot (\Lambda^{T^0} : \nabla u) = 0 \qquad (1)$$

$$\Psi^s(x) := -\tfrac{1}{2}\left(\Omega^2 x^2 - (\Omega \cdot x)^2\right) \qquad \text{(centrifugal force)} \qquad (2)$$

$$\Delta\Phi^0 = 4\pi G \rho^0 \qquad \text{(reference gravitational potential)} \qquad (3)$$

$$\Delta S(u) = -4\pi \nabla \cdot (\rho^0 u) \qquad \text{(perturbation of the gravitational potential)}$$

boundary conditions on ∂M:

$$\nu \cdot (\Lambda^{T^0} : \nabla u)|_{\partial M} = 0$$

interface conditions at Σ_{FS}:

$$[\nu \cdot (\Lambda^{T^0} : \nabla u)]_-^+ = -\nu \nabla^\Sigma \cdot (p^0 [u]_-^+) - p^0 W [u]_-^+; \quad [u \cdot \nu]_-^+ = 0 \text{ on } \Sigma_{FS}$$

- $\nabla^\Sigma \cdot =$ surface divergence
- $W =$ Weingarten operator on the interface
- $[\cdot]^\pm =$ jump across Σ_{FS} in the direction of the unit normal vector ν

Model of terrestrial planets

- $g_0' := -\nabla(\Phi^0 + \Psi^s)$
- modified stiffness tensor: $$\Lambda^{T^0}_{ijkl} = \Xi_{ijkl} + T^0_{ik}\delta_{jl} \quad (4)$$
- $T^0 =$ initial static stress
- $\Xi_{ijkl} \in L^\infty(\tilde{X}) =$ stiffness tensor of linearization of the constitutive function
- initial hydrostatic stress: $p^0 = -\frac{1}{3}T^0$
- deviatoric part of the static stress in the solid region: $\tau^0 = T^0 + p^0\mathrm{Id}$

in the fluid region Ω_F, we have

$$\nabla \cdot (\Lambda^{T^0} : \nabla u) = \nabla(\kappa\nabla \cdot u) + \nabla(\rho^0 u) \cdot g_0' - (\nabla \cdot (\rho^0 u))g_0'$$

using this formula, we see that (1) is equivalent to

$$\begin{aligned}\rho^0[\partial_t^2 u + 2\,\Omega \times \partial_t u] &= \nabla(\kappa\nabla \cdot u) + \nabla(\rho^0 u) \cdot g_0' - (\nabla \cdot (\rho^0 u))g_0' + \nabla g_0' \cdot \rho^0 u - \rho^0\nabla S(u)\\ &= \nabla(\kappa\nabla \cdot u) + \nabla(\rho^0 u \cdot g_0') - (\nabla \cdot (\rho^0 u))g_0' - \rho^0\nabla S(u).\end{aligned}$$

Model of terrestrial planets

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- initial hydrostatic stress: $p^0 = -\frac{1}{3}T^0$
- deviatoric part of the static stress in the solid region: $\tau^0 = T^0 + p^0\mathrm{Id}$

in the fluid region Ω_F, we have

$$\nabla \cdot (\Lambda^{T^0} : \nabla u) = \nabla(\kappa \nabla \cdot u) + \nabla(\rho^0 u) \cdot g_0' - (\nabla \cdot (\rho^0 u)) g_0'$$

using this formula, we see that (1) is equivalent to

$$\begin{aligned}\rho^0[\partial_t^2 u + 2\,\Omega \times \partial_t u] &= \nabla(\kappa \nabla \cdot u) + \nabla(\rho^0 u) \cdot g_0' - (\nabla \cdot (\rho^0 u)) g_0' + \nabla g_0' \cdot \rho^0 u - \rho^0 \nabla S(u) \\ &= \nabla(\kappa \nabla \cdot u) + \nabla(\rho^0 u \cdot g_0') - (\nabla \cdot (\rho^0 u)) g_0' - \rho^0 \nabla S(u).\end{aligned}$$

Function spaces and well-posedness

$$H(\mathrm{Div}, \Omega, L^2(\partial\Omega)) = \big\{ u \in L^2(\Omega) : \nabla \cdot u \in L^2(\Omega),\ \ u|_{\partial\Omega} \cdot \nu \in L^2(\partial\Omega) \big\}; \tag{5}$$

$$\begin{aligned}(u, v)_{H(\mathrm{Div},\Omega,L^2(\partial\Omega))} = \langle u, v \rangle_{L^2(\Omega)} + \langle \nabla \cdot u, \nabla \cdot v \rangle_{L^2(\Omega)} \\ + \langle u|_{\partial\Omega} \cdot \nu, v|_{\partial\Omega} \cdot \nu \rangle_{L^2(\partial\Omega)}\end{aligned} \tag{6}$$

furthermore

$$H_0(\mathrm{Div}, \Omega) = \big\{ u \in H(\mathrm{Div}, \Omega) : u|_{\partial\Omega} \cdot \nu = 0 \big\} \tag{7}$$

and

$$H_0(\mathrm{Div}\, 0, \Omega) = \big\{ u \in H(\mathrm{Div}, \Omega) : \nabla \cdot u = 0,\ \ u|_{\partial\Omega} \cdot \nu = 0 \big\} \tag{8}$$

we can modify (1) to a weak form with domain given by the next definition

Function spaces and well-posedness

Definition 3

We let

$$E = \left\{ u \in L^2(M, \rho^0 \, \mathrm{d}\, x) : \begin{cases} u|_{\Omega_S} \in H^1(\Omega_S) \\ u|_{\Omega_F} \in H\big(\mathrm{Div},\, \Omega_F,\, L^2(\partial\Omega_F)\big) \\ [u \cdot \nu]_-^+ = 0 \ \text{ along } \Sigma_{FS} \end{cases} \right\} \tag{9}$$

$$(u, v)_E := (u|_{\Omega^S}, v|_{\Omega_S})_{H^1(\Omega_S)} + (u|_{\Omega_F}, v|_{\Omega_F})_{H(\mathrm{Div}, \Omega_F, L^2(\partial\Omega_F))} .$$

we observe that

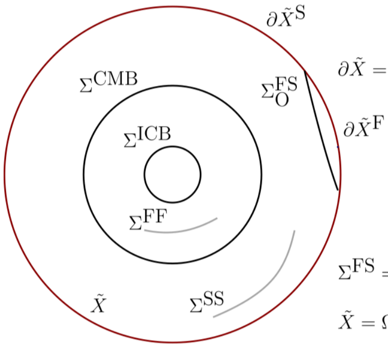
- E equipped with the inner product $(\cdot, \cdot)_E$ is a Hilbert space
- the injective inclusion of E into $H = L^2(M, \rho^0 \, \mathrm{d}\, x)$ is continuous
- E is dense in $H = L^2(M, \rho^0 \, \mathrm{d}\, x)$

as a result, we have the setting of a Hilbert triple

$$E \hookrightarrow H \hookrightarrow E'$$

weak form

Terrestrial planet geometry



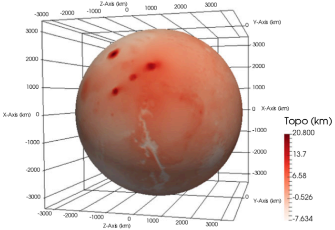
$$\partial\tilde{X} = \partial\tilde{X}^{\mathrm{S}} \cup \partial\tilde{X}^{\mathrm{F}}$$

$$\Sigma^{\mathrm{FS}} = \Sigma^{\mathrm{CMB}} \cup \Sigma^{\mathrm{ICB}} \cup \Sigma^{\mathrm{FS}}_{\mathrm{O}}$$

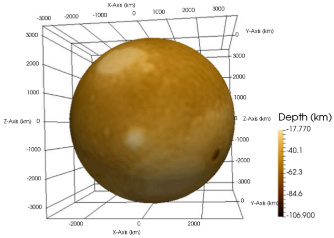
$$\tilde{X} = \Omega^{\mathrm{S}} \cup \Omega^{\mathrm{F}} \cup \Sigma \cup \partial\tilde{X}$$

Above: Conceptual geometry of a terrestrial planet

Right: Topography and crust-mantle interface of Mars using MOLA and gravity data

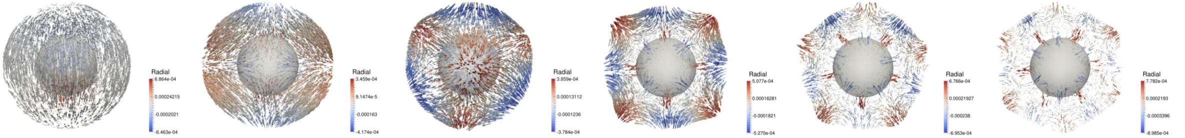


(a) Topography

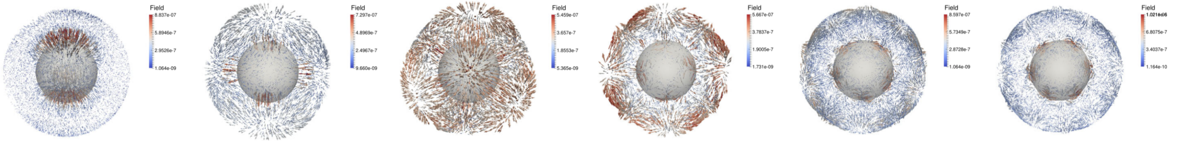


(b) Crust-mantle interface

Selected normal modes of Mars



(**a₁**) ${}_1S_1$ (**a₂**) ${}_1S_2$ (**a₃**) ${}_1S_3$ (**a₄**) ${}_1S_4$ (**a₅**) ${}_1S_5$ (**a₆**) ${}_1S_6$



(**b₁**) $\nabla S(u)$ of ${}_1S_1$ (**b₂**) $\nabla S(u)$ of ${}_1S_2$ (**b₃**) $\nabla S(u)$ of ${}_1S_3$ (**b₄**) $\nabla S(u)$ of ${}_1S_4$ (**b₅**) $\nabla S(u)$ of ${}_1S_5$ (**b₆**) $\nabla S(u)$ of ${}_1S_6$

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back

Remark 2: No compact inverse

If we additionally assume that g_0' and $\nabla\rho_0$ are parallel, which is a requirement for well-posedness of the system (1), and use the Brunt-Väisälä frequency N^2 (see (9)), the proof of Proposition 1 implies the following formulae

$$L_{12}(\lambda) = \pi_1 \left(F(\lambda) + N^2 \hat{g}_0' \hat{g}_0'^T + \nabla S \rho_0\right) \pi_2^*, \quad L_{22}(\lambda) = \pi_2 \left(F(\lambda) + N^2 \hat{g}_0' \hat{g}_0'^T + \nabla S \rho_0\right) \pi_2^*,$$
$$L_{21}(\lambda) = \pi_2 \left(F(\lambda) + N^2 \hat{g}_0' \hat{g}_0'^T + \nabla S \rho_0\right) \pi_1^* \tag{26}$$

where

$$\hat{g}_0' = \frac{g_0'}{\|g_0'\|}.$$

From these formulae and Lemma 4, $L_{22}(\lambda)$ cannot have a compact inverse. Thus by taking $u \in \mathrm{Ker}(T)$ we see that $L(\lambda)^{-1}$ cannot be compact as observed earlier.

Remark 4: Geostrophic modes

For completeness of the characterization, we briefly present how the geostropic modes (see [Dahlen and Tromp, 1999, Section 4.1.6]) appear in the analysis. Fluid motions which travel along the level surfaces of ρ_0 and preserve the density are generalized eigenfunctions of L, or geostrophic modes, corresponding to $\lambda = 0$. They are necessarily solutions to the problem

$$\begin{cases} \tilde{s} \cdot u &= 0, \\ \nabla \cdot (\rho_0 u) &= 0, \\ \nabla \cdot u|_{\partial M} &= 0. \end{cases} \tag{31}$$

Note that if $u \in H$ satisfies (31), then $u \in H_2 = \operatorname{Ker}(T)$. If $\varphi \in H^1(M)$ is such that

$$\nabla\varphi \cdot (\nabla \times \tilde{s}) = 0 \tag{32}$$

and we define u by

$$u = \rho_0^{-1} \nabla\varphi \times \tilde{s}, \tag{33}$$

then u satisfies the first and the second equations of (31) as

$$\nabla \cdot (\nabla\varphi \times \tilde{s}) = \tilde{s} \cdot (\nabla \times \nabla\varphi) - \nabla\varphi \cdot (\nabla \times \tilde{s}).$$

back

Remark 4: Geostrophic modes

Since we have also

$$\nabla \cdot (\rho_0^{-1} \nabla\varphi \times \tilde{s}) = (\nabla\varphi \times \tilde{s}) \cdot \nabla\rho_0^{-1},$$

the boundary condition in (31) is equivalent to

$$(\nabla\varphi \times \tilde{s}) \cdot \nabla\rho_0^{-1}|_{\partial M} = 0.$$

Assuming that g_0', $\nabla\rho_0$ and n are parallel on ∂M, which is required for well-posedness of the system, this boundary condition is automatically satisfied.

The geostrophic modes form a infinite-dimensional subspace of H_2. This is consistent with the fact that the essential spectrum of L corresponds with the H_2 component (i.e. $L(0)$ fails to be Fredholm because of an infinite dimensional kernel contained in H_2).

Proof of Lemma 5

- first, assume that $P_\xi^\perp \hat{g}_0' \neq 0$ and set

$$\eta = a\ P_\xi^\perp \hat{g}_0' + b\ \xi \times P_\xi^\perp \hat{g}_0' \tag{44}$$

where a and b are constants, not both equal to zero, to be determined; calculation shows

$$\begin{aligned} P_\xi^\perp(\Omega \times P_\xi^\perp(\xi \times P_\xi^\perp \hat{g}_0')) &= P_\xi^\perp(\Omega \times (\xi \times P_\xi^\perp \hat{g}_0')) \\ &= -|\xi|\Omega_\xi P_\xi^\perp \hat{g}_0' \end{aligned}$$

and

$$P_\xi^\perp(\Omega \times P_\xi^\perp \hat{g}_0') = \frac{\Omega_\xi}{|\xi|}\xi \times P_\xi^\perp \hat{g}_0'$$

- if $\lambda \in \sigma_{pt}(x,\xi)$ then for some a and b

$$\left(\lambda^2 a - 2\lambda|\xi|\Omega_\xi b + N^2 |P_\xi^\perp \hat{g}_0|^2 a\right) P_\xi \hat{g}_0' + \left(\lambda^2 b + 2\lambda \frac{\Omega_\xi}{|\xi|} a\right) \xi \times P_\xi^\perp \hat{g}_0' = 0$$

Proof of Lemma 5

- setting the two coefficients equal to zero, we see that either $\lambda = 0$ and $a = 0$ or

$$\lambda^2 = -4\Omega_\xi^2 - N^2 |P_\xi^\perp \hat{g}_0'|^2 \tag{45}$$

which completes the proof in this case

- when $P_\xi^\perp \hat{g}_0' = 0$, we choose arbitrary w orthogonal to ξ and start with

$$\eta = a\, w + b\, \xi \times w$$

instead of (44); a similar calculation gives $\lambda = 0$ or (45) in this case, and so the lemma is proven □

back

Proof of Theorem 1 (Part 1)

we begin by proving the inclusion,

$$\sigma_{ess}(L) \supset \bigcup_{x \in M,\ \pm \in \{-1,1\}} \left[-\sqrt{\max(0,-N^2)}, \sqrt{\max(0,-N^2)}\right] \cup \pm i \left[\sqrt{\max(0,\beta_-)}, \sqrt{\beta_+}\right]$$
$$= \bigcup_{(x,\xi) \in M \times \mathbb{R}^3 \setminus \{0\}} \sigma_{pt}(x,\xi) \tag{51}$$

- suppose that $\lambda \in \mathbb{C}$ is contained in $\sigma_{pt}(x_0,\xi_0)$ such that $x_0 \in M^{int}$
- there exists nonzero η orthogonal to ξ_0 such that

$$\lambda^2 P_{\xi_0}\eta + 2\lambda P_{\xi}(\Omega \times P_{\xi_0}\eta) + N^2(P_{\xi_0}\hat{g}_0' \cdot P_{\xi_0}\eta)P_{\xi_0}\hat{g}_0' = 0 \tag{52}$$

- for any $\epsilon > 0$, choose a neighbourhood $U \subset M^{int}$ of x_0 such that at all $x \in U$

$$|\lambda^2 P_{\xi_0}\eta + 2\lambda P_{\xi_0}(\Omega \times P_{\xi_0}\eta) + N^2(P_{\xi_0}\hat{g}_0' \cdot P_{\xi_0}\eta)P_{\xi_0}\hat{g}_0'| < \epsilon$$

- let $\phi \in C_c^\infty(U)$ be such that $\|\phi\|_{L^2(\rho^0\, \mathrm{d}\, x)} = 1$ and consider

$$u(x) = \eta\phi(x)e^{itx\cdot\xi_0}$$

back

Proof of Theorem 1 (Part 1)

- considering the Fourier transform, we can see that as $t \to \infty$, u converges to zero weakly
- since π_2 is a pseudodifferential operator with principal symbol given by (22), using the fact that ξ_0 is orthogonal to η, we have

$$\pi_2(u)(x) = \phi(x)e^{itx\cdot\xi_0}\eta + O\left(\frac{1}{t}\right)$$

- for t sufficiently large $\|\pi_2(u)\|_{L^2(\rho_0\,\mathrm{d}\,x)^3} > C > 0$ where C is a constant independent of t; since π_2 is continuous $\pi_2(u)$ converges weakly to zero as $t \to \infty$
- let

$$v = \frac{\pi_2(u)}{\|\pi_2(u)\|_H} \in \mathrm{Ker}(T),$$

then

$$M(\lambda)v = \frac{1}{\|\pi_2(u)\|_H}\pi_2(\lambda^2\,\mathrm{Id} + 2\lambda R_\Omega + N^2\hat{g}_0'\hat{g}_0'^T)\pi_2 u$$

and the operator on the right side is a pseudodifferential operator with principal symbol given by the map (42)

back

Proof of Theorem 1 (Part 1)

- thus

$$M(\lambda)v = \frac{1}{\|\pi_2(u)\|_H}\left(\lambda^2 P_{\xi_0}\eta + 2\lambda P_{\xi_0}(\Omega\times P_{\xi_0}\eta) + N^2(P_{\xi_0}\hat{g}_0'\cdot P_{\xi_0}\eta)P_{\xi_0}\hat{g}_0'\right)\phi(x)e^{itx\cdot\xi_0} + O\left(\frac{1}{t}\right)$$

  and so by taking t sufficiently large

$$\|M(\lambda)v\|_{L^2(\rho_0\,\mathrm{d}\,x)^3} \leq \frac{2}{\|\pi_2(u)\|_H}\epsilon$$

- since $\epsilon > 0$ was arbitrary we see that $M(\lambda)v$ converges to zero strongly and so v defines a Weyl sequence; therefore $\lambda \in \sigma_{ess}(M) = \sigma_{ess}(L)$, and this proves $\sigma_{pt}(x_0, \xi_0) \subset \sigma_{ess}(L)$ for $x_0 \in M^{int}$
- since the essential spectrum is closed and (43) is a continuous function of x once $\pm$ is chosen, for $x_0 \in \partial M$ we can take a limit from M^{int} to show $\sigma_{pt}(x_0, \xi_0) \subset \sigma_{ess}(L)$; this completes the proof of (51)

back

Proof of Theorem 1 (Part 2)

To complete the proof, we will introduce a certain system of PDEs which satisfies the **Lopatinskii conditions** if and only if λ is in the complement of the right side of (49).

- Lopatinskii satisfied $\Rightarrow$ system is a Fredholm operator $\Rightarrow M(\lambda)$ is Fredholm, $\lambda \in \sigma_{ess}(L)^c$
- the Lopatinskii conditions fail if the system is not elliptic in the interior or at the boundary
- interior ellipticity: equivalent to

$$\lambda \in \left(\bigcup_{(x,\xi)\in M\times\mathbb{R}^3\setminus\{0\}} \sigma_{pt}(x,\xi) \right)^c \tag{53}$$

  we have already shown that failure of this condition leads to existence of a Weyl sequence
- assuming interior ellipticity, we will show boundary ellipticity: equivalent to

$$\lambda \in \left(\bigcup_{x\in\partial M} i|P_n^{\perp}\hat{g}_0'| \left[-\sqrt{\max(0,N^2)}, \sqrt{\max(0,N^2)}\right] \right)^c$$

  we will show that failure of this condition also leads to existence of a Weyl sequence

Proof of Theorem 1 (Part 2)

Let us begin now deriving the PDE system:

- for any $v \in H$, consider the decomposition given by Lemma 2:

$$v = w + T^*\varphi$$

  where $w \in \mathrm{Ker}(T)$ and $\varphi \in H^1(M)$

- decompose w according the standard Helmholtz decomposition as

$$w = \nabla \times (\rho_0 w_v) + \nabla \varphi_v$$

  where $\varphi_v \in H^1(M)$ and the vector potential $\rho_0 w_v$ is in the space

$$H_{\mathrm{Curl},0}(M) = \{u \in L^2(\rho_0 \,\mathrm{d}\, x) \ : \ \nabla \times u \in L^2(\rho_0 \,\mathrm{d}\, x),\ n \times u|_{\partial M} = 0\},$$

  while also satisfying

$$\nabla \cdot (\rho_0 w_v) = 0$$

- given that M is a “ball”, a unique such decomposition exists (see [Alberti et al., 2019, Section 3])

back

Proof of Theorem 1 (Part 2)

- set $\rho_0 z_v = \nabla\varphi_v$ which must then satisfy $\nabla \times (\rho_0 z_v) = 0$
- $w \in \mathrm{Ker}(T)$ is equivalent to

$$\nabla \cdot (\rho_0 z_v) + \frac{g_0'}{c^2} \cdot \nabla \times (\rho_0 w_v) + \frac{\rho_0 g_0'}{c^2} \cdot z_v = 0, \quad n \cdot z_v|_{\partial M} = 0$$

- suppose that $u \in \mathrm{Ker}(T)$ satisfies $M(\lambda)u = f$
- as described above for v, there will be w_u and z_u such that

$$u = \nabla \times (\rho_0 w_u) + \rho_0 z_u$$

where

$$\nabla \times (\rho_0 z_u) = 0 \tag{55}$$

$$\nabla \cdot (\rho_0 w_u) = 0 \tag{56}$$

$$\nabla \cdot (\rho_0 z_u) + \frac{g_0'}{c^2} \cdot \nabla \times (\rho_0 w_u) + \frac{\rho_0 g_0'}{c^2} \cdot z_u = 0 \tag{57}$$

$$n \cdot z_u|_{\partial M} = 0 \tag{58}$$

$$n \times w_u|_{\partial M} = 0 \tag{59}$$

Proof of Theorem 1 (Part 2)

- the same equations (55)–(59) will hold for w_v and z_v constructed above for arbitrary v: let $V(\lambda) = \lambda^2 \mathrm{I} + 2\lambda R_\Omega + N^2 \hat{g}_0' \hat{g}_0'^T$ and

$$v = V(\lambda) u$$

so that $f = \pi_2 v$; (55)-(59) become

$$\nabla \times (\rho_0 w_v) + \rho_0 z_v + T^* \varphi_v = V(\lambda)(\nabla \times (\rho_0 w_u) + \rho_0 z_u) \quad (60)$$

$$f = \nabla \times (\rho_0 w_v) + \rho_0 z_v \quad (61)$$

- to make the system elliptic, we will extend it with several potentials ψ_u, ψ_v and $\widetilde{\varphi}$; setting these equal to zero, we find that the following system is satisfied:

back

Proof of Theorem 1 (Part 2)

$$\begin{pmatrix}
\frac{g_0'^T}{c^2}\nabla\times\rho_0 & \nabla\cdot\rho_0+\frac{\rho_0 g_0'^T}{c^2} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \nabla\times\rho_0 & \nabla\rho_0 & 0 & 0 & 0 & 0 & 0\\
\nabla\cdot\rho_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{g_0'^T}{c^2}\nabla\times\rho_0 & \nabla\cdot\rho_0+\frac{\rho_0 g_0'^T}{c^2} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \nabla\times\rho_0 & \nabla\rho_0 & 0 & 0\\
0 & 0 & 0 & \nabla\cdot\rho_0 & 0 & 0 & 0 & 0\\
V(\lambda)\nabla\times\rho_0 & V(\lambda)\rho_0 & 0 & -\nabla\times\rho_0 & -\rho_0 I & 0 & -T^* & 0\\
0 & 0 & 0 & \nabla\times\rho_0 & \rho_0 I & 0 & 0 & -T^*
\end{pmatrix}
\begin{pmatrix} w_u\\ z_u\\ \psi_u\\ w_v\\ z_v\\ \psi_v\\ \varphi_v\\ \widetilde{\varphi}\end{pmatrix}
=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\f\end{pmatrix} \tag{62}$$

$$n\times w_u|_{\partial M}=n\times w_v|_{\partial M}=0,\ n\cdot z_u|_{\partial M}=n\cdot z_v|_{\partial M}=\psi_u|_{\partial M}=\psi_v|_{\partial M}=0$$

back

Proof of Theorem 1 (Part 2)

- in Lemma 7, we show that the system (62) satisfies the Lopatinskii conditions when λ is in the complement of the right side of (49)
- therefore, for such λ and by when acting on $H^1(M)^{16}$ the corresponding operator is Fredholm
- considering that whenever $M(\lambda)u = f$ we have (62), we therefore conclude that $M(\lambda)$ is also Fredholm in this case
- thus these $\lambda \in \sigma_{ess}(M)^c = \sigma_{ess}(L)^c$ which shows the right inclusion for (49)

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Proof of Theorem 1 (Part 3)

key technical step in the proof of Theorem 1:

Lemma 7

Suppose that λ is in the complement of the right side of (49)*. Then the system* (62) *satisfies the Lopatinskii conditions. Furthermore, suppose*

$$\lambda \in \left(\bigcup_{x\in\partial M} i|P_n \hat{g}_0'| \left[-\sqrt{\max(0,N^2)}, \sqrt{\max(0,N^2)} \right] \right) \bigcap \left(\bigcup_{(x,\xi)\in M\times\mathbb{R}^3\setminus\{0\}} \sigma_{pt}(x,\xi) \right)^c . \quad (63)$$

Then $\lambda \in \sigma_{ess}(M)$.

Link to proof.

Lemma 7 Proof

- let the operator on the left side of (62) be labeled $\mathcal{M}(\lambda)$
- we collect the relevant operators for the boundary conditions in one large matrix

$$\mathcal{B} = \begin{pmatrix} n\times & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n\times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{64}$$

Lemma 7 Proof

- the principal symbol of $\mathcal{M}(\lambda)$ is

$$\sigma_p(\mathcal{M}(\lambda))(.,\xi) = i\rho_0 \begin{pmatrix} \frac{g_0'^T}{c^2}\xi\times & \xi^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi\times & \xi & 0 & 0 & 0 & 0 & 0 \\ \xi^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g_0'^T}{c^2}\xi\times & \xi^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi\times & \xi & 0 & 0 \\ 0 & 0 & 0 & \xi^T & 0 & 0 & 0 & 0 \\ V(\lambda)\xi\times & 0 & 0 & -\xi\times & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi\times & 0 & 0 & 0 & \xi \end{pmatrix} \tag{65}$$

- this is invertible if $V(\lambda)$ is invertible when projected onto the space orthogonal to ξ:

Lemma 7 Proof

we define

$$V_{\xi_\perp \xi_\perp}(\lambda) = P_\xi^\perp V(\lambda) P_\xi^\perp, \; V_{\xi \xi_\perp}(\lambda) = P_\xi V(\lambda) P_\xi^\perp$$

where P_ξ is the projection onto the span of ξ and $P_\xi^\perp$ the projection onto the space orthogonal to ξ

- condition (53) is equivalent to invertibility of $\widetilde{V}_\xi(\lambda) = V_{\xi_\perp \xi_\perp}(\lambda) + P_\xi$ at all points $(x, \xi) \in M \times (\mathbb{R}^3 \setminus \{0\})$

we will sometimes suppress the dependence on λ to ease the notation

Lemma 7 Proof

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we will sometimes suppress the dependence on λ to ease the notation

Lemma 7 Proof

- when it exists, the inverse of $\sigma_p(\mathcal{M}(\lambda))$ is given by

$$\sigma_p(\mathcal{M}(\lambda))(.,\xi)^{-1} = -\frac{i}{\rho_0|\xi|^2}\begin{pmatrix} 0 & 0 & \xi & 0 & 0 & 0 & -\xi\times\widetilde{V}_\xi^{-1} & -\xi\times\widetilde{V}_\xi^{-1} \\ \xi & -\xi\times & 0 & 0 & 0 & 0 & -\xi\frac{g_0'^T}{c^2}P_\xi^\perp\widetilde{V}_\xi^{-1} & -\xi\frac{g_0'^T}{c^2}P_\xi^\perp\widetilde{V}_\xi^{-1} \\ 0 & \xi^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi & 0 & -\xi\times \\ 0 & 0 & 0 & \xi & -\xi\times & 0 & 0 & -\xi\frac{g_0'^T}{c^2}P_\xi^\perp \\ 0 & 0 & 0 & 0 & \xi^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi^T(I-V_{\xi\xi_\perp})\widetilde{V}_\xi^{-1} & -\xi^T V_{\xi\xi_\perp}\widetilde{V}_\xi^{-1}P_\xi^\perp \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi^T \end{pmatrix} \tag{66}$$

back

Lemma 7 Proof

- **Lopatinskii condition** in boundary normal coordinates $(\widetilde{x}, x^3)$ where we freeze all coefficients at the central point where the Euclidean metric is the identity and write n for the inward pointing unit normal vector (WLOG the central point is the origin):

there is a unique non-zero bounded solution of the system

$$\sigma_p(\mathcal{M})(.,\widetilde{\xi} + nD_3)U = 0, \quad \mathcal{B}U = \eta \tag{67}$$

for any non-zero real $\widetilde{\xi} \in \mathbb{R}^3$ orthogonal to n and $\eta \in \mathbb{C}^8$

- assuming $\lambda \in \sigma_{pt}((\widetilde{x}, x^3), n)^c$, the ODE (67) is equivalent to

$$\frac{\mathrm{d}U}{\mathrm{d}x^3} = \underbrace{-\sigma_p(\mathcal{M})\left(.,\frac{n}{i}\right)^{-1}\sigma_p(\mathcal{M})(.,\widetilde{\xi})}_{K}\,U$$

and checking the condition amounts to analyzing the eigenvalues and eigenvectors of the matrix K on the right side of this equation

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Lemma 7 Proof

- because of (66), when the ellipticity condition is satisfied at the boundary K cannot have any eigenvalues with zero real part
- considerable calculation shows that the eigenvalues of K are $\pm|\widetilde{\xi}|$ each with algebraic multiplicity 7 and

$$\alpha_{\pm} = i|\widetilde{\xi}|\Big(n^T V_{nn_\perp} \widetilde{V}_n^{-1}\hat{\xi} + \hat{\xi}^T \widetilde{V}_n^{-1} V_{n_\perp n} n \mp \sqrt{(n^T V_{nn_\perp} \widetilde{V}_n^{-1}\hat{\xi} - \hat{\xi}^T \widetilde{V}_n^{-1} V_{n_\perp n} n)^2 - 4(\hat{\xi}\widetilde{V}_n^{-1}\hat{\xi}) n^T (V_{nn} - V_{nn_\perp}\widetilde{V}_n^{-1} V_{n_\perp n}) n}\Big)/2$$

with multiplicity 1, or possibly $\pm|\widetilde{\xi}|$ with multiplicity 8 if $\alpha_{\pm} = \pm|\widetilde{\xi}|$ (provided (53) holds, $\alpha_{\pm}$ must have non-zero real part by the ellipticity condition)

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back

Lemma 7 Proof

- we introduce the notation

$$\hat{\xi} = \frac{\widetilde{\xi}}{|\widetilde{\xi}|}, \quad n_\perp = \hat{\xi} \times n$$

- eigenvectors for $\pm|\widetilde{\xi}|$ are

$$U_{1,\pm} = \begin{pmatrix} n \pm i\hat{\xi} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, U_{2,\pm} = \begin{pmatrix} 0 \\ n \pm i\hat{\xi} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, U_{3,\pm} = \begin{pmatrix} 0 \\ n_\perp \\ \pm i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, U_{4,\pm} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ n \pm i\hat{\xi} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, U_{5,\pm} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ n \pm i\hat{\xi} \\ 0 \\ 0 \\ 0 \end{pmatrix}, U_{6,\pm} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ n_\perp \\ \pm i \\ 0 \\ 0 \end{pmatrix}$$

Lemma 7 Proof

… and there are either eigenvectors or generalized eigenvectors for $\pm|\widetilde{\xi}|$ of the form

$$U_{7,\pm} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ n_\perp \\ a_{7,\pm} n + b_{7,\pm} \hat{\xi} \\ 0 \\ \pm i \\ \mp i \end{pmatrix}$$

for some constants $a_{7,\pm}$, $b_{7,\pm} \in \mathbb{C}$

Lemma 7 Proof

... finally, either eigenvectors for $\alpha_\pm$ or generalized eigenvectors for $\pm|\tilde{\xi}|$ are given by

$$U_{8,\pm} = \begin{pmatrix} 2(\hat{\xi}^T \widetilde{V}_n^{-1} \hat{\xi}) n_\perp + a_{8,\pm} n + b_{8,\pm} \hat{\xi} \\ c_{8,\pm} n + d_{8,\pm} \hat{\xi} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{(n^T V_{nn_\perp} \widetilde{V}_n^{-1} \hat{\xi} - \hat{\xi}^T \widetilde{V}_n^{-1} V_{n_\perp n} n)}{\pm\sqrt{(n^T V_{nn_\perp} \widetilde{V}_n^{-1} \hat{\xi} - \hat{\xi}^T \widetilde{V}_n^{-1} V_{n_\perp n} n)^2 - 4(\hat{\xi}^T \widetilde{V}_n^{-1} \hat{\xi}) n^T (V_{nn} - V_{nn_\perp} \widetilde{V}_n^{-1} V_{n_\perp n}) n}} \\ 0 \end{pmatrix}$$

for some constants $a_{8,\pm}$, $b_{8,\pm}$, $c_{8,\pm}$, $d_{8,\pm} \in \mathbb{C}$

back

Lemma 7 Proof

- for the Lopatinskii condition we must restrict to the generalized eigenspace corresponding to eigenvalues with negative real part. Thus, existence of a unique bounded solution of (67) is equivalent to a unique solution $(c_1, \ldots, c_8) \in \mathbb{C}^8$ of the system

$$\mathscr{B} \sum_{j=1}^{8} c_j U_{j,-} = \eta$$

- using (64) and the equations for $U_{j,-}$ above we see that this linear system will have a unique solution if and only if $\hat{\xi}^T \widetilde{V}_n^{-1} \hat{\xi} \neq 0$; calculations show

$$\widetilde{V}_n^{-1} = P_n + \frac{1}{\lambda^4 + \lambda^2(N^2|P_n^\perp \hat{g}_0'|^2 + 4\Omega_n^2)} \Big(\lambda^2 P_n^\perp - 2\lambda\Omega_n R_n + N^2 |P_n^\perp \hat{g}_0'|^2 P_n^\perp P^\perp_{(P_n^\perp \hat{g}_0')} P_n^\perp \Big)$$

and so, since $\hat{\xi}$ is orthogonal to n,

$$\hat{\xi}^T \widetilde{V}_n^{-1} \hat{\xi} = \frac{\lambda^2 + N^2 |P_n^\perp \hat{g}_0'|^2 \hat{\xi}^T P^\perp_{(P_n^\perp \hat{g}_0')} \hat{\xi}}{\lambda^4 + \lambda^2(N^2|P_n^\perp \hat{g}_0'|^2 + 4\Omega_n^2)}$$

Lemma 7 Proof

- therefore, for λ satisfying the interior ellipticity condition (53), the Lopatinskii condition fails if and only if

$$\lambda^2 = -N^2 |P_n^\perp \hat{g}_0'|^2 \hat{\xi}^T P^\perp_{(P_n^\perp \hat{g}_0')} \hat{\xi}$$

- if $|P_n^\perp \hat{g}_0'| \neq 0$, then $\hat{\xi}^T P^\perp_{(P_n^\perp \hat{g}_0')} \hat{\xi}$ takes all values in $[0, 1]$ while if $|P_n^\perp \hat{g}_0'| = 0$ then the right side of this equation is always equal zero
- therefore, we see that the range of possible values of λ satisfying this equation is $|P_n^\perp \hat{g}_0'| [-\sqrt{-N^2}, \sqrt{-N^2}]$
- if $N^2 < 0$, this is already contained in the interior part of the essential spectrum given by the first line of (49)
- if $N^2 \geq 0$, this interval will not be contained in the interior part of the essential spectrum and is given, for a single $x \in \partial M$, by the second line in (49) (see Figure 1(a))

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Lemma 7 Proof

- it remains to show that given (63), $\lambda \in \sigma_{ess}(M)$; we will do this by showing the existence of a Weyl sequence
- by the calculations above, we see that when the Lopatinskii condition fails, for some $\widetilde{\xi}$ orthogonal to n if we set $\zeta = U_{8,-} - ib_{8,-}U_{1,-} - id_{8,-}U_{2,-}$, then we have

$$\mathscr{B}\zeta = 0$$

- since ζ is composed of eigenvectors for eigenvalues with negative real part, there will be a corresponding non-zero bounded solution U_ζ of the ODE in (67) with $U_\zeta(\widetilde{\xi}, x^3 = 0) = \zeta$
- given $\epsilon > 0$, let us choose a neighborhood Ω of x sufficiently small so that all coefficients of operator $\mathcal{M}$ vary by at most ϵ within the neighborhood, and let $\phi \in C_c^\infty(\Omega)$
- then we set

$$\mathcal{U}(x) = \phi(x) e^{it\widetilde{x}\cdot\widetilde{\xi}} U_\zeta(\widetilde{\xi}, tx^3) \tag{68}$$

which is in $H^1(M)^{16}$

Lemma 7 Proof

- with this choice of $\mathcal{U}$ we have

$$\begin{aligned}\mathcal{M}\mathcal{U} &= \mathcal{M}|_{x=0}\mathcal{U} + \epsilon\mathcal{O}(t)\\ &= it\phi(x)\sigma_p(\mathcal{M})|_{x=0}(\widetilde{\xi} + nD_3)\mathcal{U} + \epsilon\mathcal{O}(t) + \mathcal{O}(1)\\ &= \epsilon\mathcal{O}(t) + \mathcal{O}(1) \qquad \text{as } t \to \infty \text{ with norm } H^1(M)^{16}\end{aligned}$$

- now let w_u and z_u be the corresponding components of $\mathcal{U}$. Since $\hat{\xi}^T \tilde{V}_n^{-1}\hat{\xi} = 0$, in the case when $\alpha_- \neq -|\widetilde{\xi}|$ these are explicitly given by

$$\begin{aligned} w_u &= e^{t(x^3\alpha_- + i\widetilde{x}\cdot\widetilde{\xi})}(a_{8,-}n + b_{8,-}\hat{\xi}) - ib_{8,-}e^{t(-x^3|\widetilde{\xi}| + i\widetilde{x}\cdot\widetilde{\xi})}(n - i\hat{\xi}),\\ z_u &= e^{t(x^3\alpha_- + i\widetilde{x}\cdot\widetilde{\xi})}(c_{8,-}n + d_{8,-}\hat{\xi}) - id_{8,-}e^{t(-x^3|\widetilde{\xi}| + i\widetilde{x}\cdot\widetilde{\xi})}(n - i\hat{\xi}) \end{aligned} \tag{69}$$

- in the case that $\alpha_- = -|\widetilde{\xi}|$ and $U_{8,-}$ is a generalized eigenvector, these are replaced by

$$\begin{aligned} w_u &= e^{t(-x^3|\widetilde{\xi}| + i\widetilde{x}\cdot\widetilde{\xi})}(a_{8,-} - ib_{8,-})n + tx^3e^{t(-x^3|\widetilde{\xi}| + i\widetilde{x}\cdot\widetilde{\xi})}(n - i\hat{\xi}),\\ z_u &= e^{t(-x^3|\widetilde{\xi}| + i\widetilde{x}\cdot\widetilde{\xi})}(c_{8,-} - id_{8,-})n + tx^3e^{t(-x^3|\widetilde{\xi}| + i\widetilde{x}\cdot\widetilde{\xi})}(n - i\hat{\xi}) \end{aligned} \tag{70}$$

Lemma 7 Proof

- considering the first component of (62), we have $\nabla \times (\rho_0 w_u) + \rho_0 z_u \in D(T)$ and

$$T(\nabla \times (\rho_0 w_u) + \rho_0 z_u) = \epsilon \mathcal{O}(t) + \mathcal{O}(1).$$

- by the construction of π_2 described just above Lemma 3, we have

$$\pi_2(\nabla \times (\rho_0 w_u) + \rho_0 z_u) = \nabla \times (\rho_0 w_u) + \rho_0 z_u + \epsilon \mathcal{O}(t) + \mathcal{O}(1)$$

  with the H norm
- we set $u = \pi_2(\nabla \times (\rho_0 w_u) + \rho_0 z_u) \in \mathrm{Ker}(T)$
- using the last and second to last lines in (62) and the fact that most components of $\mathcal{U}$ are zero, we obtain

$$M(\lambda)u = \epsilon \mathcal{O}(t) + \mathcal{O}(1)$$

back

Lemma 7 Proof

- to construct a Weyl sequence, we need to normalize u, and so we consider $\|\nabla \times (\rho_0 w_u) + \rho_0 z_u\|_H$
- in the case $U_{8,-}$ is not a generalized eigenvector, using (69) we see

$$\nabla \times (\rho_0 w_u) + \rho_0 z_u = t e^{t(x^3 \alpha_- + i\widetilde{x}\cdot\widetilde{\xi})} \left(-\frac{\alpha_-}{|\widetilde{\xi}|} b_{8,-} + i a_{8,-} \right) |\widetilde{\xi}| n_\perp \\ + t e^{t(-x^3|\widetilde{\xi}| + i\widetilde{x}\cdot\widetilde{\xi})} i b_{8,-} |\widetilde{\xi}| n_\perp + \mathcal{O}(1)$$

- since, from the calculation constructing $U_{8,-}$, we know that $a_{8,-}$ and $b_{8,-}$ cannot simultaneously vanish, from this last formula we see that

$$\|u\|_H = \|\nabla \times (\rho_0 w_u) + \rho_0 z_u\|_H + \epsilon \mathcal{O}(t) + \mathcal{O}(t) \approx \mathcal{O}(t)$$

(by this notation, we mean that $\|u\|_H$ is bounded below by Ct as $t \to \infty$ for some constant $C > 0$)

Lemma 7 Proof

- a similar calculation beginning with (70) proves the same result when $U_{8,-}$ is a generalized eigenvector
- therefore

$$M(\lambda)\frac{u}{\|u\|_H} = \epsilon\mathcal{O}(1) + \mathcal{O}(t^{-1})$$

  and so by choosing t sufficiently large we can obtain a sequence $v_\epsilon = u/\|u\|_H \in \mathrm{Ker}(T)$ with H-norm equal to one and such that $M(\lambda)v_\epsilon \to 0$ as $\epsilon \to 0$
- because of the oscillatory nature of (68), it is also clear that v_ϵ converges weakly to zero, meaning it is a Weyl sequence and so $\lambda \in \sigma_{ess}(M)$

this completes the proof □

Terrestrial weak form

we consider the weak form of (1), which can be formulated in the setting of $E \hookrightarrow H \hookrightarrow E'$ on $(0, T)$ for $T > 0$ as follows:

$$\begin{aligned} &u \text{ is a weak solution of (1) if } u \in \mathcal{C}^0([0,T];E), \dot{u} \in \mathcal{C}^0([0,T];H) \text{ and} \\ &\forall v \in E : \quad \frac{d}{dt}(\partial_t u, v)_H + (2R_\Omega \partial_t u, v)_H + a_2(u,v) = 0 \text{ in } \mathscr{D}'(0,T) \end{aligned} \tag{12}$$

back

Terrestrial weak form

$$
\begin{aligned}
a_2(u,v) :=& \int_{\Omega_S} \Big((\Lambda^{T^0} : \nabla u) : \nabla \overline{v} + \sigma_N \nabla u : \nabla \overline{v}^T - \sigma_N (\nabla \cdot u)(\nabla \cdot \overline{v}) \Big)\, dV \\
&+ \int_{\Omega_S} \Big(-\mathfrak{S}\{(g_0' \cdot u)(\overline{v} \cdot \nabla \rho^0)\} + \mathfrak{S}\{ -(\nabla \sigma_N + \rho^0 g_0') \cdot u (\nabla \cdot \overline{v})\} \\
&+ \mathfrak{S}\{(\nabla \sigma_N - \rho^0 g_0') \cdot \nabla u \cdot \overline{v}\} \Big)\, dV \\
&+ \int_{\Omega_F} \Big(\frac{p^0 \gamma}{(\rho^0)^2} \Big(\nabla \cdot (\rho^0 u) - \tilde{s} \cdot u \Big) \Big(\nabla \cdot (\rho^0 v) - \tilde{s} \cdot \overline{v} \Big) \\
&- \tilde{s} \cdot g_0' \frac{(g_0' \cdot u)(\overline{v} \cdot g_0')}{\|g_0'\|^2} \Big)\, dV - \int_{\Sigma_{FS}} \mathfrak{S}\Big\{ (\overline{v} \cdot \nu) \Big(u_+ \cdot [\rho^0]_-^+ g_0' \Big) \Big\} d\Sigma \\
&- \frac{1}{4\pi G} \int_{\mathbb{R}^3} \nabla S(u) \cdot \nabla S(\overline{v})\, dV + \int_{\partial M} \mathfrak{S}\{\rho^0 (u \cdot g_0') \overline{v} \cdot \nu\}\, d\Sigma
\end{aligned}
\qquad (13)
$$

back

Terrestrial weak form

- $\mathfrak{S}$ denotes symmetrisation in u and v,

$$\tilde{s} = \nabla\rho^0 + \frac{g_0'(\rho^0)^2}{p^0\gamma},$$

- σ_N is a regular scalar function which is equal to $-p^0$ in Ω^F and 0 outside of a small neighborhood of Ω^F
- $\pm$ indicates limits from either side of the interface where the "$+$" side is chosen in the same direction as the unit normal vector ν to the interface
- along Σ_{FS}, ν is chosen to point from the fluid region to the solid region so that u_+ is the limit from the solid region, which is well defined since $u \in C^0([0,T];E)$

back

Terrestrial weak form

For the weak form to be well-posed we make the following assumptions.

- there exists $\mathfrak{c} > 0$ so that for all 2-tensors η_{ij}

$$\mathfrak{c}|\eta_{ij} + \eta_{ji}|^2 \leq (\Xi_{ijkl} - p^0 \delta_{ik}\delta_{jl})\eta_{kl}\overline{\eta}_{ij} \tag{14}$$

  in the solid region Ω_S
- $\Xi_{ijkl} \in L^\infty(M)$
- ρ^0 is piece-wise $W^{1,\infty}$ with discontinuities only on the boundary Σ_{FS} and is bounded away from zero
- $p^0 \in L^\infty(M)$ with p^0 bounded away from zero, and $\nabla p^0 \in L^\infty(U)$ for U a neighborhood of Σ^{FS}
- $g_0' \in L^M(\tilde{X})$ with $\|g_0'\|$ bounded away from zero
- $\gamma \in L^\infty(\Omega^F)$ with γ bounded away from zero

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back

Terrestrial weak form; coercivity

with these assumptions a_2 is bounded on E and for $\|\tau^0\|_{L^\infty(\tilde{X})}$ sufficiently small, σ_N such that

$$\|\sigma_N\|_{L^\infty(\Omega^S)} \leq \|\tau_0\|_{L^\infty(\tilde{X})}, \text{ and } \|\nabla\sigma_N\|_{L^\infty(\Omega^S)} \leq \|\nabla p^0\|_{L^\infty(U)}$$

there exist $\alpha, \beta > 0$ such that

$$a_2(u,u) \geq \alpha\|u\|_E^2 - \beta\|u\|_H^2, \ \forall u \in E \tag{15}$$